STUDIA MATHEMATICA, T. XXXVII. (1970)

A problem on Kronecker sets

by

N. TH. VAROPOULOS (Orsay)

0. Introduction and notation. Let Γ be throughout a locally compact abelian group (the model we have in mind is $\Gamma = \boldsymbol{R}$). We denote by $0 = 0_\Gamma$ the zero element of Γ, by $\mathfrak{N}_\Gamma$ the $Nh\partial$, filter of 0_Γ in Γ, and by $\hat{\Gamma}$ the group of continuous characters of Γ, i.e. the continuous homomorphisms from Γ to $\boldsymbol{T} = \boldsymbol{R}(\bmod 2\pi)$.

Let E be a locally compact space. We denote by $S(E)$ the group of continuous complex-valued functions on E whose modulus is identically equal to 1; if $\varepsilon \in E$ is some fixed point, we put

$$S_\varepsilon(E) = \{f \in S(E);\ f(\varepsilon) = 1\}.$$

When $E \subset \Gamma$, we denote by $SU(E)$ the subgroup of $S(E)$ consisting of those functions on E which are uniformly continuous (with respect to the canonical group uniformity of Γ restricted on E).

Definition 1. Let $E \subset \Gamma$ be a closed subset of Γ. We say that E is a *Kronecker set* if for every $\varepsilon > 0$ and every $f \in SU(E)$ there exists some $\chi \in \hat{\Gamma}$ such that:

$$\sup_{e \in E} |f(e) - \chi(e)| \leqslant \varepsilon.$$

Compact Kronecker sets have been extensively studied [3]. In this note we shall study some properties of Kronecker sets with special emphasis on the possible non-compactness of the set which gives rise to some additional complications.

Definition 2. We say that an arbitrary subset $E \subset \Gamma$ is *independent* if for any $n \geqslant 1$ and any choice of n distinct points $e_1, e_2, \ldots, e_n \in E$ we have

$$m_j \in \boldsymbol{Z}\ (1 \leqslant j \leqslant n), \qquad \sum_{j=1}^{n} m_j e_j = O_\Gamma \Rightarrow m_j = 0\ (1 \leqslant j \leqslant n).$$

It is well known and obvious that every Kronecker set is independent but that the converse is in general false (cf. [3] and [5]).

In this note we shall prove the following

THEOREM 1. *Let $E \subset \Gamma$ be a totally disconnected Kronecker set of the locally compact abelian group Γ. Let us suppose that Γ is metrisable and countable at infinity, and let $\gamma \in \Gamma \setminus E$ be such that $E \cup \{\gamma\}$ is an independent set of Γ. Then the set $E \cup \{\gamma\}$ is a Kronecker set of Γ.*

In [7] we have already proved a special case of Theorem 1 when E is in addition supposed compact. Here we shall combine the ideas in [7] with a technique due to Ryll-Nardzewski [6] to obtain the general result.

1. Reduction to superkronecker sets. Before we procced any further we shall make some easy remarks on Kronecker sets.

Definition 3. Let $E \subset \Gamma$ be a closed subset of the locally compact abelian group Γ such that $0 = 0_\Gamma \in E$; we say that E is a *superkronecker set* if for every $\varepsilon > 0$ and every $f \in S_0(E)$ there exists some $\chi \in \hat{\Gamma}$ such that

$$\sup_{e \in E} |f(e) - \chi(e)| \leqslant \varepsilon.$$

We prove now the following

LEMMA 1. *Let $E \subset \Gamma$ be a Kronecker subset of Γ, let us suppose that Γ is metrisable and countable at infinity and let $\{e_n \in E\}_{n=1}^\infty$ be a sequence of points of E such that $e_n \underset{n\to\infty}{\to} \infty_\Gamma$ (i.e. tends to the infinity of the locally compact space Γ); let further $\{e'_n \in E\}_{n=1}^\infty$ be another sequence of points of E and $K \subset \Gamma$ be a compact subset of Γ such that*

$$e_n - e'_n \in K \qquad \forall\, n \geqslant 1.$$

Then

$$e_n - e'_n \underset{n\to\infty}{\to} 0_\Gamma.$$

Proof. We may suppose (by extracting a subsequence if necessary) that there exists some $x \in K$ and some $N \in \mathfrak{N}_\Gamma$ such that

$$e_n - e'_n \underset{n\to\infty}{\to} x \in K,$$

(1)

$$e_{n_1} - e_{n_2} \notin N,\ e'_{n_1} - e'_{n_2} \notin N,\ e_{n_1} - e'_{n_2} \notin N, \qquad n_1, n_2 \geqslant 1,\ n_1 \neq n_2.$$

And we shall prove, under the addition assumption (1), that $x = 0_\Gamma$. This will give the lemma at once.

Towards that end let us suppose that (1) is verified and that $x \neq 0_\Gamma$. For any $\varepsilon > 0$ then, and any two sequences $\{x_n \in \boldsymbol{T};\ x'_n \in \boldsymbol{T}\}_{n=1}^\infty$ of complex numbers of modulus 1, we can find $\chi \in \hat{\Gamma}$, a character, such that

(2) $\quad |\chi(e_n) - x_n| \leqslant \varepsilon,\ |\chi(e'_n) - x'_n| \leqslant \varepsilon \quad$ for all $n \geqslant 1$ large enough;

for any fixed $\chi \in \hat{\Gamma}$, on the other hand, we have

(3) $$\frac{\chi(e_n)}{\chi(e'_n)} = \chi(e_n - e'_n) \underset{n\to\infty}{\to} \chi(x).$$

But since ε and the complex numbers x_n, x'_n are arbitrary, (2) and (3) are incompatible and this produces our required contradiction.

Definition 4. Let $E \subset \Gamma$ be a closed subset of Γ and let $S \subset E$ be closed subset of E. We then say that S is a *spine* of E if there exists $\{N_s \in \mathfrak{N}_\Gamma;\ s \in S\}$, a family of compact $Nh\partial s$, of 0_Γ in Γ and $\Omega \in \mathfrak{N}_\Gamma$ some other $Nh\partial$, of 0_Γ, such that:

(i) $$N_s \to 0_\Gamma$$

(as s runs through the filter of complements of finite subsets of S);

(ii) $$\bigcup_{s \in S} s + N_s \supset E;$$

(iii) $$(s_1 + N_{s_1}) \cap (s_2 + N_{s_2}) = \emptyset,\ s_1 \notin s_2 + \Omega, \qquad \forall\, s_1, s_2\ (s_1 \neq s_2).$$

For every $e \in E$ we denote by $s(e) \in S$ the unique element of E for which $e \in s(e) + N_{s(e)}$.

It follows from Lemma 1 that every totally disconnected Kronecker set of a metrisable and K_σ group has at least one spine.

Let now $E \subset \Gamma$ be an independent subset of Γ, let $S \subset E$ be some spine of E and let $\{N_s \in \mathfrak{N}_\Gamma;\ s \in S\}$ be compact $Nh\partial s$ of zero associated with that spine S; we then define a mapping

$$\alpha_S\colon E \to \Gamma,$$

by setting

$$\alpha_s(e) = \begin{cases} e & \text{if } e \in S, \\ e - s(e) & \text{if } e \in E \setminus S. \end{cases}$$

It is then clear that

(4) $$\overline{\alpha(E)} \subset \alpha(E) \cup \{0_\Gamma\} = E_S$$

(the bar indicates of course topological closure in Γ).

Let further $f\colon E \to \boldsymbol{T}$ be an arbitrary complex function defined on E such that $|f(e)| = 1\ \forall\, e \in E$. We define then

$$f_S\colon E_S \to \boldsymbol{T}$$

by setting

$$f_S(0) = 1,$$

$$f_S(\alpha(e)) = \begin{cases} f(e) & \text{if } e \in S, \\ \dfrac{f(e)}{f(s(e))} & \text{if } e \in E \setminus S. \end{cases}$$

Using the definition of a spine and the independence of E we see at once that

(5) $$f \in SU(E) \Leftrightarrow f_S \in S_0(E_S)$$

and that further the correspondence $f \to f_S$ is a 1-1 correspondence between $SU(E)$ and $S_0(E_S)$.

We have then

PROPOSITION 1. *Let $E \subset \Gamma$ be an independent closed subset of Γ and let S be some spine of E and let E_S be defined as in* (4). *Then E is a Kronecker set if and only if E_S is a superkronecker set.*

Proof. It is an immediate consequence of (5) and therefore left to the reader.

2. The $S(E)$ group. Let us suppose throughout in this section that E is a locally compact, totally disconnected countable at infinity space and let us fix some point $\varepsilon \in E$ of that space. We shall topologise then $S(E)$ with the uniform on compact topology and we shall identify $\boldsymbol{T}$ with the closed subgroup $T(E) = \{f \in S(E); f = \text{constant}\}$ of constant functions on E. It is then clear that $S_\varepsilon(E)$ is also a closed subgroup of $S(E)$ and that we have the topological decomposition

$$S(E) = S_\varepsilon(E) \oplus T(E)$$

which implies at once the following

PROPOSITION 2. *Every continuous character of $S_\varepsilon(E)$ (i.e. a continuous homomorphism from $S_\varepsilon(T)$ to $\boldsymbol{T}$) can be extended to a continuous character of $S(E)$.*

We have also

PROPOSITION 3. *Let $\theta\colon S(E) \to \boldsymbol{T}$ be a continuous character of $S(E)$, then there exist finitely many points $e_1, e_2, \ldots, e_p \in E$ $(p \geqslant 0)$ and finitely many integers $m_1, m_2, \ldots, m_p \in \boldsymbol{Z}$ such that*

$$\theta(f) = \big(f(e_1)\big)^{m_1} \ldots \big(f(e_p)\big)^{m_p} \quad \forall f \in S(E)$$

(a vacuous product is interpreted as 1).

Proof. A special case of proposition 3 when E is in addition supposed compact has been proved in [7]. But the general case above is an immediate consequence of the compact case of [7] by the very definition of the uniform in compacta topology; the details are left to the reader (the countability at infinity of E is not necessary here).

We have finally

PROPOSITION 4. *Let $\theta\colon S_\varepsilon(E) \to \boldsymbol{T}$ be a Borel character of $S_\varepsilon(E)$ (i.e. a character and a Borel function); then there exists finitely many points $e_1, e_2, \ldots, e_p \in E$ $(p \geqslant 0)$ and finitely many integers $m_1, m_2, \ldots, m_p \in \boldsymbol{Z}$ such that*

$$\theta(f) = \big(f(e_1)\big)^{m_1} \ldots \big(f(e_p)\big)^{m_p} \quad \forall f \in S_\varepsilon(E)$$

(a vacuous product is interpreted as 1).

Proof. Since E is countable at infinity, $S_\varepsilon(E)$ is metrisable with the uniform on compacta topology, $S_\varepsilon(E)$ is also of course complete. It thus follows from a well known theorem of Banach [5] that θ is continuous. Our proposition 4 is then an immediate corollary of propositions 2 and 3 put together.

3. Proof of the Theorem. We shall prove in this section the following

PROPOSITION 5. *Let $E \subset \Gamma$ be a totally disconnected superkronecker set of Γ. Let us suppose that Γ is metrisable and countable at infinity and let $\gamma \in \Gamma \setminus E$ be such that the set*

$$\{\gamma\} \cup (E \setminus \{0\})$$

is independent. Then the set $E \cup \{\gamma\}$ is superkronecker.

Proposition 5 then, together with proposition 1, at once imply Theorem 1.

Before we give the proof of proposition 5 we shall prove the following

LEMMA 2. *Let $E \subset \Gamma$ be a totally disconnected superkronecker set and $x \in \Gamma$ be some fixed point of Γ; let us suppose that Γ is metrisable and countable at infinity and let us suppose further that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$\chi \in \hat{\Gamma}, \quad \sup_{e \in E} |\chi(e) - 1| \leqslant \delta \Rightarrow |\chi(x) - 1| \leqslant \varepsilon.$$

Then $x \in Gp(E)$ (= the group generated in Γ by E).

Proof. Using our hypothesis we can define a (not necessarily continuous for the uniform on compacta topology) character

(6) $$\theta\colon S_0(E) \to \boldsymbol{T}$$

by setting

(7) $$\theta(f) = \lim_{n \to \infty} \chi_n(x),$$

when $\mathfrak{X} = \{\chi_n \in \hat{\Gamma}\}_{n=1}^\infty$ is a sequence of characters such that

(8) $$\sup_{e \in E} |\chi_n(e) - f(e)| \underset{n \to \infty}{\rightrightarrows} 0.$$

Indeed, using our hypothesis it is easy to verify that the limit in (7) always exists and is independent of the particular sequence $\mathfrak{X}$ chosen in (8), so that $\theta(f)$ is well defined. The verification that θ is multiplicative is immediate.

We shall prove next that θ in (6) is a Borel mapping (for the uniform on compacta topology of $S_0(E)$).

To prove this it suffices to show that G, the graph of θ in $S_0(E) \times \boldsymbol{T}$,

$$G = \{(f, t) \in S_0(E) \times \boldsymbol{T};\ t = \theta(f)\} \subset S_0(E) \times T,$$

is an analytic set (in N. Bourbaki terminology G is "un ensemble souslinien" [2]) of the metrisable complete separable space $S_0(E)\times \boldsymbol{T}$. A theorem of Kuratowski then implies that θ is a Borel mapping [4].

To prove that G is an analytic set we consider for every $\varepsilon > 0$ the set

$$\Delta_\varepsilon = \{(f, t, \chi)\in S_0(E)\times \boldsymbol{T}\times\hat{\Gamma};\ \sup_{e\in E}|f(e)-\chi(e)|\leqslant \varepsilon,\ |\chi(x)-t|\leqslant \varepsilon\}$$

which is clearly a Borel subset of the space $S_0(E)\times\boldsymbol{T}\times\hat{\Gamma}$ (topologised with the cartesian product topology of the uniform on compacta topology of $S_0(E)$ and the natural topologies on $\boldsymbol{T}$ and $\hat{\Gamma}$) which is metrisable complete and 2nd countable ("de type dénombrable" in N. Bourbaki's terminology), i.e. "un espace polonais".

Let further

$$\pi\colon\ S_0(E)\times\boldsymbol{T}\times\hat{\Gamma}\to S_0(E)\times\boldsymbol{T}$$

be the canonical projection. It is then clear that

$$G = \bigcap_{n=1}^{\infty}\pi(\Delta_{1/n})\subset S_0(E)\times\boldsymbol{T}$$

and this proves our assertion (cf. [2]).

Proposition 4 tells us then that there exist finitely many points $e_1, e_2, \ldots, e_p\in E$ $(p\geqslant 0)$ and finitely many integers $m_1, m_2, \ldots, m_p\in\boldsymbol{Z}$ such that:

$$\theta(f) = \big(f(e_1)\big)^{m_1}\ldots\big(f(e_p)\big)^{m_p} \tag{9}$$

(with the usual convention for empty products).

But if we substitute $f = \chi|_E$ for some $\chi\in\hat{\Gamma}$, we simply obtain

$$\chi(x) = \chi\Big(\sum_{j=1}^{p} m_j e_j\Big)\ \forall\ \chi\in\hat{\Gamma}$$

which implies that

$$x = \sum_{j=1}^{p} m_j e_j\in Gp(E)$$

and proves our Lemma.

Proof of Proposition 5. Clearly, to prove proposition 5 it suffices to show that for every $e^{i\theta}\in\boldsymbol{T}$ we can find a sequence $\mathfrak{X}_\theta = \{\chi_n\in\hat{\Gamma}\}_{n=1}^{\infty}$ such that

$$\sup_{e\in E}|\chi_n(e)-1|\underset{n\to\infty}{\to} 0,\qquad \chi_n(x)\underset{n\to\infty}{\to} e^{i\theta}.$$

Suppose that this was not possible. It would follow then that there exists $q\geqslant 1$ some positive integer such that for every sequence $\mathcal{J} = \{\chi_n\in\hat{\Gamma}\}_{n=1}^{\infty}$ we have

$$\sup_{e\in E}|\chi_n(e)-1|\underset{n\to\infty}{\to} 0\Rightarrow \chi_n(qx) = \big(\chi_n(x)\big)^q\underset{n\to\infty}{\to} 1,$$

i.e. that $\chi_n(x)$ "converges to $\boldsymbol{Z}(q)\subset\boldsymbol{T}$" the set of q^{th} roots of unity of $\boldsymbol{T}$ (cf. [7]). But this contradicts the hypothesis for by Lemma 2 it implies that $qx\in Gp(E)$. The proof is complete.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932, p. 23, Ch. I, Th. 4.

[2] N. Bourbaki, *Topologie générale*, Ch. 9, § 6, No. 2.

[3] J.-P. Kahane, *Approximation par des exponentielles imaginaires; ensembles de Dirichlet et ensembles de Kronecker* (to appear).

[4] C. Kuratowski, *Topologie*, Vol. I, 2nd ed, Warszawa 1948, p. 398, § 35, V.

[5] W. Rudin, *Fourier analysis on groups*, Interscience No. 12 (1962), Ch. 5.

[6] C. Ryll-Nardzewski, *Concerning almost periodic extensions of functions*, Coll. Math. 12 (1964), p. 235-237.

[7] N. Th. Varopoulos, C. R. A. S. 268 (A), p. 954-957.

Reçu par la Rédaction le 11. 10. 1969