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Extreme points in tensor products and a theorem of de Leeuw

by

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In this paper we show how an idea due to de Leeuw (see [2], lemma 3.3) can be adapted to vector-valued measures to yield a result concerning extreme points of the dual ball of a tensor product of Banach spaces (Theorem 1.1). In Section 2 we give an elementary non-measure-theoretic proof of de Leeuw's lemma which yields a stronger result than the original statement. This idea is then applied to obtain results on exposed points of the dual ball of a tensor product of Banach spaces.

1. We will denote the dual of a normed space E by E', the unit ball of E by U_E, and the set of extreme points of a convex set K by ext (K).

If E and F are normed spaces, then $E\otimes F$ denotes their (algebraic) tensor product, $E\otimes_\alpha F$ denotes $E\otimes F$ endowed with a crossnorm α (see [7]) and $E\tilde{\otimes}_\alpha F$ the completion of $E\otimes_\alpha F$.

We will restrict our attention to the crossnorm λ (see [7]) for the following reason: every extreme point of $U_{E\otimes\alpha F}$ is of the form $x'\otimes y'$, $x'\in E'$, $y'\in F'$, if and only if $\alpha=\lambda$. This follows from the definition of λ and the K^2-M^3-R theorem (see [1], p. 80).

If S is a compact Hausdorff space and E a Banach space, we let $C_E(S)$ denote the E-valued continuous functions on S with sup-norm and $C(S)$ denote the scalar-valued continuous functions. If $f\in C(S)$ and $x\in E$, $f\cdot x\colon s\to f(s)x$ defines a function in $C_E(S)$. If F is considered as a subspace of $C(S)$, then $E\otimes_\lambda F$ is isometrically embedded in $C_E(S)$ by the canonical linear mapping which sends $x\otimes f$ to $f\cdot x$.

X will denote a subspace of $C_E(S)$, and for $s\in S$, let $\Phi_s\colon X\to E$ be defined by $\Phi_s(f)=f(s)$. For each $x'\in E'$, the composition $x'\circ\Phi_s$ of x' and Φ_s is in X'.

LEMMA 1.1. $\{x'\circ\Phi_s : x'\in U_{E'},\ s\in S\}$ is weak* compact in $U_{X'}$.

Proof. If E' and X' are given their weak* topologies then the mapping $(x',s)\to x'\circ\Phi_s$ is continuous from $U_{E'}'\times S$ into X', q. e. d.

LEMMA 1.2. *Every extreme point of $U_{X'}$ is of the form $x'\circ\Phi_s$, where $x'\in U_{E'}$ and $s\in S$.*

Proof. Since $\|f\|_\infty = \sup\{|(x'\circ\Phi_s)(f)|: x'\in U_{E'}, s\in S\}$, the conclusion follows from lemma 1.1 and the K^2-M^3-R theorem (see [1], p. 80), q. e. d.

Remark. Lemma 1.2 is a vector-valued generalization of [4], lemma 6, p. 441.

PROPOSITION 1.3. *If $\Phi_s(X)$ is dense in E and if $x'\circ\Phi_s$ is an extreme point of $U_{X'}$, then $x'\in$ ext$(U_{E'})$.*

Proof. Since $\Phi_s(X)$ is dense in E, the adjoint Φ_s^* of Φ_s is a linear isometry of E' into X' and hence, an affine homeomorphism of $U_{E'}$ onto $\{x'\circ\Phi_s: x'\in U_{E'}\}$, q. e. d.

Following the technique of de Leeuw in [2], lemma 3.3, we will next establish a condition under which all functionals of the form $x'\circ\Phi_s$, $x'\in$ ext$(U_{E'})$, $s\in S$, are extreme points of the dual ball of a tensor product. First we need the following easy lemma. For notation and basic facts concerning vector-valued measures, see [3].

LEMMA 1.4. *Let $f\in C(S)$, $\|f\|_\infty = 1$. Let $m\in C_E(S)'$ (i. e., m is an E'-valued regular Borel measure on S) with $\|m\| = \tilde{m}(S) = \overline{m}(S) = 1$ (1).*

If $\|\int f\,dm\| = 1$, then $\overline{m}$ $(\{t: |f(t)|<1\})$ (and hence m $(\{t: |f(t)|<1\})$) is zero.

Proof. $\int|f|\,d\overline{m} \geq \|\int f\,dm\| = 1$. Also $0\leq|f|\leq 1$ and $\overline{m}(S) = 1$, so $\int|f|\,d\overline{m}\leq 1$. Hence $\int|f|\,d\overline{m} = 1$. Let $g = 1-|f|$. Then $0\leq g\leq 1$, $\overline{m}$ is a positive measure on S and $\int g\,d\overline{m} = 0$. Thus, $g = 0$, $\overline{m}$-a. e., q. e. d.

THEOREM 1.1. *Let A be a subspace of $C(S)$, E a Banach space, and $X = A\otimes_\lambda E\subset C_E(S)$. If there is a function in A that peaks at s relative to A, then $x'\circ\Phi_s\in$ ext$(U_{X'})$ for each $x'\in$ ext$(U_{E'})$.*

Proof. Suppose φ_1 and $\varphi_2\in U_{X'}$ with $x'\circ\Phi_s = \frac{1}{2}(\varphi_1+\varphi_2)$. By the Hahn-Banach theorem, each φ_i extends to an element of $C_E(S)'$ of norm one and is, therefore, represented by an E'-valued regular Borel measure m_i on S with $1 = \|\varphi_i\| = \tilde{m}_i(S) = \overline{m}_i(S)$. $x'\circ\Phi_s$ is represented by the measure m such that $m(T) = x'$ if $s\in T$ and $m(T) = 0$ if $s\notin T$. Thus, for each $f\in X$,

$$\int f\,dm = \tfrac{1}{2}\int f\,dm_1 + \tfrac{1}{2}\int f\,dm_2.$$

If $g\in A$ and $n = m, m_1$, or m_2, then $\langle x, \int g\,dn\rangle = \int g\cdot x\,dn$ for $x\in E$. Hence for each $g\in A$,

$$\int g\,dm = \tfrac{1}{2}\int g\,dm_1 + \tfrac{1}{2}\int g\,dm_2$$

(as elements of E'). Now suppose $h\in A$ peaks at s relative to A. Then

$$x' = h(s)x' = \int h\,dm = \tfrac{1}{2}\int h\,dm_1 + \tfrac{1}{2}\int h\,dm_2.$$

(1) By [3], prop. 4, p. 54, the variation and $\overline{m}$ semi-variation $\tilde{m}$ of m are equal since m is E'-valued.

But $x'\in$ ext$(U_{E'})$, so $\int h\,dm_i = x'$, $i = 1, 2$. Let $S_+ = \{t\in S: h(t) = 1\}$, $S_- = \{t\in S: h(t) = -1\}$, and $S_0 = \{t\in S: |h(t)|<1\} = S\sim(S_+\cup S_-)$. Then $m_i(S_0) = 0$ $(i = 1, 2)$ by lemma 1.4. But

$$x' = \int h\,dm_i = \int_{S_+} h\,dm_i + \int_{S_-} h\,dm_i + \int_{S_0} h\,dm_i$$
$$= \int_{S_+} dm_i - \int_{S_-} dm_i = m_i(S_+) - m_i(S_-), \quad i = 1, 2.$$

Now, by definition of "peaking", $S_+ = \{t\in S: g(t) = g(s)$ for all $g\in A\}$ and, since $A\otimes_\lambda E$ is dense in X, $S_+ = \{t\in S: f(t) = f(s)$ for all $f\in X\}$. Likewise, $S_- = \{t\in S: f(t) = -f(s)$ for all $f\in X\}$. Thus, for each $f\in X$, we have

$$\varphi_i(f) = \int f\,dm_i = \int_{S_+} f\,dm_i + \int_{S_-} f\,dm_i + \int_{S_0} f\,dm_i$$
$$= \langle f(s), m_i(S_+)\rangle + \langle -f(s), m_i(S_-)\rangle$$
$$= \langle f(s), m_i(S_+) - m_i(S_-)\rangle$$
$$= x'(f(s))$$
$$= x'\circ\Phi_s(f), \quad i = 1, 2,$$

q. e. d.

Remark. This characterizes the extreme points of the dual ball of lip$_E(S, d^\alpha)$, $0<\alpha<1$, under certain conditions (see [5] and [6]).

2. We now give the stronger version of de Leeuw's original result.

THEOREM 2.1. *Let A be a subspace of $C(S)$, S compact Hausdorff. Suppose there is a function f in A that peaks at $s\in S$ relative to A. If $\varphi\in U_{A'}$ and $\varphi(f) = 1$, then $\varphi = \varepsilon_s(\varepsilon_s(g) = g(s)$ for $g\in A)$. In other words, ε_s is a weak* exposed point of $U_{A'}$ and a fortiori an extreme point.*

Proof. Since f peaks at s relative to A, we have $\|f\|_\infty = f(s) = 1$ and, if $|f(t)| = 1$, then $\varepsilon_t = \pm\varepsilon_s$. Now, let $H = \{\varphi\in A': \|\varphi\| = \varphi(f) = 1\}$. It is easy to see that if $\varphi\in$ ext(H), then $\varphi\in$ ext$(U_{A'})$ and hence is of the form $\lambda\varepsilon_t$ for some $t\in S$ and $|\lambda| = 1$. But $\lambda\varepsilon_t\in H\Rightarrow\lambda f(t) = 1\Rightarrow|f(t)| = 1\Rightarrow\varepsilon_t = \pm\varepsilon_s$. Thus, every extreme point of H is of the form $\lambda\varepsilon_s$, $|\lambda| = 1$. But $\lambda\varepsilon_s\in H\Rightarrow\lambda f(s) = \lambda = 1$. Thus H has but one extreme point, ε_s. But H is a weak* compact convex set in A' and is therefore the closed convex hull of its extreme points. Hence, $H = \{\varepsilon_s\}$, q. e. d.

Remark. This result shows that every extreme point of the dual ball of lip (S, d^α), $0<\alpha<1$, is a weak* exposed point (see [2] and [5]).

The idea of the above proof of theorem 2.1 applies to subspaces X of $C_E(S)$ as follows:

THEOREM 2.2. *Suppose that, given any $x\in E$ with $\|x\| = 1$, there is an $f\in X$ such that $\|f\|_\infty = 1$, $f(s) = x$, and $\|f(t)\| = 1\Rightarrow\Phi_t = \pm\Phi_s$. If x'*

is a weak exposed point of $U_{E'}$, then $x' \circ \Phi_s$ is a weak* exposed point of $U_{X'}$.*

Proof. Since x' is a weak* exposed point of $U_{E'}$, we can find $x \in E$, $\|x\| = 1$, such that $\{y: \|y'\| = y'(x) = 1\} = \{x'\}$. Let $f \in X$ satisfy the hypotheses above. Define $H = \{\varphi: \varphi(f) = \|\varphi\| = 1\}$. If $y' \circ \Phi_t \in H$, then $\|y'\| \leqslant 1$ and $y'(f(t)) = 1$, so $\|f(t)\| = 1$. Hence, $\Phi_t = \pm\Phi_s$ and $f(t) = \pm f(s) = \pm x$. Thus, $y'(x) = \pm 1$, so $y' = \pm x'$. Hence, the only extreme point of H is $x' \circ \Phi_s$.

Again, since H is the weak* closed convex hull of $\mathrm{ext}(H)$, $H = \{x' \circ \Phi_s\}$.

COROLLARY 2.3. *Let A be a subspace of $C(S)$ and $X = A \otimes_\lambda E$ (or $A \tilde{\otimes}_\lambda E$). If there is a function in A that peaks at s relative to A and if x' is a weak* exposed point of $U_{E'}$, then $x' \circ \Phi_s$ is a weak* exposed point of $U_{X'}$.*

Proof. Let $x \in E$, $\|x\| = 1$. Let $g \in A$ peaks at s relative to A. Then $f = g \cdot x$ satisfies the hypotheses of theorem 2.2, q. e. d.

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On positive functionals on a group algebra multiplicative on a subalgebra

by

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This paper was motivated by two independent facts. One, observed by Thoma [13], was that if G is a discrete group in which every element has finitely many conjugates and $\mathfrak{z}$ is the center of the l_1-group algebra of G, then a class-function which defines a multiplicative functional on $\mathfrak{z}$ is positive-definite because $\mathfrak{z}$ is symmetric. The other fact, observed by M. Moskovitz (oral communication), was that if G is a locally compact group, K a compact subgroup of G, then any bounded K-spherical function on G is positive-definite, if $L_1(G)$ is symmetric.

Of course, if $L_1(G)$ is symmetric, then any Banach *-subalgebra of it is symmetric. Thus if one knows that $L_1(G)$ is symmetric, one can establish the positive-definiteness of certain functions by means of the facts revealed above. However, to decide whether $L_1(G)$ is symmetric may be difficult even for such simple groups as the groups of motions (cf. [1]). The aim of this note is to propose a property which resembles symmetry of a Banach *-algebra and which, on one hand, is much easier to prove for $L_1(G)$ for a large and natural (cf. [2], [5], [8], [9], and [14]) class of locally compact groups G and, on the other hand, implies the positiveness of multiplicative functionals on a *-subalgebra of $L_1(G)$ in which the functions with compact support are dense.

This will lead to two theorems in section 4, one of which asserts that if G is $[FC^-]$, then the set of extreme positive-definite, normalized class functions is equal to the set of the multiplicative functionals on the center of $L_1(G)$. Under a more restrictive assumption a similar result has been recently obtained on another way by H. Kaniuth. The other implies e. g. that the spherical functions on a group which is an extension of a nilpotent group by a compact group are positive-definite.

The paper is organized as follows. Section 1 is devoted to a theorem on Banach *-algebras and the crucial property (A) which implies that multiplicative functionals are positive. In section 2 we turn to group