and, for $n = 2, 3, \ldots$

$$d_n \geqslant d_{n-1} - g_{n-1} + g_n \quad \text{and} \quad M(d_n) \leqslant M(d_{n-1} - g_{n-1} + g_n) + \varepsilon/2^n,$$

from which it follows that $d_1 \leqslant d_2 \leqslant \ldots$ and, for $n = 1, 2, \ldots$

$$(14) \qquad d_n \geqslant g_n \quad \text{and} \quad M(d_n) - M(g_n) \leqslant \varepsilon\left(\frac{1}{2} + \ldots + \frac{1}{2^n}\right) \leqslant \varepsilon.$$

For all $x \in \operatorname{ex} X$, $\lim_n d_n(x) \geqslant \lim_n g_n(x) \geqslant 0$ hence, from (d), $\lim_n \inf d_n(X) \geqslant 0$ and so $\lim_n M(d_n) \geqslant 0$. Combining this with (14), $\lim_n M(g_n) \geqslant -\varepsilon$ and the required result follows since ε is arbitrary.

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UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA

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On the function g_λ^* and the heat equation

by

C. SEGOVIA (Princeton, N. J.) and R. L. WHEEDEN (New Brunswick, N. J.)

INTRODUCTION AND NOTATIONS

In the present paper, a function analogous to the g_λ^* function of Littlewood, Paley, Zygmund, Stein (see [13] and [10]) is introduced for functions $u(x, t, y)$ which are solutions of the boundary problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial y^2}, \qquad y > 0$$

and

$$\lim_{y\to 0} u(x, t, y) = f(x, t).$$

The definition of g_λ^* is given in section 2, (2.1), and its properties concerning the preservation of L^p classes are discussed in theorems (2.2), (2.3), and (2.4). The method used here is an adaptation to the parabolic case of the one found in C. L. Fefferman's doctoral dissertation [2]. In section 3, theorem (3.1), the function g_λ^* is applied to obtain a characterization of the $\mathscr{L}_a^p$ spaces introduced by B. F. Jones in [4] and [5]. This characterization is suggested by those given by Hirschman [3] and Stein [9]. Also, a generalization of the g-function of Littlewood–Paley involving fractional derivatives is considered (theorem (2.25)). For an analogue in the case of analytic and harmonic functions, see [3] and [8].

We shall denote by E_{n+1} the set of all $(n+1)$-tuples $(x_1, \ldots, x_n, t) = (x, t)$ of real numbers, with the explicit intention of distinguishing the last variable. E_{n+2}^+ denotes the set of all $(n+2)$-tuples $(x_1, \ldots, x_n, t, y)$ of real numbers with $y > 0$. By $|x|$ we denote the absolute value of $(x_1, \ldots, x_n)$, which is given by $\left(\sum_1^n x_i^2\right)^{1/2}$. The complement of a set A is denoted by A' and its Lebesgue measure by $|A|$. The definition of Fourier transform of a function $f(x,t)\in L^1(E_{n+1})$ we use is

$$\hat{f}(x,t) = \int_{E_{n+1}} \exp\{-i(x_1z_1+\ldots+x_nz_n+ts)\}f(z,s)\,dz\,ds.$$

Therefore, Plancherel's theorem reads as follows:

$$\int_{E_{n+1}} \bar{f}(x,t)g(x,t)\,dx\,dt = \frac{1}{(2\pi)^{n+1}}\int_{E_{n+1}} \bar{\hat{f}}(x,t)\hat{g}(x,t)\,dx\,dt.$$

By $\mathscr{S}$ we denote the space of all infinitely differentiable functions defined on E_{n+1} with derivatives decreasing at infinity faster than any polynomial. The dual space $\mathscr{S}'$ is the so-called space of tempered distributions.

SECTION 1

PRELIMINARIES

Let us consider the kernel $\Gamma(x,t,y)$, $y>0$, introduced by B. F. Jones in [4], and defined as

$$\Gamma(x,t,y) = \begin{cases} \dfrac{y}{(4\pi)^{(n+1)/2}t^{(n+3)/2}}\exp\left\{-\dfrac{|x|^2+y^2}{4t}\right\}, & t>0,\\ 0, & t\leqslant 0.\end{cases}$$

We list here some properties of this kernel that will be used later.

1. The kernel $\Gamma(x,t,y)$ is positive and

$$\int_{E_{n+1}} \Gamma(x,t,y)\,dx\,dt = 1$$

for every $y>0$.

2. If y_1 and y_2 are positive numbers then

$$\Gamma(x,t,y_1)*\Gamma(x,t,y_2) = \Gamma(x,t,y_1+y_2).$$

3. If $f(x,t)\in L^p(E_{n+1})$, $1\leqslant p\leqslant\infty$, then

$$u(x,t,y) = \int_{E_{n+1}} \Gamma(x-z,t-s,y)f(z,s)\,dz\,ds$$

satisfies

(a) $\|u(x,t,y)\|_p\leqslant\|f\|_p$, for every $y>0$, and

(b) $\lim_{y\to 0}u(x,t,y) = f(x,t)$ a.e. and in norm.

4. If $p' = p/(p-1)$, then

(1.0) $$\Big(\int_{E_{n+1}} \Gamma(x,t,y)^p\,dx\,dt\Big)^{1/p}\leqslant A_{n,p}y^{-(n+2)/p'}$$

for $y>0$.

5. The Fourier transform of $\Gamma(x,t,y)$ in the variables x,t is given by

$$\hat{\Gamma}(x,t,y) = \exp\{-y\sqrt{|x|^2+it}\}$$

for $y>0$. The branch of $z^{1/2}$ chosen is that assigning positive values to $z^{1/2}$ for positive values of the variable z.

6. For the partial derivatives of $\Gamma(x,t,y)$ we have the estimates:

(1.1) $$|\partial_x^\mu\partial_t^a\partial_y^b\Gamma(x,t,y)|\leqslant ct^{-(n+2+|\mu|+2a+b)/2}\exp\{-(|x|^2+y^2)/8t\},$$

and

$$|\partial_x^\mu\partial_t^a\partial_y^b\Gamma(x,t,y)|\leqslant cy^{-(|\mu|+2a+b)}\Gamma(x,2t,y).$$

7. The function

(1.2) $$(|x|+|t|^{1/2}+y)^r\frac{1}{|t|^{r/2}}\exp\{-(|x|^2+y^2)/8|t|\}$$

is bounded on E_{n+2}^+ for every $r>0$.

For the proofs of these and other properties of the kernel $\Gamma(x,t,y)$, see [4] and [5].

The following change of coordinates was introduced by E. Fabes and N. Riviere in [1]. Let

(1.3) $$\begin{aligned} x_n &= \varrho\sin\varphi_1\ldots\sin\varphi_{n-1}\sin\varphi_n,\\ x_{n-1} &= \varrho\sin\varphi_1\ldots\sin\varphi_{n-1}\cos\varphi_n,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ x_1 &= \varrho\sin\varphi_1\cos\varphi_2,\\ t &= \varrho^2\cos\varphi_1,\end{aligned}$$

where $0\leqslant\varphi_n\leqslant 2\pi$ and $0\leqslant\varphi_i\leqslant\pi$ for $i=1,\ldots,n-1$. We shall refer to this as the parabolic polar coordinates on E_{n+1}. Its Jacobian satisfies

$$\left|\frac{\partial(x,t)}{\partial(\varrho,\varphi)}\right| = \varrho^{n+1}(1+\cos^2\varphi_1)(\sin\varphi_1)^{n-1}\ldots(\sin\varphi_{n-1}),$$

or also

$$dx\,dt = \varrho^{n+1}(1+\cos^2\varphi_1)\,d\varrho\,d\Omega,$$

where $d\Omega$ denotes the element of area of the unit sphere in E_{n+1}. From the definition of parabolic polar coordinates it follows that

$$\frac{|x|^2}{\varrho^2}+\frac{|t|^2}{\varrho^4} = 1,$$

which implies that

$$\tfrac{1}{2}\varrho\leqslant\sqrt{|x|^2+|t|}\leqslant 2\varrho.$$

We define the function $\mathcal{J}^a(x,t)$ as:

$$\mathcal{J}^a(x,t) = \begin{cases} \dfrac{1}{(4\pi)^{n/2}\Gamma(a/2)} t^{(a-n-2)/2}\exp\{-|x|^2/4t\}, & t>0, \\ 0, & t\leqslant 0 \end{cases}$$

for $0<a<n+2$.

By (1.2), this function is locally integrable, and if $f(x,t)$ is a well behaved function, say belonging to $\mathcal{S}$, the integral

$$(1.4)\qquad \int_{E_{n+1}} \mathcal{J}^a(x-z,t-s)f(z,s)\,dz\,ds$$

is absolutely convergent and defines an operator that will be denoted by $\mathcal{J}_a(f)$ and called the *parabolic fractional integral of order* a. It is easy to see that the function $\mathcal{J}^a(x,t)$ defines a distribution belonging to $\mathcal{S}'$ and we shall compute its Fourier transform. Let $\varphi(x,t)\in\mathcal{S}$. We have

$$\int_{E_{n+1}} \bar{\varphi}(x,t)\Gamma(x,t,y)\,dx\,dt = \frac{1}{(2\pi)^{n+1}}\int_{E_{n+1}} \bar{\hat{\varphi}}(x,t)\exp\{-y\sqrt{|x|^2+it}\}\,dx\,dt.$$

Multiplying both members by y^{a-1}, integrating y from 0 to ∞ and changing the order of integration, we get

$$\int_{E_{n+1}} \bar{\varphi}(x,t)\,dx\,dt\int_0^\infty y^{a-1}\Gamma(x,t,y)\,dy$$

$$= \frac{1}{(2\pi)^{n+1}}\int_{E_{n+1}} \bar{\hat{\varphi}}(x,t)\,dx\,dt\int_0^\infty y^{a-1}\exp\{-y\sqrt{|x|^2+it}\}\,dy,$$

or since

$$\int_0^\infty \Gamma(x,t,y)y^{a-1}\,dy = \frac{2^a\Gamma((a+1)/2)}{(4\pi)^{(n+1)/2}}t^{(a-n-2)/2}\exp\{-|x|^2/4t\},\qquad t>0,$$

$$(1.5)\qquad \frac{2^a\Gamma((a+1)/2)}{(4\pi)^{(n+1)/2}}\int_{E_{n+1}} t^{(a-n-2)/2}\exp\{-|x|^2/4t\}\bar{\varphi}(x,t)\,dx\,dt$$

$$= \frac{\Gamma(a)}{(2\pi)^{n+1}}\int_{E_{n+1}} (\sqrt{|x|^2+it})^{-a}\bar{\hat{\varphi}}(x,t)\,dx\,dt.$$

The change of the order of integration is justified by showing that the resulting integrals are absolutely convergent. In order to do this, it is enough to prove that $(\mathrm{Re}\sqrt{|x|^2+it})^{-a}$ and $t^{(a-n-2)/2}\exp\{-|x|^2/4t\}$

are locally integrable functions. Passing to parabolic polar coordinates and observing $\mathrm{Re}\sqrt{|x|^2+it}/|\sqrt{|x|^2+it}|$ is bounded below by a positive constant, we have

$$\int_{\varrho\leqslant M} |\sqrt{|x|^2+it}|^{-a}\,dx\,dt\leqslant c\int_0^M \varrho^{-a+n+1}\,d\varrho<\infty$$

if $a<n+2$. Similarly,

$$\int_{\varrho\leqslant M} t^{(a-n-2)/2}\exp\{-|x|^2/4t\}\,dx\,dt\leqslant c\int_0^M \varrho^{a-1}\,d\varrho<\infty$$

if $a>0$. Therefore (1.5) holds, and since

$$\Gamma\left(\frac{a+1}{2}\right)/\Gamma(a) = 2\pi^{1/2}/2^a\Gamma\left(\frac{a}{2}\right),$$

we get

$$\hat{\mathcal{J}}^a(x,t) = (\sqrt{|x|^2+it})^{-a}.$$

This expression for the Fourier transform of $\mathcal{J}^a$ and the next theorem justify calling (1.4) the parabolic fractional integral of order a.

Parabolic version of the Sobolev theorem on fractional integrals (see also [7]):

(1.6) THEOREM. *Let $f(x,t)\in L^p(E_{n+1})$ and $1/q = 1/p-a/(n+2)$, $0<a<n+2$, $1<p,q<\infty$. The parabolic fractional integral operator of order a satisfies*

$$\|\mathcal{J}_a(f)\|_q\leqslant c_{p,a,n}\|f\|_p$$

where the constant $c_{p,a,n}$ does not depend on f.

Proof. We shall show that $\mathcal{J}_a$ is an operator of weak type (p,q) for every p satisfying $1\leqslant p<(n+2)/a$. Let K_0 and K_∞ be the restrictions of $\mathcal{J}^a(x,t)$ to the sets $\varrho\leqslant\mu$ and $\varrho\geqslant\mu$ respectively. Then

$$(1.7)\qquad \|K_0\|_1\leqslant c\int_0^\mu \varrho^{a-1}\,d\varrho = c\mu^a,$$

$$(1.8)\qquad \|K_\infty\|_{p'}\leqslant c\Big(\int_\mu^\infty \varrho^{(a-n-2)p'+n+1}\,d\varrho\Big)^{1/p'} = c\mu^{-(n+2)/q},$$

where $p' = p/(p-1)$, and

$$(1.9)\qquad \|K_\infty\|_\infty\leqslant c\mu^{a-n-2}.$$

Moreover,

$$(1.10)\qquad |\{(x,t):\ |\mathcal{J}_a(f)(x,t)|>2\lambda\}|$$

$$\leqslant |\{(x,t):\ |K_0*f(x,t)|>\lambda\}|+|\{(x,t):\ |K_\infty*f(x,t)|>\lambda\}|.$$

Let us consider first the case $p = 1$, $\|f\|_1 = 1$. If we choose μ to be a constant times $\lambda^{-1/(n+2-a)}$ then (1.9) implies that the second term on the right of (1.10) is zero. By Young's theorem and (1.7), the first term is majorized by $c\mu^a\lambda^{-1} = c\lambda^{-(n+2)/(n+2-a)}$. Therefore,

$$\left|\{(x,t)\colon\ |\mathscr{I}_a(f)(x,t)| > 2\lambda\}\right| \leqslant c\lambda^{-(n+2)/(n+2-a)},$$

which shows that $\mathscr{I}_a$ is an operator of weak type $\big(1, (n+2)/(n+2-a)\big)$.

The case $1 < p < (n+2)/a$ is similar if we choose μ to be constant times $\lambda^{-q/(n+2)}$ and use (1.8) instead of (1.9). The theorem then follows from the Marcinkiewicz interpolation theorem. (See [14].)

Let us consider now the function $\mathscr{G}^a(x,t)$, $a > 0$, defined as

$$\mathscr{G}^a(x,t) = \begin{cases} \dfrac{1}{(4\pi)^{n/2}\Gamma(a/2)}\, t^{(a-n-2)/2}\exp\{-t-(|x|^2/4t)\}, & t > 0,\\ 0, & t \leqslant 0.\end{cases}$$

It is shown in [4] that $\mathscr{G}^a(x,t)$ is integrable over E_{n+1} with integral 1, and that its Fourier transform is $(1+|x|^2+it)^{-a/2}$. For $f \in L^p(E_{n+1})$, $1 \leqslant p \leqslant \infty$, the operator

$$\mathscr{G}_a(f)(x,t) = \int_{E_{n+1}} \mathscr{G}^a(x-z, t-s) f(z,s)\,dz\,ds$$

is therefore well-defined and maps $L^p(E_{n+1})$ into $L^p(E_{n+1})$.

(1.11) Definition. The *space* $\mathscr{L}^p_a(E_{n+1})$, $a > 0$ and $1 \leqslant p \leqslant \infty$, is defined as the image under $\mathscr{G}_a$ of $L^p(E_{n+1})$; that is, $f(x,t) \in \mathscr{L}^p_a$ if and only if

$$f(x,t) = \mathscr{G}_a(\varphi)(x,t)$$

for some $\varphi \in L^p(E_{n+1})$. The *norm in* $\mathscr{L}^p_a$ is defined by

$$\|f\|_{p,a} = \|\varphi\|_p.$$

Now, let $f \in L^p(E_{n+1})$ and denote by $M(f)$ the function

$$M(f)(x,t) = \sup_I \left\{\frac{1}{|I|}\int_I |f(z,s)|\,dz\,ds\right\}$$

where I denotes a parallelepiped of the form

$$I = \{(z,s)\colon\ |z_1-c_1| < h, \ldots, |z_n-c_n| < h,\ |s-c_0| < h^2\}$$

containing the point (x,t). The transformation from f to $M(f)$ is of weak type (p,p) for $1 \leqslant p \leqslant \infty$. (See [12].)

We now state a theorem whose proof is similar to that of theorem 6 of [0] and will therefore be omitted.

(1.12) Theorem. *Let $k(x,t)$ be a function defined on E_{n+1} which satisfies $|k(x,t)| \leqslant \varphi(\varrho)$, where $\varphi(s)$, $s > 0$, is non-negative, decreasing and such that*

$$\int_0^\infty \varphi(s)\,s^{n+1}\,ds < \infty.$$

Then the function $g_\varepsilon(x,t)$ given by

$$g_\varepsilon(x,t) = \frac{1}{\varepsilon^{n+2}}\int_{E_{n+1}} f(z,s)\,k\left(\frac{x-z}{\varepsilon}, \frac{t-s}{\varepsilon^2}\right)dz\,ds$$

satisfies

$$\sup_{\varepsilon>0}|g_\varepsilon(x,t)| \leqslant cM(f)(x,t),$$

where the constant c is independent of f.

The function $g(f)(x,t)$ defined as

$$g(f)(x,t) = \left(\int_0^\infty y\left|\frac{\partial u}{\partial y}(x,t,y)\right|^2 dy\right)^{1/2},$$

where $u(x,t,y) = f(x,t) * \Gamma(x,t,y)$, was introduced in [5]. In that paper, it was shown that if $f \in L^p(E_{n+1})$, $1 < p < \infty$, then

(1.13) $$c_1\|f\|_p \leqslant \|g(f)\|_p \leqslant c_2\|f\|_p,$$

the constants being positive and independent of f.

SECTION 2

PARABOLIC ANALOGUE OF THE g^*_λ FUNCTION OF LITTLEWOOD, PALEY, ZYGMUND, STEIN

We now define our analogue of the g^* function and study its properties regarding the preservation of L^p. The methods we use are those developed by C. L. Fefferman in his thesis [2]. We also discuss a variant of the g-function which involves derivatives of a fractional order (see [3] and [8]) which will be needed in section 3.

Let $u(x,t,y)$ be the convolution of a function $f \in L^p(E_{n+1})$, $1 < p < \infty$, and $\Gamma(x,t,y)$. For $\lambda > 1$, we define $g^*_\lambda(f)(x,t)$ as

(2.1) $g^*_\lambda(f)(x,t)$

$$= \left(\int_{E^+_{n+2}} \left(\frac{y}{|x-z|+|t-s|^{1/2}+y}\right)^{(n+2)\lambda}\left|\frac{\partial u}{\partial y}(z,s,y)\right|^2 \frac{dz\,ds\,dy}{y^{n+1}}\right)^{1/2}.$$

We shall prove

(2.2) THEOREM. *Let $1 < p < \infty$ and $p > 2/\lambda$. There exist two positive constants c_1 and c_2 such that*

$$c_1\|f\|_p \leqslant \|g_\lambda^*(f)\|_p \leqslant c_2\|f\|_p$$

for $f \in L^p(E_{n+1})$. The constants c_1, c_2 depend only on λ, p and n.

The part of the theorem asserting that $c_1\|f\|_p \leqslant \|g_\lambda^*(f)\|_p$ will be discussed at the end of this chapter. The part $\|g_\lambda^*(f)\|_p \leqslant c_2\|f\|_p$ is a consequence of the Marcinkiewicz interpolation theorem (see [14]) and the following two theorems.

(2.3) THEOREM. *Let $\lambda > 1$ and $p \geqslant 2$. There exists a constant c such that*

$$\|g_\lambda^*(f)\|_p \leqslant c\|f\|_p$$

for every $f \in L^p(E_{n+1})$. The constant c depends only on λ, p and n.

(2.4) THEOREM. *Let $1 < p < 2$ and $\lambda = 2/p$. There exists a constant $c > 0$ such that*

$$|\{(x, t):\ g_\lambda^*(f)(x, t) > \mu\}| \leqslant c\mu^{-p}\|f\|_p \tag{2.5}$$

for every $\mu > 0$ and $f \in L^p(E_{n+1})$. The constant c depends only on n and p.

Proof of theorem (2.3). For $p \geqslant 2$, let $q = p/(p-2)$ and $h(x, t) \geqslant 0$ and $h \in L^q(E_{n+1})$. From the definition of $g_\lambda^*(f)(x, t)$ and a change in the order of integration, we have

$$\int_{E_{n+1}} h(x, t)[g_\lambda^*(f)(x, t)]^2\,dx\,dt$$
$$= \int_{E_{n+2}^+} \left|\frac{\partial u}{\partial y}(z, s, y)\right|^2 \frac{dz\,ds\,dy}{y^{n+1}} \int_{E_{n+1}} \left(\frac{y}{|x-z|+|t-s|^{1/2}+y}\right)^{(n+2)\lambda} h(x, t)\,dx\,dt.$$

Since $(|z|+|s|^{1/2}+1)^{-(n+2)\lambda}$ is integrable over E_{n+1} for $\lambda > 1$, we obtain from theorem (1.12) that

$$\sup_{y>0}\left\{\frac{1}{y^{n+2}}\int_{E_{n+1}}\left(\frac{y}{|x-z|+|t-s|^{1/2}+y}\right)^{(n+2)\lambda} h(x, t)\,dx\,dt\right\} \leqslant cM(h)(z, s).$$

Thus

$$\int_{E_{n+1}} h(x, t)[g_\lambda^*(f)(x, t)]^2\,dx\,dt \leqslant c \int_{E_{n+2}^+} y\left|\frac{\partial u}{\partial y}(z, s, y)\right|^2 M(h)(z, s)\,dz\,ds\,dy$$
$$= c\int_{E_{n+1}} M(h)(z, s)[g(f)(z, s)]^2\,dz\,ds.$$

The last member above is less than or equal to

$$c\|h\|_q\|g(f)\|_p^2,$$

and since h is an arbitrary non-negative function in $L^q(E_{n+1})$, it follows that

$$\|g_\lambda^*(f)\|_p^2 \leqslant c\|g(f)\|_p^2.$$

Applying (1.13) we get that $\|g_\lambda^*(f)\|_p \leqslant c\|f\|_p$, which is the statement of the theorem.

Proof of theorem (2.4). Let $I(x, t)$ be any parallelepiped containing (x, t), not necessarily as its center, of the type

$$I_{(x,t)} = \{(z, s):\ |z_1 - c_1| \leqslant h, \ldots, |z_n - c_n| \leqslant h, |s - c_0| \leqslant h^2\},$$

and let $\bar{f}(x, t)$ be the function

$$\bar{f}(x, t) = \sup_{I_{(x,t)}}\left\{\frac{1}{|I_{(x,t)}|}\int_{I_{(x,t)}} |f(z, s)|^p\,dz\,ds\right\}$$

where $f \in L^p(E_{n+1})$. Since $|f(z, s)|^p$ belongs to $L^1(E_{n+1})$, we have from section 1 that

$$|\{(x, t):\ \bar{f}(x, t) > \mu^p\}| \leqslant \frac{A}{\mu^p}\int_{E_{n+1}} |f(z, s)|^p\,dz\,ds.$$

Let us denote $\{(x, t):\ \bar{f}(x, t) > \mu^p\}$ by Ω. Since Ω is open and has finite measure the distance from a point $(x, t) \in \Omega$ to Ω' is bounded. Denote by G_k, $-\infty < k < \infty$, the grid of non-overlapping parallelepipeds of the form

$$\left\{(x, t):\ \frac{m_i}{2^k} \leqslant x_i \leqslant \frac{m_i+1}{2^k},\ \frac{m}{4^k} \leqslant t \leqslant \frac{m+1}{4^k}\right\}$$

where m_i and m are arbitrary integers. These parallelepipeds have sides of lengths $1/2^k$ in the x_i directions and $1/4^k$ in the t direction. In general, if I denotes a parallelepiped with sides of lengths h in the x_i directions and h^2 in the t direction, then $\tilde{I}$, I^*, and I^{**} denote parallelepipeds with the same center as I and sides parallel to those of I with lengths $2nh$, $4n^2h^2$; $5nh$, $25n^2h^2$; and $10nh$, $100n^2h^2$ respectively. We define Π_k to be the family of all $I \in G_k$ not contained in a parallelepiped belonging to a $\Pi_{k'}$ with $k' < k$, and such that $I^* \subset \Omega$ but $I^{**} \not\subset \Omega$. Observe that all the Π_k with indices less than a certain number are empty. We claim that the parallelepipeds in all the Π_k, $-\infty < k < \infty$, cover Ω. For if $(\bar{x}, \bar{t}) \in \Omega$ and is not covered, then for every k we have at least one parallelepiped $I_k \in G_k$ such that $(\bar{x}, \bar{t}) \in I_k$ and $I_k^{**} \not\subset \Omega$. Pick a point $(x_k, t_k) \in \Omega' \cap I_k^{**}$. Since the diameter of I_k^{**} tends to zero, the sequence (x_k, t_k) converges to $(\bar{x}, \bar{t})$, and hence $(\bar{x}, \bar{t}) \in \Omega'$, which is a contradiction. Clearly the parallelepipeds belonging to all the Π_k can be ordered in a sequence, which we shall denote by $\{I_i\}$. These parallelepipeds are non-overlapping and

$$\sum_i |I_i| = |\Omega| \leqslant \frac{A}{\mu^p}\|f\|_p^p. \tag{2.6}$$

If $(x,t)\notin\Omega$ then $\bar{f}(x,t)\leqslant\mu^p$, so, by differentiation, we get

$$|f(x,t)|\leqslant\mu \quad \text{a.e.}$$

for $(x,t)\notin\Omega$.

Since $I_i^{**}\not\subset\Omega$, there exists $(x,t)\in I_i^{**}\cap\Omega'$. Thus,

$$\frac{1}{|I_i^{**}|}\int_{I_i^{**}}|f(z,s)|^p\,dz\,ds\leqslant\mu^p$$

so

$$\frac{1}{|I_i^{**}|}\int_{I_i}|f(z,s)|^p\,dz\,ds\leqslant\mu^p,$$

and since $|I_i^{**}| = (10n)^{n+2}|I_i|$, we obtain

$$\frac{1}{|I_i|}\int_{I_i}|f(z,s)|^p\,dz\,ds\leqslant A\mu^p. \tag{2.7}$$

Let l_i and l_i^2 be the lengths of the sides of I_i. Then, if $(x,t)\in\tilde{I}_i$ and s, s^2 are the lengths of the sides of the largest parallelepiped with center at (x,t) contained in Ω, there exist finite constants c_1, c_2 which are independent of i and such that

$$c_1 l_i\leqslant s\leqslant c_2 l_i. \tag{2.8}$$

In particular, if $\tilde{I}_j$ intersects $\tilde{I}_i$ we get the fact that the ratio of l_i and l_j is bounded above and below by positive constants which are independent of i and j.

To show (2.8), observe that since $\tilde{I}_i\subset I_i^*\subset\Omega$, then $s\geqslant c_1 l_i$. Also, since $I_i^{**}\not\subset\Omega$, we have $s\leqslant c_2 l_i$.

We define $f'(x,t)$ as

$$f'(x,t)=\begin{cases}\dfrac{1}{|I_i|}\displaystyle\int_{I_i}f(z,s)\,dz\,ds, & \text{if}\quad (x,t)\in I_i,\\ f(x,t), & \text{if}\quad (x,t)\notin\Omega,\end{cases}$$

and $f''(x,t) = f(x,t)-f'(x,t)$. Note that $|f'(x,t)|\leqslant A\mu$ a.e. (by (2.7)),

$$\|f'\|_p\leqslant\|f\|_p \quad\text{and}\quad \|f''\|_p\leqslant 2\|f\|_p, \tag{2.9}$$

f'' is supported on Ω,

$$\frac{1}{|I_i|}\int_{I_i}|f''(z,s)|^p\,dz\,ds\leqslant A\mu^p \quad (\text{by } (2.7)), \tag{2.10}$$

$$\int_{I_i}f''(z,s)\,dz\,ds = 0. \tag{2.11}$$

To prove the theorem, it is enough to show that (2.5) holds for both f' and f'' instead of f. For f' we have from theorem (2.3) that

$$|\{(x,t):\ g_\lambda^*(f')(x,t)>\mu\}|\leqslant\frac{A}{\mu^2}\|f'\|_2^2\leqslant\frac{A}{\mu^2}\mu^{2-p}\|f'\|_p^p,$$

and by (2.9), the last member is less than or equal to $\dfrac{A}{\mu^p}\|f\|_p^p$.

Consider now f''. Let

$$u''(x,t,y) = f''(x,t)*\Gamma(x,t,y)$$

and

$$\frac{\partial u''}{\partial y} = f''*\frac{\partial\Gamma}{\partial y} = \sum_i (f''\cdot\chi_{I_i})*\frac{\partial\Gamma}{\partial y},$$

where χ_{I_i} denotes the characteristic function of I_i. Hence denoting $(f''\cdot\chi_{I_i})*\dfrac{\partial\Gamma}{\partial y}$ by h_i, we have

$$\frac{\partial u''}{\partial y} = \sum_i h_i$$

and

$$g_\lambda^*(f'')(x,t) = \Bigg(\int_{E_{n+2}^+}\Big(\frac{y}{|x-z|+|t-s|^{1/2}+y}\Big)^{(n+2)\lambda}\Big|\sum_i h_i(z,s,y)\Big|^2\frac{dz\,ds\,dy}{y^{n+1}}\Bigg)^{1/2}$$
$$\leqslant g^{(1)}(x,t)+g^{(2)}(x,t),$$

where

$$g^{(1)}(x,t) = \Bigg(\int_{E_{n+2}^+}\Big(\frac{y}{|x-z|+|t-s|^{1/2}+y}\Big)^{(n+2)\lambda}\Big|\sum_{(z,s)\in\tilde{I}_i}h_i(z,s,y)\Big|^2\frac{dz\,ds\,dy}{y^{n+1}}\Bigg)^{1/2}$$

and

$$g^{(2)}(x,t) = \Bigg(\int_{E_{n+2}^+}\Big(\frac{y}{|x-z|+|t-s|^{1/2}+y}\Big)^{(n+2)\lambda}\Big|\sum_{(z,s)\notin\tilde{I}_i}h_i(z,s,y)\Big|^2\frac{dz\,ds\,dy}{y^{n+1}}\Bigg)^{1/2}.$$

Consider first $g^{(1)}(x,t)$. We claim that

$$\Big|\sum_{(z,s)\notin\tilde I_i} h_i(z,s,y)\Big| \leqslant \frac{A\mu}{y}. \tag{2.12}$$

To prove this, we note

$$\Big|\sum_{(z,s)\notin\tilde I_i} h_i(z,s,y)\Big| \leqslant \sum_{(z,s)\notin\tilde I_i}\int_{I_i}|f''(\xi,\eta)|\Big|\frac{\partial\Gamma}{\partial y}(z-\xi,s-\eta,y)\Big|d\xi\,d\eta$$
$$\leqslant \sum_{(z,s)\notin\tilde I_i}\Big\{\sup_{(\xi,\eta)\in I_i}\Big|\frac{\partial\Gamma}{\partial y}(z-\xi,s-\eta,y)\Big|\Big\}\int_{I_i}|f''(\xi,\eta)|\,d\xi\,d\eta,$$

which by (2.10) is less than

$$A\mu\sum_{(z,s)\notin\tilde I_i}\sup_{(\xi,\eta)\in I_i}\Big|\frac{\partial\Gamma}{\partial y}(z-\xi,s-\eta,y)\Big|\,|I_i|. \tag{2.13}$$

We shall show later that for $(z,s)\notin\tilde I_i$, the inequality

$$\sup_{(\xi,\eta)\in I_i}\Big|\frac{\partial\Gamma}{\partial y}(z-\xi,s-\eta,y)\Big| \leqslant \frac{A}{|I_i|}\int_{I_i}|s-\omega|^{-(n+3)/2}\exp\Big\{-c\,\frac{|z-u|^2+y^2}{|s-\omega|}\Big\}du\,d\omega \tag{2.13a}$$

holds for positive constants A and c not depending on i. If so, our last sum (2.13) is at most equal to

$$A\mu\int_{E_{n+1}}|\omega|^{-(n+3)/2}\exp\Big\{-c\,\frac{|u|^2+y^2}{|\omega|}\Big\}du\,d\omega = \frac{A\mu}{y},$$

which proves (2.12).

Hence,

$$[g^{(1)}(x,t)]^2 \leqslant A\mu\int_{E^+_{n+2}}\Big(\frac{y}{|x-z|+|t-s|^{1/2}+y}\Big)^{(n+2)\lambda}\Big|\sum_{(z,s)\notin\tilde I_i}h_i(z,s,y)\Big|\frac{dz\,ds\,dy}{y^{n+2}}$$
$$= A\mu J(x,t).$$

In order to show that

$$|\{(x,t):\ g^{(1)}(x,t)>\mu\}| \leqslant \frac{A}{\mu^p}\|f\|_p^p, \tag{2.14}$$

it is enough to prove that

$$|\{(x,t):\ J(x,t)>\mu\}| \leqslant \frac{A}{\mu^p}\|f\|_p^p. \tag{2.15}$$

We claim that $\|J\|_1\leqslant \frac{A}{\mu^{p-1}}\|f\|_p^p$. If so, we get

$$|\{(x,t):\ J(x,t)>\mu\}| \leqslant \frac{A}{\mu}\|J\|_1 \leqslant \frac{A}{\mu^p}\|f\|_p^p,$$

which proves (2.15). That $\|J\|_1\leqslant\frac{A}{\mu^{p-1}}\|f\|_p^p$ can be seen as follows:

$$\|J\|_1 = \int_{E^+_{n+2}}\Big|\sum_{(z,s)\notin\tilde I_i}h_i(z,s,y)\Big|\frac{dz\,ds\,dy}{y^{n+2}}\int_{E_{n+1}}\Big(\frac{y}{|x-z|+|t-s|^{1/2}+y}\Big)^{(n+2)\lambda}dx\,dt$$
$$= A\int_{E^+_{n+2}}\Big|\sum_{(z,s)\notin\tilde I_i}h_i(z,s,y)\Big|\,dz\,ds\,dy \leqslant A\sum_i\int_{(z,s)\notin\tilde I_i}|h_i(z,s,y)|\,dz\,ds\,dy.$$

Now,

$$\int_{(z,s)\notin\tilde I_i}|h_i(z,s,y)|\,dz\,ds\,dy = \int_{(z,s)\notin\tilde I_i}dz\,ds\,dy\Big|\int_{I_i}f''(u,\omega)\frac{\partial\Gamma}{\partial y}(z-u,s-\omega,y)\,du\,d\omega\Big|$$

which, by (2.11), is

$$\int_{(z,s)\notin\tilde I_i}\Big|\int_{I_i}f''(u,\omega)\Big\{\frac{\partial\Gamma}{\partial y}(z-u,s-\omega,y)-\frac{\partial\Gamma}{\partial y}(z-u_i,s-\omega_i,y)\Big\}du\,d\omega\Big|\,dz\,ds\,dy \tag{2.16}$$

where (u_i,ω_i) is the center of I_i.

We shall prove later that for $(u,\omega)\in I_i$ the integral

$$\int_{(z,s)\notin\tilde I_i}\Big|\frac{\partial\Gamma}{\partial y}(z-u,s-\omega,y)-\frac{\partial\Gamma}{\partial y}(z-u_i,s-\omega_i,y)\Big|\,dz\,ds\,dy \tag{2.17}$$

is less than a constant independent of i. If so, changing the order of integration in (2.16) we have that

$$A\int_{I_i}|f''(u,\omega)|\,du\,d\omega \leqslant A|I_i|\Big(\frac{1}{|I_i|}\int_{I_i}|f''(u,\omega)|^p\,du\,d\omega\Big)^{1/p}$$

majorizes (2.16). By (2.10), this is smaller than $A\mu|I_i|$. Hence

$$\|J\|_1 \leqslant A\mu \sum |I_i| = A\mu|\Omega| \leqslant \frac{A}{\mu^{p-1}}\|f\|_p^p$$

which proves our claim and therefore also (2.14).

Consider now $g^{(2)}(x,t)$. We shall show that

$$|\{(x,t):\ g^{(2)}(x,t) > \mu\}| \leqslant \frac{A}{\mu^p}\|f\|_p^p \tag{2.18}$$

where $p = 2/\lambda$. By (2.6) we may consider only points $(x,t)\notin\Omega$. Then,

$$[g^{(2)}(x,t)]^2 = \int_0^\infty \int_\Omega \left(\frac{y}{|x-z|+|t-s|^{1/2}+y}\right)^{(n+2)\lambda} \Big|\sum_{(z,s)\in\tilde I_i} h_i(z,s,y)\Big|^2 \frac{dz\,ds\,dy}{y^{n+1}}$$

$$\leqslant \sum_j \int_0^\infty \int_{I_j} \frac{y^{(n+2)\lambda-n-1}}{(|x-z|+|t-s|^{1/2})^{(n+2)\lambda}} \Big|\sum_{(z,s)\in\tilde I_i} h_i(z,s,y)\Big|^2 dz\,ds\,dy.$$

Now, for $(x,t)\notin\Omega$ (so $(x,t)\notin\tilde I_j$) and $(z,s)\in I_j$, we have

$$|x-z|+|t-s|^{1/2} \geqslant c\{|x-z_j|+|t-s_j|^{1/2}\}$$

where (z_j, s_j) is the center of I_j. Hence

$$[g^{(2)}(x,t)]^2 \leqslant c\sum_j \frac{1}{(|x-z_j|+|t-s_j|^{1/2})^{(n+2)\lambda}} \times$$

$$\times \int_0^\infty \int_{I_j} y^{(n+2)\lambda-n-1}\Big|\sum_{(z,s)\in\tilde I_i} h_i(z,s,y)\Big|^2 dz\,ds\,dy.$$

For the inner sum, we have

$$\Big|\sum_{(z,s)\in\tilde I_i} h_i(z,s,y)\Big| \leqslant \sum_{(z,s)\in\tilde I_i}\int_{I_i}\Big|f''(u,w)\frac{\partial\Gamma}{\partial y}(z-u, s-w, y)\Big|du\,dw. \tag{2.19}$$

If $(z,s)\in I_j\cap\tilde I_i$, we have from (2.8) that $I_i\subset I_j^*$. Hence, denoting by D_j the union of those I_i which are contained in I_j^*, we obtain that (2.19) is less than or equal to

$$\int_{D_j}|f''(u,w)|\Big|\frac{\partial\Gamma}{\partial y}(z-u,s-w,y)\Big|du\,dw$$

$$\leqslant c\int_{s-w>0} f^{(j)}(u,w)(s-w)^{-(n+3)/2}\exp\Big\{-\frac{|z-u|^2+y^2}{8(s-w)}\Big\}du\,dw,$$

where $f^{(j)}(u,w)$ denotes the restriction of $|f''(u,w)|$ to D_j. This last integral is equal to a constant times

$$\frac{1}{y}(f^{(j)}*\Gamma)(z,s,y).$$

Thus,

$$\int_0^\infty\int_{I_j} y^{(n+2)\lambda-n-1}\Big|\sum_{(z,s)\in\tilde I_i} h_i(z,s,y)\Big|^2 dz\,ds\,dy$$

$$\leqslant c\int_{E_{n+2}^+} y^{(n+2)\lambda-n-3}|(f^{(j)}*\Gamma)(z,s,y)|^2 dz\,ds\,dy.$$

By Plancherel's theorem, the last expression is equal to

$$c\int_0^\infty y^{(n+2)\lambda-n-3}dy\int_{E_{n+1}}|\widehat{f^{(j)}}(z,s)|^2|\exp\{-y\sqrt{|z|^2+is}\}|^2 dz\,ds$$

$$= c\int_{E_{n+1}}|\widehat{f^{(j)}}(z,s)|^2 dz\,ds\int_0^\infty y^{(n+2)\lambda-n-3}\exp\{-2y\,\mathrm{Re}\sqrt{|z|^2+is}\}dy$$

$$= c\int_{E_{n+1}}|\widehat{f^{(j)}}(z,s)|^2(\mathrm{Re}\sqrt{|z|^2+is})^{-(n+2)\lambda+n+2}dz\,ds.$$

But since $\mathrm{Re}\sqrt{|z|^2+is}\leqslant|\sqrt{|z|^2+is}|\leqslant 2\,\mathrm{Re}\sqrt{|z|^2+is}$, the last integral is majorized by

$$c\int_{E_{n+1}}\Big|\widehat{f^{(j)}}(z,s)\big(\sqrt{|z|^2+is}\big)^{-\frac{(n+2)\lambda-n-2}{2}}\Big|^2 dz\,ds$$

$$= c\|[\mathscr{I}_{\left(\frac{(n+2)\lambda-n-2}{2}\right)}(f^{(j)})]^\wedge(z,s)\|_2^2.$$

Now, for $p = 2/\lambda$ we have the identity

$$\frac{1}{2} = \frac{1}{p} - \frac{[(n+2)\lambda-n-2]/2}{n+2}.$$

Therefore, by theorem (1.6), we have

$$\|[\mathscr{I}_{\left(\frac{(n+2)\lambda-n-2}{2}\right)}(f^{(j)})]^\wedge\|_2^2 \leqslant c\|f^{(j)}\|_p^2. \tag{2.20}$$

But by (2.10),

$$\|f^{(j)}\|_p \leqslant \Big(\int_{D_j}|f''(z,s)|^p dz\,ds\Big)^{1/p} \leqslant A\Big(\sum_{I_i\subset I_j^*}\mu^p|I_i|\Big)^{1/p}$$

which by (2.8) is majorized by

$$A\mu|I_j|^{1/p}.$$

Therefore, from (2.20) we get

$$|g^{(2)}(x,t)|^2 \leqslant A\mu^2 \sum_j \frac{|I_j|^{2/p}}{(|x-z_j|+|t-s_j|^{1/2})^{(n+2)\lambda}} = A\mu^2 H(x,t),$$

where (z_j, s_j) is the center of I_j and $(x,t)\notin\Omega$.

To complete the proof of (2.18) it is enough to show that

$$|\{(x,t)\colon\ H(x,t)>1\}| \leqslant \frac{A}{\mu^p}\|f\|_p^p.$$

This will follow if we can prove that

$$\int_{\Omega'} H(x,t)\,dx\,dt \leqslant \frac{A}{\mu^p}\|f\|_p^p.$$

Now

$$\int_{\Omega'} H(x,t)\,dx\,dt = \sum_j |I_j|^{2/p}\int_{\Omega'}\frac{dx\,dt}{(|x-z_j|+|t-s_j|^{1/2})^{(n+2)\lambda}} \leqslant \sum_j |I_j|^{2/p}\int_{\tilde I_j'}\frac{dx\,dt}{(|x-z_j|+|t-s_j|^{1/2})^{(n+2)\lambda}}.$$

If l_j and l_j^2 are the lengths of the sides of I_j, we get

$$\int_{\tilde I_j'}\frac{dx\,dt}{(|x-z_j|+|t-s_j|^{1/2})^{(n+2)\lambda}} \leqslant A l_j^{-(n+2)(\lambda-1)} \leqslant A|I_j|^{-(\lambda-1)},$$

which implies that

$$\int_{\Omega'} H(x,t)\,dx\,dt \leqslant A\sum_j |I_j|^{2/p}\cdot|I_j|^{-(\lambda-1)} = A\sum |I_j| = A|\Omega| \leqslant \frac{A}{\mu^p}\|f\|_p^p.$$

This completes the proof of (2.18).

To complete the proof of the theorem we must show that (2.13a) and (2.17) are valid. Changing variables, we see it is enough to prove the following two lemmas.

(2.21) Lemma. *Let I denote a parallelepiped with center at the origin and edges parallel to the axes with lengths h and h^2 for the z and s directions respectively. Then, if $(z,s)\notin\tilde I$, we have*

$$\sup_{(\xi,\eta)\in I}\left\{|s-\eta|^{-(n+3)/2}\exp\left\{-a\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\}\right\} \leqslant \frac{c}{|I|}\int_I |s-\eta|^{-(n+3)/2}\exp\left\{-b\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\}d\xi\,d\eta,$$

where b and c are constants depending on a and n only.

Proof. If $(z,s)\notin\tilde I$, then $|z_i|>nh$ for at least one i or else $|s|>2n^2h^2$. This implies $|z|>nh$ or $|s|>2h^2$. We shall consider three cases: 1) $|z|>nh$ and $|s|>2h^2$; 2) $|z|>nh$ and $|s|\leqslant 2h^2$; and 3) $|z|\leqslant nh$ and $|s|>2h^2$.

We observe that under the assumption $(\xi,\eta)\in I$, the conditions $|z|>nh$ and $|s|>2h^2$ imply that for a constant $c>1$

$$\text{(2.22)}\qquad \frac{|z|}{c}<|z-\xi|<c|z| \quad\text{and}\quad \frac{|s|}{c}<|s-\eta|<c|s|,$$

respectively. Also, we remark that the function $t^{-(n+3)/2}e^{-\mu/t}$, $t>0$, has its absolute maximum at $t=2\mu/(n+3)$, and that the value of this maximum is

$$\left(\frac{n+3}{2\mu}\right)^{(n+3)/2}e^{-(n+3)/2}.$$

Case 1. Since $|z|>nh$ and $|s|>2h^2$ we have

$$|s-\eta|^{-(n+3)/2}\exp\left\{-a\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\} \leqslant \frac{c^{(n+3)/2}}{s^{(n+3)/2}}\exp\left\{-\frac{a}{c^3}\cdot\frac{|z|^2+y^2}{|s|}\right\}.$$

Also,

$$|s-\eta|^{-(n+3)/2}\exp\left\{-b\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\} \geqslant \frac{1}{c^{(n+3)/2}s^{(n+3)/2}}\exp\left\{-bc^3\frac{|z|^2+y^2}{|s|}\right\},$$

which proves the lemma in this case.

Case 2. Since $|z|>nh$, we have

$$|s-\eta|^{-(n+3)/2}\exp\left\{-a\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\} \leqslant |s-\eta|^{-(n+3)/2}\exp\left\{-\frac{a}{c^2}\frac{|z|^2+y^2}{|s-\eta|}\right\}.$$

Now, since $|s-\eta|\leqslant|s|+|\eta|\leqslant 2h^2+h^2/2<3h^2$, our second remark shows that the last expression is less than

$$\frac{1}{(3h^2)^{(n+3)/2}}\exp\left\{-\frac{a}{3c^2}\frac{|z|^2+y^2}{h^2}\right\} \quad\text{if}\quad 3h^2\leqslant\frac{2a}{(n+3)c^2}(|z|^2+y^2),\ \text{and}$$

$$\text{(2.23)}\qquad \left(\frac{n+3}{2}\frac{c^2}{a}\right)^{(n+3)/2}e^{-(n+3)/2}(|z|^2+y^2)^{-(n+3)/2} \quad\text{if}\quad 3h^2>\frac{2a}{(n+3)c^2}(|z|^2+y^2).$$

Also, we have that

$$|s-\eta|^{-(n+3)/2}\exp\left\{-b\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\} \geqslant \frac{1}{|s-\eta|^{(n+3)/2}}\exp\left\{-bc^2\frac{|z|^2+y^2}{|s-\eta|}\right\} \geqslant \frac{1}{(3h^2)^{(n+3)/2}}\exp\left\{-bc^2\frac{|z|^2+y^2}{|s-\eta|}\right\}.$$

Integrating this inequality over I, we get

$$\int_I |s-\eta|^{-(n+3)/2}\exp\left\{-b\,\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\}d\xi\,d\eta$$

$$\geqslant \frac{1}{(3h^2)^{(n+3)/2}}\int_I \exp\left\{-bc^2\frac{|z|^2+y^2}{|s-\eta|}\right\}d\xi\,d\eta$$

$$\geqslant \frac{1}{(3h^2)^{(n+3)/2}}\int_{I\cap\{(\xi,\eta):|s-\eta|>h^2/4\}} \exp\left\{-bc^2\frac{|z|^2+y^2}{|s-\eta|}\right\}d\xi\,d\eta$$

$$\geqslant \frac{1}{(3h^2)^{(n+3)/2}}\exp\left\{-4bc^2\frac{|z|^2+y^2}{h^2}\right\}\left|I\cap\left\{(\xi,\eta):\ |s-\eta|>\frac{h^2}{4}\right\}\right|.$$

Since $|I\cap\{(\xi,\eta):\ |s-\eta|>h^2/4\}|$ is greater than a constant times $|I|$, we obtain

$$\frac{1}{|I|}\int_I |s-\eta|^{-(n+3)/2}\exp\left\{-b\,\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\}d\xi\,d\eta$$

$$\geqslant \frac{A}{(3h^2)^{(n+3)/2}}\exp\left\{-4bc^2\frac{|z|^2+y^2}{h^2}\right\}.$$

If $3h^2<\dfrac{2a}{(n+3)c^2}(|z|^2+y^2)$, the last expression multiplied by a constant is greater than (2.23), provided that the constant and b are suitably chosen. If $3h^2>\dfrac{2a}{(n+3)c^2}(|z|^2+y^2)$, then

$$\frac{A}{(3h^2)^{(n+3)/2}}\exp\left\{-4bc^2\frac{|z|^2+y^2}{h^2}\right\}\geqslant\frac{A}{(3h^2)^{(n+3)/2}}\exp\left\{-6(n+3)\frac{bc^4}{a}\right\}.$$

But since $|z|>h$, (2.23) is less than

$$\left(\frac{n+3}{2}\,\frac{c^2}{a}\right)^{(n+3)/2}e^{-(n+3)/2}h^{-n-3},$$

and the lemma follows in this case too.

Case 3. For this case we have

$$|s-\eta|^{-(n+3)/2}\exp\left\{-a\,\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\}\leqslant\left(\frac{c}{|s|}\right)^{(n+3)/2}\exp\left\{-\frac{a}{c}\,\frac{y^2}{|s|}\right\}.$$

Also,

$$|s-\eta|^{-(n+3)/2}\exp\left\{-b\,\frac{|z-\xi|^2+y^2}{|s-\eta|}\right\}\geqslant\frac{1}{(c|s|)^{(n+3)/2}}\exp\left\{-4bcn^2\frac{h^2+y^2}{|s|}\right\}$$

$$\geqslant\frac{\exp\{-2bcn^2\}}{c^{(n+3)/2}}\,\frac{1}{|s|^{(n+3)/2}}\exp\left\{-4bcn^2\frac{y^2}{|s|}\right\},$$

which shows the validity of the lemma for case 3.

(2.24) LEMMA. *Let* $I=\{(z,s):\ |z_i|<1,\ i=1,\dots,n,$ *and* $|s|<1\}$. *Then, for every* $(u,w)\in I$ *we have*

$$\int_{I'}\left|\frac{\partial\Gamma}{\partial y}(z-u,s-w,y)-\frac{\partial\Gamma}{\partial y}(z,s,y)\right|dz\,ds\,dy\leqslant c<\infty.$$

Proof. Write

$$\frac{\partial\Gamma}{\partial y}(z-u,s-w,y)-\frac{\partial\Gamma}{\partial y}(z,s,y)=\int_0^1\frac{d}{dr}\left[\frac{\partial\Gamma}{\partial y}(z-ru,s-rw,y)\right]dr.$$

Taking absolute values and using $(u,w)\in I$, we get

$$\left|\frac{\partial\Gamma}{\partial y}(z-u,s-w,y)-\frac{\partial\Gamma}{\partial y}(z,s,y)\right|$$

$$\leqslant\sum_1^n\int_0^1\left|\frac{\partial^2\Gamma}{\partial x_i\partial y}(z-ru,s-rw,y)\right|dr+\int_0^1\left|\frac{\partial^2\Gamma}{\partial t\partial y}(z-ru,s-rw,y)\right|dr.$$

Integrating this inequality, changing variables and enlarging the domain of integration, we obtain

$$\int_{I'}\left|\frac{\partial\Gamma}{\partial y}(z-u,s-w,y)-\frac{\partial\Gamma}{\partial y}(z,s,y)\right|dz\,ds\,dy$$

$$\leqslant\sum_1^n\int_{I'}\left|\frac{\partial^2\Gamma}{\partial x_i\partial y}(z,s,y)\right|dz\,ds\,dy+\int_{I'}\left|\frac{\partial^2\Gamma}{\partial t\partial y}(z,s,y)\right|dz\,ds\,dy.$$

From the estimates (1.1) for the derivatives of Γ we have that the second member above is less than a constant times

$$\int_{I'}|s|^{-(n+4)/2}\exp\left\{-\frac{|z|^2+y^2}{8|s|}\right\}dz\,ds\,dy+\int_{I'}|s|^{-(n+5)/2}\exp\left\{-\frac{|z|^2+y^2}{8|s|}\right\}dz\,ds\,dy.$$

These two integrals are finite, and therefore the proof of the lemma is complete.

We shall discuss next a generalization of the g-function in (1.13). (See [3] and [8].)

Let $f\in L^p(E_{n+1})$ and $u(x,t,y)=f(x,t)*\Gamma(x,t,y)$. We define the derivative of order β of $u(x,t,y)$ (β real and positive) as

$$u^{(\beta)}(x,t,y)=\frac{e^{-i\gamma\pi}}{\Gamma(\gamma)}\int_0^\infty\frac{\partial^m u}{\partial y^m}(x,t,y+\eta)\eta^{\gamma-1}d\eta$$

where m is an integer, $0 < \gamma \leqslant 1$ and $\gamma+\beta = m$. To show that $u^{(\beta)}(x, t, y)$ is well defined for $y > 0$, we use the estimate

$$\frac{\partial^m \Gamma}{\partial y^m}(x, t, y+\eta) \leqslant c(y+\eta)^{-m}\Gamma(x, 2t, y+\eta)$$

(see (1.1)) together with (1.0) to obtain

$$\left|\frac{\partial^m u}{\partial y^m}(x, t, y+\eta)\right| \leqslant c\|f\|_p (y+\eta)^{-\left(m+\frac{n+2}{p}\right)}.$$

Hence,

$$\int_0^\infty \left|\frac{\partial^m u}{\partial y^m}(x, t, y+\eta)\right| \eta^{\gamma-1} d\eta \leqslant c\|f\|_p y^{-\beta-\frac{n+2}{p}},$$

which shows that $u^{(\beta)}(x, t, y)$ is finite for $y > 0$.

(2.25) THEOREM. *Let $f \in L^p(E_{n+1})$, $1 < p < \infty$, $\beta > 0$ and $u(x, t, y) = f(x, t) * \Gamma(x, t, y)$. There exist two positive constants c_1, c_2 such that the function*

$$g_\beta(f)(x, t) = \left(\int_0^\infty y^{2\beta-1}|u^{(\beta)}(x, t, y)|^2 dy\right)^{1/2}$$

satisfies

$$c_1\|f\|_p \leqslant \|g_\beta(f)\|_p \leqslant c_2\|f\|_p.$$

The constants do not depend on f.

This function $g_\beta(f)(x, t)$ may be called the parabolic Littlewood–Paley function of fractional order β. To prove the theorem we need the following lemma.

(2.26) LEMMA. *Let $0 < \beta < \beta'$. Then $g_\beta(f)(x, t) \leqslant c g_{\beta'}(f)(x, t)$, where the constant c is independent of f.*

Proof. It is easy to see that

$$u^{(\beta)}(x, t, y) = \int_0^\infty u^{(\beta')}(x, t, y+\eta)\eta^{\beta'-\beta-1} d\eta.$$

Hence, using Schwarz's inequality and changing variables, we have

$$|u^{(\beta)}(x, t, y)|^2 \leqslant \int_y^\infty |u^{(\beta')}(x, t, \eta)|^2 \eta^\mu (\eta-y)^{\beta'-\beta-1} d\eta \int_y^\infty \eta^{-\mu}(\eta-y)^{\beta'-\beta-1} d\eta$$

$$= c y^{-\mu+\beta'-\beta} \int_y^\infty |u^{(\beta')}(x, t, \eta)|^2 \eta^\mu (\eta-y)^{\beta'-\beta-1} d\eta,$$

provided that $\beta'-\beta < \mu$. Multiplying both members by $y^{2\beta-1}$ and integrating with respect to y we get

$$\int_0^\infty y^{2\beta-1}|u^{(\beta)}(x, t, y)|^2 dy \leqslant c\int_0^\infty y^{\beta+\beta'-\mu-1} dy \int_y^\infty |u^{(\beta')}(x, t, \eta)|^2 \eta^\mu (\eta-y)^{\beta'-\beta-1} d\eta.$$

Interchanging the order of integration, we obtain

$$\int_0^\infty y^{2\beta-1}|u^{(\beta)}(x, t, y)|^2 dy \leqslant c\int_0^\infty |u^{(\beta')}(x, t, \eta)|^2 \eta^\mu d\eta \int_0^\eta y^{\beta+\beta'-\mu-1}(\eta-y)^{\beta'-\beta-1} dy.$$

Now, if $\beta+\beta' > \mu$, the inner integral on the right equals $\eta^{2\beta'-\mu-1}$ times the value of $\int_0^1 y^{\beta+\beta'-\mu-1}(1-y)^{\beta'-\beta-1} dy$. Therefore, we get

$$[g_\beta(f)(x, t)]^2 = \int_0^\infty y^{2\beta-1}|u^{(\beta)}(x, t, y)|^2 dy$$

$$\leqslant c\int_0^\infty \eta^{2\beta'-1}|u^{(\beta')}(x, t, \eta)|^2 d\eta = c[g_{\beta'}(f)(x, t)]^2,$$

and the lemma is proved.

Proof of theorem (2.25). By lemma (2.26), we have

$$g_\beta(f)(x, t) \leqslant c g_m(f)(x, t),$$

where $m > \beta+2n+2$. Consider the inequality,

$$\left|\frac{\partial^m u}{\partial y^m}(x, t, y)\right| \leqslant \int_{E_{n+1}} \left|\frac{\partial u}{\partial y}(\xi, \eta, y/2)\frac{\partial^{m-1}\Gamma}{\partial y^{m-1}}(x-\xi, t-\eta, y/2)\right| d\xi\, d\eta.$$

By (1.1), we can write

$$\left|\frac{\partial^m u}{\partial y^m}(x, t, y)\right|$$

$$\leqslant c\int_{E_{n+1}} \left|\frac{\partial u}{\partial y}(\xi, \eta, y/2)\right| |t-\eta|^{-(n+1+m)/2} \exp\left\{-\frac{|x-\xi|^2+y^2/4}{8|t-\eta|}\right\} d\xi\, d\eta.$$

Applying Schwarz's inequality, we obtain

$$\left|\frac{\partial^m u}{\partial y^m}(x, t, y)\right|^2$$

$$\leqslant c\int_{E_{n+1}} \left|\frac{\partial u}{\partial y}(\xi, \eta, y/2)\right|^2 |t-\eta|^{-(n+1+m)/2} \exp\left\{-\frac{|x-\xi|^2+y^2/4}{8|t-\eta|}\right\} d\xi\, d\eta \times$$

$$\times \int_{E_{n+1}} |t-\eta|^{-(n+1+m)/2} \exp\left\{-\frac{|x-\xi|^2+y^2/4}{8|t-\eta|}\right\} d\xi\, d\eta,$$

where the last integral converges to a constant times y^{-m+1}. Then, multiplying by y^{2m-1} and integrating y from 0 to ∞ we get

$$\int_0^\infty y^{2m-1}\left|\frac{\partial^m u}{\partial y^m}(x,t,y)\right|^2 dy$$
$$\leqslant c\int_{E_{n+2}^+}\frac{y^m}{|t-\eta|^{(n+1+m)/2}}\exp\left\{-\frac{|x-\xi|^2+y^2/4}{8|t-\eta|}\right\}\left|\frac{\partial u}{\partial y}(\xi,\eta,y/2)\right|^2 d\xi\, d\eta .$$

By (1.2), the second member is less than

$$c\int_{E_{n+2}^+}\left(\frac{y}{|x-\xi|+|t-\eta|^{1/2}+y}\right)^{n+1+m}\left|\frac{\partial u}{\partial y}(\xi,\eta,y)\right|^2\frac{d\xi\, d\eta\, dy}{y^{n+1}}$$
$$= c[g^*_{\left(\frac{n+1+m}{n+2}\right)}(f)(x,t)]^2 .$$

Since $\lambda = \dfrac{n+1+m}{n+2} > \dfrac{3n+3}{n+2} \geqslant \dfrac{2n+4}{n+2} = 2$, theorem (2.2) implies

$$\|g_\beta(f)\|_p \leqslant c\|g_m(f)\|_p \leqslant c\|g^*_\lambda(f)\|_p \leqslant c\|f\|_p$$

for every $1<p<\infty$.

To complete the proofs of theorems (2.2) and (2.25), it remains for us to show

$$c_1\|f\|_p \leqslant \|g^*_\lambda(f)\|_p \quad \text{and} \quad c_1\|f\|_p \leqslant \|g_\beta(f)\|_p .$$

Let φ and $\psi \in C_0^\infty(E_{n+1})$ and $u = \varphi * \Gamma$, $v = \psi * \Gamma$. Consider the expression

$$\int_{E_{n+1}} dx\, dt\int_0^\infty y^{2\beta-1}u^{(\beta)}(x,t,y)v^{(\beta)}(x,t,y)\, dy .$$

Changing the order of integration and applying Plancherel's theorem, we see this is

$$c\int_0^\infty y^{2\beta-1}dy\int_{E_{n+1}}\bar{\hat{\varphi}}(x,t)\hat{\psi}(x,t)\left|\sqrt{|x|^2+it}\right|^2\exp\{-2y\operatorname{Re}\sqrt{|x|^2+it}\}\, dx\, dt .$$

Changing the order of integration once more, we get that this equals

$$c\int_{E_{n+1}}\bar{\hat{\varphi}}(x,t)\hat{\psi}(x,t)\left(\frac{\left|\sqrt{|x|^2+it}\right|}{\operatorname{Re}\sqrt{|x|^2+it}}\right)^{2\beta}dx\, dt,$$

and since $\left(\dfrac{\left|\sqrt{|x|^2+it}\right|}{\operatorname{Re}\sqrt{|x|^2+it}}\right)^{2\beta}$ is the symbol of an invertible parabolic singular integral (see [1]), we obtain that for a positive constant c,

$$\int_{E_{n+1}} dx\, dt\int_0^\infty y^{2\beta-1}u^{(\beta)}(x,t,y)v^{(\beta)}(x,t,y)\, dy = c\int_{E_{n+1}}\tilde{\varphi}(x,t)\psi(x,t)\, dx\, dt .$$

On the other hand, we have by Schwarz's inequality

$$\int_{E_{n+1}} dx\, dt\int_0^\infty y^{2\beta-1}u^{(\beta)}(x,t,y)v^{(\beta)}(x,t,y)\, dy \leqslant \int_{E_{n+1}} g_\beta(\varphi)(x,t)g_\beta(\psi)(x,t)\, dx\, dt .$$

Holder's inequality plus the part of theorem (2.25) already proved shows that if $p' = p/(p-1)$ then

$$\int_{E_{n+1}} dx\, dt\int_0^\infty y^{2\beta-1}u^{(\beta)}(x,t,y)v^{(\beta)}(x,t,y)\, dy$$
$$\leqslant \|g_\beta(\varphi)\|_p\|g_\beta(\psi)\|_{p'} \leqslant c\|g_\beta(\varphi)\|_p\|\psi\|_{p'} .$$

We conclude that for every $\varphi, \psi \in C_0^\infty(E_{n+1})$

$$\int_{E_{n+1}}\tilde{\varphi}(x,t)\psi(x,t)\, dx\, dt \leqslant c\|g_\beta(\varphi)\|_p\|\psi\|_{p'},$$

which implies $\|\tilde{\varphi}\|_p \leqslant c\|g_\beta(\varphi)\|_p$, and so $\|\varphi\|_p \leqslant c\|g_\beta(\varphi)\|_p$. The case of an arbitrary φ follows by continuity.

The proof for g^*_λ is similar and will not be given.

SECTION 3

CHARACTERIZATION OF $\mathcal{L}^p_\alpha$

This chapter is devoted to a characterization of the $\mathcal{L}^p_\alpha$ spaces defined in [5]. Our characterization is suggested by those given in [3] and [9]. See also theorem 3 of [4].

Let $\varphi(x,t)$ be an infinitely differentiable function belonging to L^r for every $1 \leqslant r \leqslant \infty$, together with all its derivatives, and let $\varphi_\alpha(x,t)$ denote its parabolic fractional integral of order α, $0<\alpha<1$; that is, $\varphi_\alpha(x,t) = \mathcal{J}_\alpha(\varphi)(x,t)$. We shall be concerned first with the function $T_\alpha(\varphi)(x,t)$ defined by

$$T_\alpha(\varphi)(x,t) = \left(\int_{E_{n+1}}\frac{|\varphi_\alpha(x-z,t-s)-\varphi_\alpha(x,t)|^2}{(|z|+|s|^{1/2})^{2\alpha+n+2}}\, dz\, ds\right)^{1/2},$$

and will prove the following theorem:

(3.1) THEOREM. *Let $\varphi(x,t)$ be an infinitely differentiable function which together with its derivatives of all orders belongs to all $L^r(E_{n+1})$, $1 \leqslant r \leqslant \infty$, and let $0<\alpha<1$. There exist two positive constants c_1, c_2 such that*

$$c_1\|\varphi\|_p \leqslant \|T_\alpha(\varphi)\|_p \leqslant c_2\|\varphi\|_p$$

provided that $p > 2(n+2)/(2a+n+2)$. *The constants* c_1, c_2 *depend on* a, p *and* η, *but not on* φ.

Proof. Let

$$u(x,t,y) = \varphi(x,t)*\Gamma(x,t,y) \text{ and } u_a(x,t,y) = \varphi_a(x,t)*\Gamma(x,t,y).$$

We denote $|z|+|s|^{1/2}$ by ϱ and observe that this ϱ is equivalent to the one introduced in the parabolic change of coordinates (1.3). Adding and subtracting $u_a(x,t,\varrho)$, we have

$$(3.2)\qquad |\varphi_a(x-z,t-s)-\varphi_a(x,t)| \leqslant |u_a(x-z,t-s,0)-u_a(x,t,\varrho)|+|u_a(x,t,\varrho)-u_a(x,t,0)|.$$

For the first term on the right we have

$$u_a(x-z,t-s,0)-u_a(x,t,\varrho) = -\int_0^\varrho \frac{d}{dr}\left[u_a\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right]dr.$$

Integrating by parts, we obtain

$$u_a(x-z,t-s,0)-u_a(x,t,\varrho) = -r\frac{d}{dr}\left[u_a\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right]\Bigg|_0^\varrho + \frac{r^2}{2}\frac{d^2}{dr^2}\left[u_a\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right]\Bigg|_0^\varrho - \int_0^\varrho \frac{r^2}{2}\frac{d^3}{dr^3}\left[u_a\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right]dr.$$

Performing the indicated differentiations, we obtain

$$|u_a(x-z,t-s,0)-u_a(x,t,\varrho)| \leqslant c\Bigg\{\sum_{|h|+k+l=1,2} \varrho^{|h|+2k+l}\left|\frac{\partial^{|h|+k+l}u_a}{\partial x^h\partial t^k\partial y^l}(x,t,\varrho)\right| + \sum_{|h|+k+l=3}\int_0^\varrho r^{k+2}\left|\frac{\partial^3 u_a}{\partial x^h\partial t^k\partial y^l}\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right|dr + \sum_{|h|+k+l=1}\int_0^\varrho r^{k+2}\left|\frac{\partial^2 u_a}{\partial x^h\partial t^{1+k}\partial y^l}\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right|dr\Bigg\}.$$

For the second term on the right of (3.2), we have

$$u_a(x,t,\varrho)-u_a(x,t,0) = \int_0^\varrho \frac{\partial u_a}{\partial y}(x,t,r)\,dr,$$

or, integrating by parts,

$$u_a(x,t,\varrho)-u_a(x,t,0) = \varrho\frac{\partial u_a}{\partial y}(x,t,\varrho)-\frac{\varrho^2}{2}\frac{\partial^2 u_a}{\partial y^2}(x,t,\varrho)+\int_0^\varrho \frac{r^2}{2}\frac{\partial^3 u_a}{\partial y^3}(x,t,r)\,dr.$$

Therefore,

$$|u_a(x,t,\varrho)-u_a(x,t,0)| \leqslant \varrho\left|\frac{\partial u_a}{\partial y}(x,t,\varrho)\right| + \frac{\varrho^2}{2}\left|\frac{\partial^2 u_a}{\partial y^2}(x,t,\varrho)\right| + \int_0^\varrho \frac{r^2}{2}\left|\frac{\partial^3 u_a}{\partial y^3}(x,t,r)\right|dr.$$

The Fourier transform of $\dfrac{\partial^{|h|+k+l}u_a}{\partial x^h\partial t^k\partial y^l}$ is a constant times

$$\exp\{-y\sqrt{|x|^2+it}\}\,(\sqrt{|x|^2+it})^{l-a}t^k x^h\hat{\varphi}(x,t).$$

Hence if we denote by $\bar{\varphi}$ the function whose Fourier transform is

$$\hat{\bar{\varphi}}(x,t) = \frac{t^k x^h}{(\sqrt{|x|^2+it})^{|h|+2k}}\,\hat{\varphi}(x,t),$$

then

$$(3.3)\qquad \frac{\partial^{|h|+k+l}u_a}{\partial x^h\partial t^k\partial y} = \frac{\partial^{|h|+2k+l}\bar{u}_a}{\partial y^{|h|+2k+l}},$$

where $\bar{u}_a = \mathscr{J}_a(\bar{\varphi})(x,t)*\Gamma(x,t,y)$. It can easily be seen that $\bar{\varphi}$ is an infinitely differentiable function which together with its derivatives of any order belongs to $L^r(E_{n+1})$ for every $1 < r \leqslant \infty$. Moreover, $\|\bar{\varphi}\|_p \leqslant c_p\|\varphi\|_p$ for $1 < p < \infty$ (see [1]).

With the expression (3.3) for the partial derivatives of $u_a(x,t,y)$ and the estimates we got above, we can write

$$|\varphi_a(x-z,t-s)-\varphi_a(x,t)|^2 \leqslant c\Bigg\{\sum_{k=1}^4 \varrho^{2k}\left|\frac{\partial^k \bar{u}_a}{\partial y^k}(x,t,\varrho)\right|^2 + \sum_{k=0}^3\left(\int_0^\varrho r^{2+k}\left|\frac{\partial^{3+k}\bar{u}_a}{\partial y^{3+k}}\left(x-z+r\frac{z}{\varrho},\, t-s+r^2\frac{s}{\varrho^2},\, r\right)\right|dr\right)^2 + \left(\int_0^\varrho r^2\left|\frac{\partial^3 u_a}{\partial y^3}(x,t,r)\right|dr\right)^2\Bigg\}.$$

Multiplying both members by $\varrho^{-(2a+n+2)} = (|z|+|s|^{1/2})^{-(2a+n+2)}$ and integrating in (z, s) over E_{n+1}, we obtain

$$(3.4)\quad [T_a(\varphi)(x,t)]^2$$

$$= \int_{E_{n+1}} \frac{|\varphi_a(x-z, t-s)-\varphi_a(z,s)|^2}{(|z|+|s|^{1/2})^{2a+n+2}}\,dz\,ds$$

$$\leqslant c\Big\{\sum_{k=1}^{4}\int_{E_{n+1}} \varrho^{2k-2a-n-2}\left|\frac{\partial^k \bar{u}_a}{\partial y^k}(x,t,\varrho)\right|^2 dz\,ds +$$

$$+\sum_{k=0}^{3}\int_{E_{n+1}} \varrho^{-2a-n-2}\left(\int_0^{\varrho} r^{2+k}\left|\frac{\partial^{3+k}\bar{u}_a}{\partial y^{3+k}}\left(x-z+r\frac{z}{\varrho}, t-s+r^2\frac{s}{\varrho^2}, r\right)\right|dr\right)^2 dz\,ds$$

$$+\int_{E_{n+1}} \varrho^{-2a-n-2}\left(\int_0^{\varrho} r^2\left|\frac{\partial^3 u_a}{\partial y^3}(x,t,r)\right|dr\right)^2 dz\,ds\Big\}.$$

For the first group of terms on the right, we pass to parabolic polar coordinates and obtain

$$\int_{E_{n+1}} \varrho^{2k-2a-n-2}\left|\frac{\partial^k \bar{u}_a}{\partial y^k}(x,t,\varrho)\right|^2 dz\,ds$$

$$\leqslant c\int_0^{\infty} \varrho^{2(k-a)-1}\left|\frac{\partial^k \bar{u}_a}{\partial y^k}(x,t,\varrho)\right|^2 d\varrho = c[g_{(k-a)}(\bar{\varphi})(x,t)]^2.$$

By theorem (2.25), $g_{(k-a)}(\bar{\varphi})$ has L^p norm less than a constant times $\|\bar{\varphi}\|_p \leqslant c_p\|\varphi\|_p$.

For the second group of terms on the right of (3.4), we apply Schwarz's inequality and obtain the majorization

$$(3.5)\quad \int_{E_{n+1}} \varrho^{-2a-n-2+\varepsilon}\,dz\,ds \times$$

$$\times \int_0^{\varrho} r^{5+2k-\varepsilon}\left|\frac{\partial^{k+3}\bar{u}_a}{\partial y^{k+3}}\left(x-z+r\frac{z}{\varrho}, t-s+r^2\frac{s}{\varrho^2}, r\right)\right|^2 dr,$$

where ε is an arbitrary number between 0 and $2a$.

We claim that

$$(3.6)\quad \left|\frac{\partial^{k+3}\bar{u}_a}{\partial y^{k+3}}(u,w,r)\right|$$

$$\leqslant c\int_{E_{n+1}}\left|\frac{\partial \bar{u}}{\partial y}(\xi,\eta,r/2)\right| |w-\eta|^{-\frac{n+k+4-a}{2}} \exp\left\{-\frac{|u-\xi|^2+r^2/4}{8|w-\eta|}\right\} d\xi\,d\eta.$$

This can be seen as follows. By definition of fractional integration, we have

$$\left|\frac{\partial^{k+3}\bar{u}_a}{\partial y^{k+3}}(u,w,r)\right|$$

$$\leqslant \int_0^{\infty}\left|\frac{\partial^{k+3}\bar{u}}{\partial y^{k+3}}(u,w,r+s)\right| s^{a-1}\,ds$$

$$\leqslant \int_0^{\infty}\int_{E_{n+1}}\left|\frac{\partial \bar{u}}{\partial y}(\xi,\eta,r/2)\right|\left|\frac{\partial^{k+2}\Gamma}{\partial y^{k+2}}(u-\xi, w-\eta, r/2+s)\right| s^{a-1}\,d\xi\,d\eta\,ds$$

which, by (1.1), is smaller than a constant times

$$\int_{E_{n+1}}\left|\frac{\partial \bar{u}}{\partial y}(\xi,\eta,r/2)\right|\Big(\int_0^{\infty}|w-\eta|^{-\frac{n+4+k}{2}}\times$$

$$\times\exp\left\{-\frac{|u-\xi|^2+r^2/4+s^2}{8|w-\eta|}\right\}s^{a-1}\,ds\Big)\,d\xi\,d\eta$$

$$= \int_{E_{n+1}}\left|\frac{\partial \bar{u}}{\partial y}(\xi,\eta,r/2)\right| |w-\eta|^{-(n+4+k)/2}\exp\left\{-\frac{|u-\xi|^2+r^2/4}{8|w-\eta|}\right\}\times$$

$$\times\left(\int_0^{\infty}\exp\left\{-\frac{s^2}{8|w-\eta|}\right\}s^{a-1}\,ds\right)d\xi\,d\eta$$

$$= c\int_{E_{n+1}}\left|\frac{\partial \bar{u}}{\partial y}(\xi,\eta,r/2)\right| |w-\eta|^{-(n+4+k-a)/2}\exp\left\{-\frac{|u-\xi|^2+r^2/4}{8|w-\eta|}\right\}d\xi\,d\eta,$$

which proves (3.6).

Now let $0 < a, b, a+b = 2$, $a > \dfrac{2a+n+2-\varepsilon}{n+4+k-a}$ and $b > \dfrac{n+2}{n+4+k-a}$.

We can choose such a and b since

$$(3.7)\quad \frac{2a+n+2-\varepsilon}{n+4+k-a}+\frac{n+2}{n+4+k-a} < \frac{2n+6}{n+3} = 2.$$

Applying Schwarz's inequality in (3.6), we get

$$\left|\frac{\partial^{k+3}\bar{u}_a}{\partial y^{k+3}}(u,w,r)\right|^2$$

$$\leqslant \int_{E_{n+1}}\left|\frac{\partial \bar{u}}{\partial y}(\xi,\eta,r/2)\right|^2 |w-\eta|^{-a(n+k-a)/2}\exp\left\{-a\frac{|u-\xi|^2+r^2/4}{8|w-\eta|}\right\}d\xi\,d\eta\times$$

$$\times\int_{E_{n+1}}|w-\eta|^{-b(n+k+4-a)/2}\exp\left\{-b\frac{|u-\xi|^2+r^2/4}{8|w-\eta|}\right\}d\xi\,d\eta.$$

The last integral is a constant times $r^{-b(n+k+4-a)+n+2}$. Therefore (3.5) is less than a constant times

$$(3.8)\quad \int_{E_{n+1}} \varrho^{-2a-n-2+\varepsilon}\,dz\,ds\int_0^{\varrho} r^{7+n+2k-\varepsilon-b(n+k+4-a)}\,dr\,\times$$
$$\times\int_{E_{n+1}}\left|\frac{\partial\bar u}{\partial y}(\xi,\eta,r/2)\right|^2\left|t-s+r^2\frac{s}{\varrho^2}-\eta\right|^{-a(n+k+4-a)/2}\times$$
$$\times\exp\left\{-a\frac{\left|x-z+r\dfrac{z}{\varrho}-\xi\right|^2+r^2/4}{8\left|t-s+r^2\dfrac{s}{\varrho^2}-\eta\right|}\right\}d\xi\,d\eta.$$

A change of variables in ξ, η gives

$$\int_{E_{n+1}} \varrho^{-2a-n-2+\varepsilon}\,dz\,ds\int_0^{\varrho} r^{9+2n+2k-\varepsilon-b(n+k+4-a)}\,dr\,\times$$
$$\times\int_{E_{n+1}}\left|\frac{\partial\bar u}{\partial y}(x-r\xi,t-r^2\eta,r/2)\right|^2\left|r^2\eta-s+s\frac{r^2}{\varrho^2}\right|^{-a(n+4+k-a)/2}\times$$
$$\times\exp\left\{-a\frac{|r\xi-z+rz/\varrho|^2+r^2/4}{8\left|r^2\eta-s+r^2\dfrac{s}{\varrho^2}\right|}\right\}d\xi\,d\eta.$$

Changing the order of integration and then changing variables in z, s, we obtain

$$(3.9)\quad \int_0^{\infty} r\,dr\int_{E_{n+1}}\left|\frac{\partial\bar u}{\partial y}(x-r\xi,t-r^2\eta,r/2)\right|^2 d\xi\,d\eta\,\times$$
$$\times\int_{\varrho>1}\varrho^{-2a-n-2+\varepsilon}\left|\eta-s+\frac{s}{\varrho^2}\right|^{-a(n+4+k-a)/2}\times$$
$$\times\exp\left\{-a\frac{\left|\xi-z+\dfrac{z}{\varrho}\right|^2+1/4}{8\left|\eta-s+\dfrac{s}{\varrho^2}\right|}\right\}dz\,ds.$$

We claim that the innermost integral in (3.9) is smaller than a constant times

$$(3.10)\quad (|\xi|+|\eta|^{1/2}+1)^{-(2a+n+2-\varepsilon)}.$$

We consider two cases: (1) $|\xi|+|\eta|^{1/2}\leqslant 4m$, and (2) $|\xi|+|\eta|^{1/2}>4m$ where m is a large positive constant to be chosen.

Case 1. Since the function

$$|s|^{-a(n+4k-a)/2}e^{-a(|z|^2+1/4)/8|s|}$$

is bounded on E_{n+1}, we see that the integral under consideration is less than

$$c\int_{\varrho>1}\varrho^{-2a-n-2+\varepsilon}\,dz\,ds<\infty,$$

and since the function $(|\xi|+|\eta|^{1/2}+1)^{-2a-n-2+\varepsilon}$ is bounded below by a positive number for $|\xi|+|\eta|^{1/2}\leqslant 4m$, our claim follows in this case.

Case 2. We write our integral as

$$\int_{\varrho>1}=\int_{1<\varrho<\frac14(|\xi|+|\eta|^{1/2})}+\int_{\varrho\geqslant\frac14(|\xi|+|\eta|^{1/2})}=A+B.$$

Consider first the integral A. If $|\xi|\geqslant|\eta|^{1/2}$ then $|\xi|+|\eta|^{1/2}\leqslant 2|\xi|$ and since $1<\varrho<\frac14(|\xi|+|\eta|^{1/2})$, we obtain $|z|<\frac12|\xi|$. Thus

$$\left|\xi-z+\frac{z}{\varrho}\right|\geqslant|\xi|-|z|\geqslant\frac12|\xi|\geqslant\frac15(|\xi|+|\eta|^{1/2}+1).$$

If $|\xi|<|\eta|^{1/2}$ an analogous analysis gives that

$$\left|\eta-s+\frac{s}{\varrho^2}\right|\geqslant\frac{1}{25}(|\xi|+|\eta|^{1/2}+1)^2.$$

Hence if $1<\varrho<\frac14(|\xi|+|\eta|^{1/2})$ and $|\xi|+|\eta|^{1/2}\geqslant 8$, we have

$$\left|\xi-z+\frac{z}{\varrho}\right|+\left|\eta-s+\frac{s}{\varrho^2}\right|^{1/2}+1\geqslant\frac15(|\xi|+|\eta|^{1/2}+1.$$

With this inequality and the fact that

$$(|\xi|+|\eta|^{1/2}+1)^{a(n+4+k-a)}|\eta|^{-a(n+4+k-a)/2}\exp\left\{-a\frac{|\xi|^2+1/4}{8|\eta|}\right\}$$

is bounded on E_{n+1} (see (1.2)) we obtain

$$A\leqslant c(|\xi|+|\eta|^{1/2}+1)^{-a(n+4+k-a)}\int_{\varrho>1}\varrho^{-2a-n-2+\varepsilon}\,dz\,ds$$
$$=c(|\xi|+|\eta|^{1/2}+1)^{-a(n+4+k-a)}\leqslant c(|\xi|+|\eta|^{1/2}+1)^{-2a-n-2+\varepsilon}.$$

The last inequality is true since by (3.7)

$$a(n+4+k-a)>2a+n+2-\varepsilon.$$

The value of B is less than

(3.11) $$\frac{c}{(|\xi|+|\eta|^{1/2}+1)^{2a+n+2-\varepsilon}}\times$$
$$\times\int_{\varrho>m}\left|\eta-s+\frac{s}{\varrho^2}\right|^{-a(n+4+k-a)/2}\exp\left\{-a\frac{\left|\xi-z+\frac{z}{\varrho}\right|^2+\frac{1}{4}}{8\left|\eta-s+\frac{s}{\varrho^2}\right|}\right\}dz\,ds.$$

Consider the change of variables

$$z'=z\left(1-\frac{1}{\varrho}\right),\qquad s'=s\left(1-\frac{1}{\varrho^2}\right)$$

for $\varrho>m$. We shall prove that this is one-to-one and that the absolute value of its Jacobian is greater than a positive constant. Let (z_1,s_1) and (z_2,s_2) be two points such that $\varrho_1>\varrho_2>m$. If $z_1'=z_2'$ and $s_1'=s_2'$, then

$$|z_1|\left(1-\frac{1}{\varrho_1}\right)=|z_2|\left(1-\frac{1}{\varrho_2}\right)$$

and since $1-\frac{1}{\varrho_1}>1-\frac{1}{\varrho_2}$, it follows that $|z_1|<|z_2|$. In the same way we obtain $|s_1|<|s_2|$ and therefore $\varrho_1<\varrho_2$, which is a contradiction. Thus, if two points have the same image then $\varrho_1=\varrho_2$, which in turn implies $z_1=z_2$ and $s_1=s_2$.

To estimate the Jacobian $\frac{\partial(z',s')}{\partial(z,s)}$, we observe that

$$\frac{\partial z_j'}{\partial z_j}=1-\frac{1}{\varrho}+\frac{z_j^2}{|z|\varrho^2}=1+O\left(\frac{1}{\varrho}\right),\quad \frac{\partial s'}{\partial s}=1+\frac{1}{\varrho^2}+\frac{|s|^{1/2}}{\varrho^3}=1+O\left(\frac{1}{\varrho}\right),$$
$$\frac{\partial z_j'}{\partial z_i}=\frac{z_iz_j}{|z|\varrho^2}=O\left(\frac{1}{\varrho}\right)\quad\text{for}\quad i\neq j,$$
$$\frac{\partial z_j'}{\partial s}=\frac{z_j\,\mathrm{sign}\,s}{2|s|^{1/2}\varrho^2}\quad\text{and}\quad\frac{\partial s'}{\partial z_i}=\frac{2sz_i}{|z|\varrho^3}.$$

In particular,

$$\frac{\partial z_j'}{\partial s}\frac{\partial s'}{\partial z_i}=\frac{|s|^{1/2}z_iz_j}{|z|\varrho^5}=O\left(\frac{1}{\varrho}\right).$$

Therefore, in the expansion of the Jacobian, the term which arises from the product down the diagonal is $1+O(1/\varrho)$ and every other term is $O(1/\varrho)$. If we choose m sufficiently large, it follows from the condition $\varrho>m$ that the Jacobian exceeds a positive constant.

Therefore, changing variables and enlarging the domain of integration, we see that the integral in (3.11) turns out to be less than a constant times

$$\int_{E_{n+1}}|s|^{-a(n+4+k-a)/2}\exp\left\{-a\frac{|z|^2+1/4}{8|s|}\right\}dz\,ds.$$

This converges since $a>\frac{2a+n+2-\varepsilon}{n+4+k-a}$, $0<\varepsilon<2a$, and the proof of case 2 is complete.

We see that (3.10) implies that (3.9) is less than a constant times

$$\int_0^\infty\int_{E_{n+1}}\frac{r}{(|\xi|+|\eta|^{1/2}+1)^{2a+n+2-\varepsilon}}\left|\frac{\partial\bar u}{\partial y}(x-r\xi,t-r^2\eta,r/2)\right|^2d\xi\,d\eta\,dr.$$

A change of variables shows this is

$$\int_{E_{n+2}^+}\left(\frac{r}{|x-\xi|+|t-\eta|^{1/2}+r}\right)^{2a+n+2-\varepsilon}\left|\frac{\partial\bar u}{\partial y}(\xi,\eta,r)\right|^2\frac{d\xi\,d\eta\,dr}{r^{n+1}}=[g_\lambda^*(\bar\varphi)(x,t)]^2,$$

where $\lambda=\frac{2a+n+2-\varepsilon}{n+2}$.

By theorem (2.2) the L^p-norm of the square root of the second term on the right of (3.4) is less than a constant times $\|\varphi\|_p$, provided that $p>\frac{2(n+2)}{2a+n+2-\varepsilon}$. But since ε can be chosen arbitrarily small, the condition becomes $p>\frac{2(n+2)}{2a+n+2}$.

Finally, for the third term on the right of (3.4), we have

$$\int_{E_{n+1}}\varrho^{-2a-n-2}\left(\int_0^\varrho r^2\left|\frac{\partial^3u_a}{\partial y^3}(x,t,r)\right|dr\right)^2dz\,ds$$
$$\leqslant c\int_{E_{n+1}}\varrho^{-2a-n-2+\varepsilon}\left(\int_0^\varrho r^{5-\varepsilon}\left|\frac{\partial^3u_a}{\partial y^3}(x,t,r)\right|^2dr\right)dz\,ds,$$

which by (3.6) is smaller than

$$c\int_{E_{n+1}}\varrho^{-2a-n-2+\varepsilon}dz\,ds\int_0^\varrho r^{5-\varepsilon}dr\times$$
$$\times\left(\int_{E_{n+1}}\left|\frac{\partial u}{\partial y}(\xi,\eta,r/2)\right||t-\eta|^{-(n+4-a)/2}\exp\left\{-\frac{|x-\xi|^2+r^2/4}{8|t-\eta|}\right\}d\xi\,d\eta\right)^2.$$

Let $a > \frac{2a+n+2-\varepsilon}{n+4-a}$, $b > \frac{n+2}{n+4-a}$, $a+b=2$. By Schwarz's inequality, the last integral is smaller than

$$c\int_{E_{n+1}} \varrho^{-2a-n-2+\varepsilon}\,dz\,ds\int_0^{\varrho} r^{7+n-\varepsilon-b(n+4-a)}\,dr\times$$
$$\times\int_{E_n}\left|\frac{\partial u}{\partial y}(\xi,\eta,r/2)\right|^2|t-\eta|^{-a(n+4-a)/2}\exp\left\{-a\frac{|x-\xi|^2+r^2/4}{8|t-\eta|}\right\}d\xi\,d\eta.$$

Changing the order of integration and applying (1.2), we get

$$c\int_{E^+_{n+2}}\left(\frac{r}{|x-\xi|+|t-\eta|^{1/2}+r}\right)^{a(n+4-a)}\left|\frac{\partial u}{\partial y}(\xi,\eta,r)\right|^2\frac{d\xi\,d\eta\,dr}{r^{n+1}} = c[g^*_\lambda(\varphi)(x,t)]^2,$$

where $\lambda = \frac{a(n+4-a)}{n+2}$.

Now, since $a > \frac{2a+n+2-\varepsilon}{n+4-a}$, it follows that

$$\lambda = \frac{a(n+4-a)}{n+2} > \frac{2a+n+2-\varepsilon}{n+2}.$$

Therefore, the square root of the third integral on the right of (3.4) has L^p-norm less than a constant times $\|\varphi\|_p$, provided that $p > \frac{2(n+2)}{2a+n+2}$, as before.

To complete the proof of theorem (3.1), we must show that $c_1\|\varphi\|_p \leqslant \|T_a(\varphi)\|_p$. In order to prove this, we observe that

$$\int_{E_{n+1}}\frac{\partial\Gamma}{\partial y}(z,s,y)\,dz\,ds = 0 \quad \text{for} \quad y>0.$$

Therefore

$$\frac{\partial u_a}{\partial y}(x,t,y) = \int_{E_{n+1}}[\varphi_a(x-z,t-s)-\varphi_a(x,t)]\frac{\partial\Gamma}{\partial y}(z,s,y)\,dz\,ds.$$

By (1.1), we have

$$\left|\frac{\partial u_a}{\partial y}(x,t,y)\right|$$
$$\leqslant c\int_{E_{n+1}}|\varphi_a(x-z,t-s)-\varphi_a(x,t)|\,|s|^{-(n+3)/2}\exp\left\{-\frac{|z|^2+y^2}{8|s|}\right\}dz\,ds.$$

If a and b satisfy

$$\frac{n+2}{n+3} < b < \frac{n+4-2a}{n+3}, \qquad a = 2-b,$$

then, by Schwarz's inequality,

$$\left|\frac{\partial u_a}{\partial y}(x,t,y)\right|^2$$
$$\leqslant c\int_{E_{n+1}}|\varphi_a(x-z,t-s)-\varphi_a(x,t)|^2\,|s|^{-a\frac{n+3}{2}}\exp\left\{-a\frac{|z|^2+y^2}{8|s|}\right\}dz\,ds\times$$
$$\times\int_{E_{n+1}}|s|^{-b\frac{n+3}{2}}\exp\left\{-b\frac{|z|^2+y^2}{8|s|}\right\}dz\,ds.$$

The last integral converges to a constant times $y^{-b(n+3)+n+2}$. Therefore

$$\int_0^\infty y^{1-2a}\left|\frac{\partial u_a}{\partial y}(x,t,y)\right|^2dy$$
$$\leqslant c\int_{E_{n+1}}|\varphi_a(x-z,t-s)-\varphi_a(x,t)|^2|s|^{-a\frac{n+3}{2}}\exp\left\{-a\frac{|z|^2}{8|s|}\right\}dz\,ds\times$$
$$\times\int_0^\infty y^{-(b-1)(n+3)-2a}\exp\left\{-\frac{y^2}{8|s|}\right\}dy$$
$$= c\int_{E_{n+1}}|\varphi_a(x-z,t-s)-\varphi_a(x,t)|^2|s|^{-(n+2+2a)/2}\exp\left\{-a\frac{|z|^2}{8|s|}\right\}dz\,ds.$$

Applying (1.2), we get

$$[g_{(1-a)}(\varphi)(x,t)]^2 = \int_0^\infty y^{1-2a}\left|\frac{\partial u_a}{\partial y}(x,t,y)\right|^2dy$$
$$\leqslant c\int_{E_{n+1}}\frac{|\varphi_a(x-z,t-s)-\varphi_a(x,t)|^2}{(|z|+|s|^{1/2})^{2a+n+2}}\,dz\,ds$$
$$= c[T_a(\varphi)(x,t)]^2.$$

Theorem (2.25) gives the inequality

$$c_1\|\varphi\|_p \leqslant \|T_a(\varphi)\|_p,$$

which completes the proof of theorem (3.1).

Let $P_a(f)$ be the function

$$(3.12)\qquad P_a(f)(x,t) = \left(\int_{E_{n+1}} \frac{|f(x-z,t-s)-f(x,t)|^2}{(|z|+|s|^{1/2})^{2a+n+2}}\,dz\,ds\right)^{1/2}.$$

We shall prove the following theorem, which is the main result of this section.

(3.13) THEOREM. *For* $0<a<1$ *and* $\frac{2(n+2)}{2a+n+2}<p<\infty$, *the following two conditions are equivalent:*

(3.14) 1) $f\in L^p(E_{n+1})$ *and* $P_a(f)\in L^p(E_{n+1})$,

(3.15) 2) $f\in \mathscr{L}^p_a(E_{n+1})$.

Moreover, if $f\in\mathscr{L}^p_a(E_{n+1})$, *there exist two positive constants* c_1, c_2 *independent of* f *such that*

$$(3.16)\qquad c_1\|f\|_{p,a}\leqslant \|f\|_p+\|P_a(f)\|_p\leqslant c_2\|f\|_{p,a}.$$

To prove this theorem we will need two lemmas.

(3.17) LEMMA. *For* $0<a<4$, *there exist three finite measures* $\mu_a^{(1)}$, $\mu_a^{(2)}$ *and* $\mu_a^{(3)}$ *on* E_{n+1} *satisfying*

1) $(|x|^2+it)^{a/2} = \widehat{\mu_a^{(1)}}(x,t)(1+|x|^2+it)^{a/2}$,

and

2) $(1+|x|^2+it)^{a/2} = \widehat{\mu_a^{(2)}}(x,t)+\widehat{\mu_a^{(3)}}(x,t)(|x|^2+it)^{a/2}$.

Proof. The proof of a similar lemma in [11] applies without change, provided that $0<a<4$.

(3.18) LEMMA. *Let* $\varphi\in L^r(E_{n+1})$ *for every* $1\leqslant r<\infty$ *and let* $0<a<n+2$. *Then*

$$(3.19)\qquad \mathscr{J}_a(\mu_a^{(1)}*\varphi) = \mathscr{G}_a(\varphi)$$

almost everywhere on E_{n+1}.

Proof. The term on the left of (3.19) is well defined and belongs to every L^p with $p>\frac{n+2}{n+2-a}$. For $\psi(x,t)\in\mathscr{S}$

$$(3.20)\qquad \int_{E_{n+1}} \psi(x,t)\,\mathscr{J}_a(\mu_a^{(1)}*\varphi)(x,t)\,dx\,dt$$

$$= \int_{E_{n+1}}\Big[\int_{E_{n+1}} \mathscr{J}^a(x-u,t-w)\,\psi(x,t)\,dx\,dt\Big](\mu_a^{(1)}*\varphi)(u,w)\,du\,dw.$$

We observed in section 1 that

$$\Gamma(a)^{-1}\int_0^\infty \Gamma(x,t,y)\,y^{a-1}\,dy = \mathscr{J}^a(x,t).$$

Thus, if we denote by $v(x,t,y)$ the convolution of $\psi(-x,-t)$ with $\Gamma(x,t,y)$, we get that the integrals in (3.20) are equal to

$$\Gamma(a)^{-1}\int_0^\infty y^{a-1}\,dy\int_{E_{n+1}} v(-u,-w,y)(\mu_a^{(1)}*\varphi)(u,w)\,du\,dw.$$

By Plancherel's theorem, this is

$$\frac{\Gamma(a)^{-1}}{(2\pi)^{n+1}}\int_0^\infty y^{a-1}\,dy\,\times$$

$$\times\int_{E_{n+1}} \bar{\hat{\psi}}(u,w)\exp\{-y\sqrt{|u|^2+iw}\}\left(\frac{|u|^2+iw}{1+|u|^2+iw}\right)^{a/2}\hat{\varphi}(u,w)\,du\,dw.$$

Changing the order of integration and integrating in y, we get

$$\frac{1}{(2\pi)^{n+1}}\int_{E_{n+1}} \bar{\hat{\psi}}(u,w)(1+|u|^2+iw)^{-a/2}\hat{\varphi}(u,w)\,du\,dw,$$

which by another application of Plancherel's theorem gives

$$\int_{E_{n+1}} \psi(x,t)\,\mathscr{J}_a(\mu_a^{(1)}*\varphi)(x,t)\,dx\,dt = \int_{E_{n+1}} \psi(x,t)\,\mathscr{G}_a(\varphi)(x,t)\,dx\,dt.$$

This proves the lemma.

Proof of theorem (3.13). Let $f\in\mathscr{L}^p_a$; that is, $f=\mathscr{G}^a*\varphi$, where $\varphi\in L^p(E_{n+1})$. Choose a sequence $\{\varphi_k\}$ of functions in $\mathscr{S}$ which converge to φ in L^p and pointwise a.e. Since $\mu_a^{(1)}*\varphi_k$ belongs to all $L^r(E_{n+1})$, $1\leqslant r\leqslant\infty$, we have by theorem (2.2)

$$(3.21)\qquad c_1\|\mu_a^{(1)}*\varphi_k\|_p\leqslant \|T_a(\mu_a^{(1)}*\varphi_k)\|_p\leqslant c_2\|\mu_a^{(1)}*\varphi_k\|_p.$$

By lemma (3.18), however, $T_a(\mu_a^{(1)}*\varphi_k)$ coincides a.e. with $P_a(f_k)$, where $f_k=\mathscr{G}_a(\varphi_k)=\mathscr{G}^a*\varphi_k$. In particular,

$$\|P_a(f_k)\|_p\leqslant c_2\|\mu_a^{(1)}*\varphi_k\|_p\leqslant c_2\|\mu_a^{(1)}\|\|\varphi_k\|_p,$$

and since f_k converges pointwise to f we obtain from Fatou's lemma

$$(3.22)\qquad \|P_a(f)\|_p\leqslant c\|\varphi\|_p.$$

This inequality and the sublinearity of $P_a(f)$ as a function of φ imply that if φ_k converges to φ in L^p then $\|P_a(f_k)\|_p$ converges to $\|P_a(f)\|_p$. From the first inequality (3.21) we then obtain

$$c_1\|\mu_a^{(1)}*\varphi\|_p \leqslant \|P_a(f)\|_p. \tag{3.23}$$

Since the identity (see lemma (3.17))

$$1 = \widehat{\mu_a^{(2)}}(x,t)(1+|x|^2+it)^{-a/2}+\widehat{\mu_a^{(3)}}(x,t)\widehat{\mu_a^{(1)}}(x,t)$$

implies

$$\varphi(x,t) = (\mu_a^{(2)}*f)(x,t)+(\mu_a^{(3)}*\mu_a^{(1)}*\varphi)(x,t) \quad \text{a.e.},$$

we have

$$\|f\|_{p,a} = \|\varphi\|_p \leqslant c(\|f\|_p+\|\mu_a^{(1)}*\varphi\|_p) \leqslant c\,(\|f\|_p+\|P_a(f)\|_p).$$

On the other hand, by (3.1), we have

$$\|f\|_p+\|P_a(f)\|_p \leqslant c\|\varphi\|_p,$$

which proves

$$c_1\|f\|_{p,a} \leqslant \|f\|_p+\|P_a(f)\|_p \leqslant c_2\|f\|_{p,a}$$

for $f\in \mathscr{L}_a^p(E_{n+1})$.

To complete the proof of the theorem we have only to show that (3.14) implies (3.15). If $V_{a,y}(x,t)$, $y>0$, is the function defined by

$$\hat{V}_{a,y}(x,t) = (1+|x|^2+it)^{a/2}\exp\{-y\sqrt{|x|^2+it}\},$$

then $V_{a,y}$ is integrable and

$$V_{a,y}*\mathscr{G}^a = \Gamma(x,t,y).$$

Let f and $P_a(f)$ belong to $L^p(E_{n+1})$. We have

$$u(x,t,y) = (f*\Gamma)(x,t,y) = \big(\mathscr{G}^a*(V_{a,y}*f)\big)(x,t),$$

which shows that $u(x,t,y)$ belongs to $\mathscr{L}_a^p$ for every $y>0$. Hence, by (3.16),

$$c_1\|V_{a,y}*f\|_p \leqslant \big\|P_a\big(u(x,t,y)\big)\big\|_p+\|u(x,t,y)\|_p.$$

Since

$$P_a\big(u(x,t,y)\big) \leqslant P_a(f)*\Gamma(x,t,y)$$

by Minkowski's inequality, we get

$$c_1\|V_{a,y}*f\|_p \leqslant \|P_a(f)\|_p+\|f\|_p \leqslant c$$

for every $y>0$. This shows that the family of functions $V_{a,y}*f$, $y>0$, is bounded in norm. By a theorem of Banach and Saks (see [6]) there is a sequence $y_k\to 0$ such that

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m} V_{a,y_k}*f = \varphi$$

for some $\varphi\in L^p$, convergence being in L^p. But since $\mathscr{G}^a*V_{a,y_k}*f(x,t) = u(x,t,y_k)$ and $u(x,t,y_k)$ converges in norm to $f(x,t)$, we obtain

$$f(x,t) = \lim_{k\to\infty}u(x,t,y_k) = \lim_{m\to\infty}\frac{1}{m}\sum_{1}^{m}u(x,t,y_k)$$

$$= \lim_{m\to\infty}\mathscr{G}^a*\frac{1}{m}\sum_{1}^{m}(V_{a,y_k}*f) = \mathscr{G}^a*\varphi,$$

which shows that $f\in \mathscr{L}_a^p$ and completes the proof of the theorem.

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PRINCETON UNIVERSITY
RUTGERS UNIVERSITY

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