STUDIA MATHEMATICA, T. XXXVII. (1971)

An algebraic view of distributions and operators*

by

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Dedicated to the memory of John W. Cell

In this paper we obtain an algebraic characterization of Schwartzian distributions of one variable and establish some intimate connections between these distributions and Mikusiński operators. The writer is indebted to Kwangil Koh for many helpful discussions in connection with this work.

Let C^∞ denote the vector space of infinitely differentiable, complex-valued functions φ on $-\infty < t < \infty$, and let D denote the subspace consisting of those functions with compact support. If $\varphi \in C^\infty$ and $\psi \in D$, then the convolution $\sigma = \varphi * \psi$ is the function

$$\sigma(t) = \int_{-\infty}^{\infty} \varphi(\tau)\psi(t-\tau)\,d\tau$$

and belongs to C^∞. If $\varphi, \psi \in D$, then $\sigma \in D$. Moreover, under addition and convolution, D becomes a commutative ring and, according to Titchmarsh's theorem on convolution [6], it is devoid of zero divisors.

A mapping $F\colon D \to C^\infty$ will be called an *operator homomorphism* if it satisfies

(i) $F(\varphi * \psi) = F(\varphi) * \psi$ for all $\varphi, \psi \in D$.

Let f be a distribution on $-\infty < t < \infty$ [7] and let $\varphi \in D$. Then the convolution $f * \varphi$ of f and φ is the function $\sigma(t) = \langle f(\tau), \varphi(t-\tau)\rangle$ and belongs to C^∞. Moreover, the mapping $f\colon \varphi \to \sigma$ satisfies (i) and, thus, we have the result

LEMMA 1. *A distribution f, acting on D under convolution, is an operator homomorphism. If f_1 and f_2 are distinct distributions, then $f_1 * \varphi \neq f_2 * \varphi$ for some $\varphi \in D$ and so f_1 and f_2 are distinguished by their action on D.*

If f is a distribution with support bounded on the left or has compact

* Work supported by the U. S. Army Research Office, Durham.

support and $\varphi \in D$, then $\sigma = f*\varphi$ has support bounded on the left or has compact support. Thus, we also have

LEMMA 2. *If f is a distribution with support bounded on the left or has compact support, then as an operator homomorphism, f maps D into C_R^∞ or D, where C_R^∞ denotes the subspace of C^∞ consisting of all functions with supports bounded on the left (right-sided functions). Moreover, these range subspaces determine the character of the support of the corresponding distributions.*

The space of all distributions will be denoted by D', the subspace of all distributions with supports bounded on the left (right-sided distributions) by D'_R and the subspace of all distributions with compact supports (compact distributions) by E'.

We now prove an easy, but rather surprising result, which seems to have been overlooked.

LEMMA 3. *If F is an operator homomorphism, then F satisfies* (ii), *$rF(\varphi)+sF(\psi) = F(r\varphi+s\psi)$ for all complex r and s and all $\varphi, \psi \in D$. Moreover, there exists a distribution f such that $f*\varphi = F(\varphi)$ for all $\varphi \in D$.*

PROOF. We prove the latter conclusion first and then the former follows immediately from the properties of the convolution for distributions. Let F be an operator homomorphism and let $\{\delta_n\}$ be any "delta-function" sequence in D. Because F satisfies (i), for any $\varphi \in D$ and any n we have $F(\varphi)*\delta_n = F(\varphi*\delta_n) = F(\delta_n*\varphi) = F(\delta_n)*\varphi$ and so, $F(\varphi) = \lim F(\delta_n)*\varphi$. When these latter functions are evaluated at zero, the sequence $F(\delta_n)$ (considered as a sequence of regular distributions) defines a linear functional f on D in the usual weak sense for distributions. Such weak limits are known [7] to be distributions. Clearly, this limit distribution f satisfies $F(\varphi) = f*\varphi$ for all $\varphi \in D$ and thus, in particular, the operator homomorphism F satisfies (ii).

Because property (i) implies property (ii), we have taken the liberty of using the standard terminology "homomorphism" in this context and we shall hereafter denote the vector space of all such homomorphisms by the standard symbol for it, $\mathrm{Hom}_D(D, C^\infty)$, i. e. the vector space of all homomorphisms of the D-module D into the D-module C^∞ [2]. The reason this lemma is surprising is that, together with Lemma 1, it yields a completely *algebraic* characterization of distributions. We summarize with a statement of the first main result of this paper.

THEOREM 1. *Each distribution is (uniquely) characterized as a mapping of D into C^∞ which commutes with convolution. Moreover, the vector space D' of all distributions is isomorphic to the vector space $\mathrm{Hom}_D(D, C^\infty)$ of all operator homomorphisms.*

The vector space D' of distributions is normally endowed with the topology of pointwise convergence (for numerical-valued sequences) on D. Similarly, we may endow the vector space $\mathrm{Hom}_D(D, C^\infty)$ of homomorphisms with the topology of pointwise convergence, wherein a sequence in C^∞ will be said to be *convergent* in C^∞ if the sequence, together with each of its derived sequences, converges uniformly on compact subsets of the real line. It is not difficult to show that the two vector spaces are then homeomorphic as well as isomorphic and that each operator homomorphism is a continuous map [7]. (A sequence converges in D if it converges in C^∞ and the supports are uniformly bounded.) This brings us to the second main result, which we now consider.

THEOREM 2. *If F is an operator homomorphism, then F satisfies* (iii), *$\lim \varphi_n = \varphi$ in D implies $\lim F(\varphi_n) = F(\varphi)$ in C^∞. Moreover, the vector spaces D' and $\mathrm{Hom}_D(D, C^\infty)$ endowed with the topologies of pointwise convergence are homeomorphic.*

PROOF. Using the correspondence (algebraic isomorphism) for which $F(\varphi) = f*\varphi$, property (iii) follows readily from (ii) and the local boundedness property for distributions. Using this same correspondence, it is clear that $F_n \to F$ in $\mathrm{Hom}_D(D, C^\infty)$ implies $f_n \to f$ in D'. To prove the converse, we assume that we have a sequence of distributions such that $f_n \to f$ in D' and $F_n \nrightarrow F$ in $\mathrm{Hom}_D(D, C^\infty)$, for the corresponding homomorphisms. Then for some $\varphi \in D$ we have $F_n(\varphi) \nrightarrow F(\varphi)$ in C^∞ and so there exists a natural number k and a compact subset K of real numbers such that the τ-function sequence $\{\langle f_n(t), \varphi^{(k)}(\tau-t)\rangle\}$ does not converge uniformly on K to the τ-function $\langle f(t), \varphi^{(k)}(\tau-t)\rangle$. (Here, $\varphi^{(k)}$ denotes the ordinary k^{th} derivative of φ and we have suppressed the superfluous sign factor $(-1)^k$ in the k^{th} derivative of the τ-functions.) Hence for some $\varepsilon > 0$, there exists a subsequence $\{\nu_n\}$ of natural numbers and a sequence $\{\tau_n\}$ in K such that

$$|\langle f_{\nu_n}(t), \varphi^{(k)}(\tau_n-t)\rangle - \langle f(t), \varphi^{(k)}(\tau_n-t)\rangle| = |\langle f_{\nu_n}(t)-f(t), \varphi^{(k)}(\tau_n-t)\rangle| \geqslant \varepsilon \quad \text{for all } n.$$

Since K is compact we can assume that $\tau_n \to \tau \in K$ and it readily follows that

$$\psi_n(t) = [\varphi^{(k)}(\tau-t)-\varphi^{(k)}(\tau_n-t)] \to 0 \quad \text{in } D.$$

Accordingly, we have,

$$\begin{aligned} 0 &= \lim \langle f_{\nu_n}(t)-f(t), \psi_n(t)\rangle \\ &= \lim \langle f_{\nu_n}(t)-f(t), \varphi^{(k)}(\tau-t)\rangle + \lim \langle f_{\nu_n}(t)-f(t), \varphi^{(k)}(\tau_n-t)\rangle \\ &= \lim \langle f_{\nu_n}(t)-f(t), \varphi^{(k)}(\tau_n-t)\rangle, \end{aligned}$$

and this contradicts the above ε-inequality.

Hereafter, whenever distributions are considered as homomorphic mappings they shall be called *distributional operators*. These are the most important operators in the operational calculus. They include all the ordinary functions, the differentiation, the integration and the translation operators. It is now an easy step to obtain the field of operators introduced by Mikusiński. Before doing this, however, we shall first describe another operator field which arises in a more natural manner here.

Let us consider an arbitrary commutative ring P without divisors of zero. In the following manner we can associate with P a field $E(P)$ isomorphic to the field of quotients for the ring P. A mapping $\mathfrak{f}\colon I \to P$ with domain a non-zero ideal $I \subseteq P$ is called a *partial homomorphism* iff $\mathfrak{f}(a\cdot c) = \mathfrak{f}(a)\cdot c$ and $\mathfrak{f}(a-b) = \mathfrak{f}(a) - \mathfrak{f}(b)$ for all $a, b \in I$ and $c \in P$. Every non-zero partial homomorphism $\mathfrak{f}$ is an injective mapping whose image is also a non-zero ideal. If $\mathfrak{f}$ is not the restriction to I of any partial homomorphism defined on a (properly) larger ideal of P, then $\mathfrak{f}$ is said to be *maximal*. A partial homomorphism $\mathfrak{f}$ on I is readily extended to a (unique) maximal one $\mathfrak{f}'$ by defining $\mathfrak{f}'(b) = c$ whenever $\mathfrak{f}(b\cdot a) = c\cdot a$ for all $a \in I$. In the collection $E(P)$ of maximal partial homomorphisms of the ring P, we can define two internal operations for which $E(P)$ has a field structure. Namely, $\mathfrak{f}+g = h'$, where h is the ordinary sum of the restrictions of $\mathfrak{f}$ and g to (domain $\mathfrak{f}$)$\cdot$(domain g) and $\mathfrak{f}\cdot g = k'$, where k is the ordinary composition of $\mathfrak{f}$ and the restriction of g to (domain $\mathfrak{f}$)$\cdot$(domain g). These definitions are proper since the product of two non-zero ideals is a non-zero ideal. It is easily seen that $(E(P), +, \cdot)$ is a field containing an isomorphic image of P (the extended centralizer of P over the regular P-module P [4]).

If we apply the above construction to the ring D we obtain a field which we denote by M_0. According to Lemma 2, the space E' of compact distributional operators represents a subalgebra (wherein multiplication corresponds to convolution) of the field M_0 which, according to Lemma 3, consists of those operators each of whose domain is the entire ring D, i. e., $\mathrm{Hom}_D(D, D)$. The field M_0 also contains the differentiation, integration and translation operators and is isomorphic to a proper subfield of the Mikusiński operator field.

To obtain the latter we apply the above construction to the ring C_R^∞ of right-sided, infinitely differentiable functions, which is closed under convolution and is also without divisors of zero. The resulting field, i. e., the extended centralizer $E(C_R^\infty)$ of C_R^∞ over the regular C_R^∞-module C_R^∞, is isomorphic to the Mikusiński operator field M [9]. The latter is usually interpreted [6] as the field of quotients for the ring C of continuous functions on $[0, \infty]$ under addition and convolution. The right-sided distributions can be defined (using convolution) as operators on C_R^∞ and in this way D'_R is imbedded as a subalgebra in the Mikusiński field M. This subalgebra consists of those operators each of whose domain is the entire ring C_R^∞ [9], i. e., $\mathrm{Hom}_{C_R^\infty}(C_R^\infty, C_R^\infty)$. If we view M as the field of quotients of C_R^∞ and M_0 as the field of quotients of D, then since $D \subset C_R^\infty$, there is a natural imbedding of M_0 in M. However, there are elements of M (for example, any step function on the half line $[0, \infty)$ with ever increasing length of steps to the right) which correspond to no elements of M_0 under this natural imbedding. An alternative procedure is to use the commutative ring D'_R, which is closed under convolution and is also devoid of zero divisors, and to obtain a representation of the Mikusiński operator field M as the collection of all maximal partial homomorphisms in the ring D'_R, i. e., $E(D'_R)$. It seems appropriate at this point to summarize these latter observations with a statement of the third main result.

THEOREM 3. *The algebras* $\mathrm{Hom}_D(D, D)$ *and* $\mathrm{Hom}_{C_R^\infty}(C_R^\infty, C_R^\infty)$ *of homomorphisms are, respectively, isomorphic to the algebras* E' *and* D'_R *of compact and right-sided distributions wherein multiplication corresponds to convolution. The fields* $E(C_R^\infty)$ *and* $E(D'_R)$ *of maximal partial homomorphisms in* C_R^∞ *and* D'_R *are isomorphic to each other and to the Mikusiński operator field* M. *The field* $M_0 = E(D)$ *of maximal partial homomorphisms in* D *is isomorphic to a proper subfield of* M.

It may be of some interest to point out that we can easily recover (construct) the space D' of all distributions starting with the Mikusiński operator field M. Indeed, we need only have at our disposal those Mikusiński operators represented by the subalgebra D'_R of right-sided distributions, i. e. those operators each of whose domain is all of C_R^∞. To obtain any arbitrary distributional operator $f \in D'$, we select $\psi_n(t) \in C_R^\infty$ with support in $(-n-1, \infty)$ and having the value one on $[-n, \infty)$ for $n = 1, 2, \ldots$ Then, in the usual distributional sense, for each n, ψ_n is a multiplier in D', and the corresponding right-sided distribution $f_n = \psi_n f$ agrees with f on $(-n, \infty)$. It follows that for each $\varphi \in D$ and each compact subset K of real numbers, the infinitely differentiable functions $f_n * \varphi$ and $f*\varphi$ agree on K provided n is sufficiently large. Thus the distributional operator $f \in D'$ can be characterized as the weak limit in $\mathrm{Hom}_D(D, C^\infty)$ of the above sequence $\{f_n\}$ of distributional operators in D'_R.

The interesting question arises as to what topological structures can be imposed upon the operator field M. Mikusiński has defined [6] a sequential convergence in M which has recently been generalized to nets and filters [10]. However, it turns out that there is no topology on M for which convergence of nets and filters is precisely this generalized Mikusiński convergence. Here we shall introduce a sequential convergence which is topological and, with respect to which, the algebra D'_R of right-sided distributions becomes imbedded in M topologically as well as algebraically. Our definition depends upon the interpretation of operators as functions from ideals in C_R^∞ into C_R^∞ and is, thus, function-theoretic in nature. However, the setting is unusual in that the functions are not defined on a single common domain. On the other hand, it readily follows from a theorem due to Boehme ([1], Theorem 3), that the collection of domains of any sequence of operators always possesses a non-trivial intersection. Thus we are led to the following definition. (For an operator f, $\operatorname{dom} f = $ domain of f.)

A sequence $\{\mathfrak{f}_n\}$ of operators is said to be *convergent in M*, if for each subsequence $\{\mathfrak{f}_i\}$ (not necessarily proper) and each $\varphi \in \cap \operatorname{dom} \mathfrak{f}_i$, the corresponding function sequence $\{\mathfrak{f}_i(\varphi)\}$ converges in C_R^∞. (A sequence converges in C_R^∞ if it converges in C^∞ and the supports are uniformly bounded on the left.)

It is easy to see that a convergent sequence of operators determines a unique operator limit. Moreover, a "constant" sequence is convergent and converges to the "constant" and every subsequence of a convergent sequence is convergent and converges to the same operator. On the other hand, if an operator sequence $\{\mathfrak{f}_n\}$ fails to converge in M to an operator $\mathfrak{f}$, then for some subsequence $\{\mathfrak{f}_i\}$ and some $\varphi \in \cap \operatorname{dom} \mathfrak{f}_i$, either we have $\varphi \notin \operatorname{dom} \mathfrak{f}$ or $\varphi \in \operatorname{dom} \mathfrak{f}$ and the function sequence $\{\mathfrak{f}_i(\varphi)\}$ fails to converge in C_R^∞ to the function $\mathfrak{f}(\varphi)$. Thus, in any event, if an operator sequence fails to converge in M to an operator $\mathfrak{f}$, there exists a subsequence $\{\mathfrak{f}_m\}$ such that every subsequence of the latter also fails to converge in M to the operator $\mathfrak{f}$. These four properties guarantee that convergence in M is topological [3], [5], i. e., there is a unique topology on M for which convergence of sequences is precisely convergence in M and a set in M is closed iff it contains all limits of M-convergent sequences of its members. (For still another definition of convergence of operators which is topological, see [8].)

Whenever a sequence of operators possesses the common domain C_R^∞ it is necessarily a sequence of right-sided distributional operators and for it, convergence in M becomes pointwise convergence on all of C_R^∞. But by Theorem 2 (suitably specialized to right-sided distributions, for which the domains D and C_R^∞ may be interchanged [7]) pointwise convergence in the operational and in the distributional sense are equivalent. We have established, therefore, the final main result.

THEOREM 4. *Convergence in M characterizes a topology T for the Mikusiński operator field, with respect to which, the algebra D'_R of right-sided distributions becomes imbedded in M topologically* (*as a closed subset*) *as well as algebraically.*

Remark. Appropriate versions of Theorems 1 and 2 remain valid for distributions of several variables. The same proofs apply.

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Reçu par la Rédaction le 1. 8. 1969