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Minimal sublinear functionals *

by

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0. INTRODUCTION

In Section 1 we consider a class $\mathscr{P}$ of sublinear functionals on a real linear space E and show that $\mathscr{P}$ contains elements minimal with respect to the pointwise ordering on R^E. The general existence theorem is Theorem 15 and involves the definition of a "boundary" for $\mathscr{P}$ in Notation 13.

In Section 2 we give conditions for an element of $\mathscr{P}$ to dominate a unique minimal element of $\mathscr{P}$.

In Section 3 we give a Shilov theorem for sublinear functionals on E.

Under certain conditions (Theorem 12, Notation 23 and Lemma 27(b)) the minimal elements of $\mathscr{P}$ coincide with the linear elements of $\mathscr{P}$. In Section 6 we deduce various forms of the Hahn–Banach theorem and generalizations of results of Kelley and Sikorski (see Remark 29).

In Section 7 we deduce, with a number of improvements over the known results, Shilov theorems and conditions for the existence and uniqueness of balayages defined by a cone in $\mathscr{C}(X)$ (X compact Hausdorff) (see Remark 32). There is also a short discussion of the Choquet boundary of a subspace of $\mathscr{C}(X)$ (see Remark 35).

In Section 8 we suppose that X is a compact convex set in a Hausdorff locally convex space and deduce, with a number of improvements, results of Milman, Bauer and Choquet–Meyer (see Remark 38) as well as the Choquet–Bishop–deLeeuw theorem.

We use mainly linear space techniques — the only places where any measure theory is mentioned are Theorem 30(g), Theorem 33(a) and Theorem 36(e). In Section 9 we apply our results to a "non-$\mathscr{C}(X)$" situation, replacing $\mathscr{C}(X)$ by the set of continuous affine functions on a compact convex set (in a Hausdorff locally convex space).

In Section 10 we make some further observations about the uniqueness problem.

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Sections 1–10 of this paper are at quite a high level of abstraction. In Section 11 we present a proof of the linear space part of the Choquet–Bishop–deLeeuw theorem that uses the same ideas as our general results but is completely self-contained. This proof does not use the Hahn–Banach theorem or Tychonoff's theorem. Section 11 might well be read before Sections 1–10 to provide an insight into the techniques we use.

1. THE EXISTENCE OF CERTAIN SUBLINEAR FUNCTIONALS

1. Notation. We suppose that E is a nonzero real linear space. We say that F $(\subset E)$ is a *cone* if $0\in F$, $F+F\subset F$ and, for all $\lambda\geqslant 0$, $\lambda F\subset F$. If F $(\subset E)$ is a cone we say that ψ is *sublinear* (resp. *linear*) *on* F if $\psi\in R^F$ and for all $f_1,f_2\in F$, $\lambda_1,\lambda_2\geqslant 0$, $\psi(\lambda_1 f_1+\lambda_2 f_2)\leqslant$ (resp. $=$) $\lambda_1\psi(f_1)+\lambda_2\psi(f_2)$. We write

$$\mathscr{S}=\{S\colon S \text{ is sublinear on } E\}.$$

We suppose that $\vdash$ is a relation on E such that, for all $g\in E$, there exists $d\in E$ such that $d\vdash g$, for all $d_1,d_2,g_1,g_2\in E$, $\lambda_1,\lambda_2\geqslant 0$, $d_1\vdash g_1$ and $d_2\vdash g_2$ imply that $\lambda_1 d_1+\lambda_2 d_2\vdash\lambda_1 g_1+\lambda_2 g_2$ and, for all $d,g\in E$, $d\vdash g$ implies that $d\vdash d$. We write

$$D=\{d\colon d\in E, \text{ there exists } g\in E \text{ such that } d\vdash g\}.$$

D is a cone (the set of "dominators"). We write

$$\mathscr{P}=\big\{S\colon S\in\mathscr{S}, \text{ for all } g,d\in E,\ d\vdash g \text{ implies that } S(g-d)\leqslant 0,$$
$$\text{and, for all } g\in E,\ S(g)=\inf\{S(d)\colon d\in E,\ d\vdash g\}\big\}.$$

We write $\preccurlyeq$ for the pointwise ordering on R^E and, if $\emptyset\neq F\subset E$, $\preccurlyeq_F$ for the pointwise ordering on R^F. If $S\in\mathscr{S}$ (resp. $S\in\mathscr{P}$) we write $\mathscr{S}_S$ (resp. $\mathscr{P}_S$) for $\{T\colon T\in\mathscr{S}$ (resp. $\mathscr{P}$), $T\preccurlyeq S\}$. We observe that if $S\in\mathscr{S}$ and $P\in\mathscr{S}_S$ then

$$g,h\in E \text{ and } S(g-h)\leqslant 0 \text{ imply that } P(g)\leqslant P(h)$$

and

$$g\in E \text{ implies that } P(g)\geqslant -S(-g).$$

2. Example. If "$d\vdash g$" means "$d=g$" then $D=E$ and $\mathscr{S}=\mathscr{P}$.

3. Example. X is a compact Hausdorff space, $E=\mathscr{C}(X)$, C is a cone in E containing the positive constants and $C-C$ is norm-dense in E. "$d\vdash g$" means "$d\geqslant g$ and $d\in C$". Then $D=C$. If $g\in E$ we write $S(g)=0\vee\sup g(X)$. Then $S\in\mathscr{P}$.

4. Example. X is a compact convex subset of a real Hausdorff locally convex space V, with dual V', $E=\mathscr{C}(X)$ and $C=\{d\colon d\in E,\ d$ is concave$\}$. From the Stone–Weierstrass theorem, $C-C$ is norm-dense in E. The remainder of the notation is as in Example 3.

5. Example. X is as in Example 4, $E=\{g\colon g\in\mathscr{C}(X),\ g$ is affine$\}$, C is a cone in E containing the positive constants and $C-C$ is norm dense in E. The remainder of the notation is as in Example 3.

6. Lemma. *Let* $P,Q\in\mathscr{P}$. *Then* $P\preccurlyeq Q\Leftrightarrow P|D\preccurlyeq_D Q|D$.

Proof. ($\Rightarrow$) is trivial and ($\Leftarrow$) follows from the definition of $\mathscr{P}$.

7. Remark. Lemma 6 shows that in Example 4 the ordering induced by $\preccurlyeq$ on $\mathscr{P}$ coincides *formally* with that usually used in the proof of Choquet's theorem. The behavior here is different because the functions we are considering are not necessarily linear.

8. Lemma. *We suppose that* $S\in\mathscr{P}$ *and* F $(\subset D)$ *is a cone. We say that* ψ *is* $S-F$*-admissible if* ψ *is sublinear on* F *and, for all* $f\in F$, $\psi(f)\geqslant -S(-f)$. *If this is the case then, for all* $g\in E$, *we write*

$$S*\psi(g)=\inf\{S(g-f)+\psi(f)\colon f\in F\}.$$

Then the following results are true.

(1) (a) $S*\psi\in\mathscr{P}_S$,
(b) $S*\psi|F\preccurlyeq_F\psi$,
(c) *if* $Q\in\mathscr{S}$, $Q\preccurlyeq S*\psi\Leftrightarrow$

$$Q\preccurlyeq S \quad\text{and}\quad Q|F\preccurlyeq_F\psi. \tag{2}$$

Proofs. These results all follow from routine computations with infs. We give in detail the only nontrivial one that involves $\vdash$, the proof, in (a), that $S*\psi(g)\geqslant\inf\{S*\psi(d)\colon d\vdash g\}$. If $g\in E$, $f\in F$ and $h\vdash(g-f)$ then, since $f\vdash f$, $(h+f)\vdash g$. If we write $d=h+f$ then $d\vdash g$ and $h=d-f$. Hence

$$S*\psi(g)=\inf\{S(g-f)+\psi(f)\colon f\in F\}=\inf\{S(h)+\psi(f)\colon f\in F,\ h\vdash(g-f)\}$$
$$\geqslant\inf\{S(d-f)+\psi(f)\colon f\in F,\ d\vdash g\}\geqslant\inf\{S*\psi(d)\colon d\vdash g\}.$$

9. Lemma. (a) *If* $0\leqslant\alpha\leqslant 1$, S, F *are as in Lemma* 8 *and* ψ,ψ' *are* $S-F$*-admissible then so also is* $\alpha\psi+(1-\alpha)\psi'$ *and then*

$$S*\big(\alpha\psi+(1-\alpha)\psi'\big)\preccurlyeq\alpha(S*\psi)+(1-\alpha)(S*\psi').$$

(b) *If* $S\in\mathscr{P}$ *and* $Q\in\mathscr{S}_S$ *then* $Q|D$ *is* $S-D$*-admissible and* $Q\preccurlyeq S*(Q|D)$. *Further,* $Q=S*(Q|D)\Leftrightarrow Q\in\mathscr{P}$.

Proofs. (a) is immediate.

In (b), it follows from Lemma 8(c)($\Leftarrow$) that $Q\preccurlyeq S*(Q|D)$. If $Q=S*(Q|D)$ then, from Lemma 8(a), $Q\in\mathscr{P}$. If, conversely, $Q\in\mathscr{P}$ then, from Lemma 6 and Lemma 8(a), (b), $S*(Q|D)\preccurlyeq Q$.

10. Lemma. *We suppose that $S \in \mathscr{P}$.*

(a) *If $\emptyset \neq B \subset D$,*

(3) *for all $b, b' \in B$ there exists $b'' \in B$ such that $S(b-b') \leqslant 0$ and $S(b'-b'') \leqslant 0$,*

$$\beta = \sup -S(-B) < \infty,$$

F is the cone in E generated by B and, for all $f \in F$,

$$\psi(f) = \inf\{(\lambda_1+\ldots+\lambda_n)\beta\colon\ n \geqslant 1,\ \lambda_1, \ldots, \lambda_n \geqslant 0,\ b_1, \ldots, b_n \in B,\ \lambda_1 b_1+\ldots+\lambda_n b_n = f\}$$

then ψ is $S-F$-admissible and, for all $b \in B$, $S*\psi(b) \leqslant \beta$.

(b) *If $d \in D$ then there exists $S_d \in \mathscr{P}_S$ such that $S_d(d) \leqslant -S(-d)$.*

Proofs. (a) If $f = \lambda_1 b_1+\ldots+\lambda_n b_n$ we choose $b \in B$ such that, for all $i = 1, \ldots, n$, $S(b-b_i) \leqslant 0$. Then

$$-S(-f) = -S(-\lambda_1 b_1-\ldots-\lambda_n b_n) \leqslant -S(-\lambda_1 b-\ldots-\lambda_n b) = (\lambda_1+\ldots+\lambda_n)\big(-S(-b)\big) \leqslant (\lambda_1+\ldots+\lambda_n)\beta.$$

It follows that $\psi(f) \geqslant -S(-f)$. It is immediate that ψ is sublinear on F. Finally, if $b \in B$ then, from Lemma 8(b), $S*\psi(b) \leqslant \psi(b) \leqslant \beta$.

(b) follows from (a) with $B = \{d\}$ and $S_d = S*\psi$.

11. Notation. We write $\mathscr{M}$ for the set of all minimal elements of $(\mathscr{P}, \preccurlyeq)$ and, if $S \in \mathscr{P}$, $\mathscr{M}_S$ for $\{M\colon M \in \mathscr{M}, M \preccurlyeq S\}$.

12. Theorem. *If $M \in \mathscr{P}$ the following conditions are equivalent:*

(a) *For all $d \in D$, $M(d) \leqslant -M(-d)$.*

(b) *$M|D-D$ is linear on (the subspace) $D-D$.*

(c) *$M \in \mathscr{M}$.*

Proof. It is immediate that (a) $\Leftrightarrow$ (b). If (a) is true and $Q \in \mathscr{P}_M$ then, for all $d \in D$, $Q(d) \geqslant -M(-d) \geqslant M(d)$ hence, from Lemma 6, $M \preccurlyeq Q$. We have proved that (a) $\Rightarrow$ (c). If $d \in D$ then, as in Lemma 10(b), $M_d \in \mathscr{P}_M$ so, if (c) is true, $M \preccurlyeq M_d$ hence $M(d) \leqslant M_d(d) \leqslant -M(-d)$. We have proved that (c) $\Rightarrow$ (a).

13. Notation. If $\mathscr{A}$ is a convex set in a real linear space, we say that $\mathscr{B}$ is a *face* of $\mathscr{A}$ if $\emptyset \neq \mathscr{B} \subset \mathscr{A}$, $\mathscr{B}$ is convex and $A, A' \in \mathscr{A}$, $0 < a < 1$ and $aA+(1-a)A' \in \mathscr{B}$ imply that $A, A' \in \mathscr{B}$. If $Q, S \in \mathscr{S}$ we say that Q *edges* S if $\mathscr{S}_Q$ is a face of $\mathscr{S}_S$ in R^E (which implies that $Q \preccurlyeq S$). We write

$$\Delta(S) = \{M\colon M \in \mathscr{M}, M \text{ edges } S\}.$$

14. Lemma. *We suppose that $S, M \in \mathscr{P}$.*

(a) *If B, F, ψ are as in Lemma 10(a) then $S*\psi$ edges S.*

(b) *If $d \in D$ then S_d edges S.*

(c) *If $d \in D$ and M edges S then M_d edges S.*

Proofs. (a) If $Q, Q' \in \mathscr{S}_S$, $0 < a < 1$ and $aQ+(1-a)Q' \preccurlyeq S*\psi$ then, given $b, b' \in B$, we choose $b'' \in B$ as in (3). Then

$$\beta \geqslant S*\psi(b'') \geqslant aQ(b'')+(1-a)Q'(b'') \geqslant aQ(b)+(1-a)Q'(b') \geqslant aQ(b)-(1-a)S(-b').$$

Taking the sup over b' yields that $\beta \geqslant aQ(b)+(1-a)\beta$ hence

(4) $$\text{for all } b \in B,\ Q(b) \leqslant \beta.$$

If now $f = \lambda_1 b_1+\ldots+\lambda_n b_n$ then, from (4), $Q(f) \leqslant (\lambda_1+\ldots+\lambda_n)\beta$ and, taking the inf, $Q(f) \leqslant \psi(f)$. We have proved that $Q|F \preccurlyeq_F \psi$ and so, from Lemma 8(c) ($\Leftarrow$), $Q \preccurlyeq S*\psi$. A similar argument shows that $Q' \preccurlyeq S*\psi$. Hence $S*\psi$ edges S.

(b) is a special case of (a).

(c) follows from (b) and the transitivity of the relation "is a face of".

15. Theorem. *We suppose that $S \in \mathscr{P}$.*

(a) *If F is as in Lemma 8 and ψ is $S-F$-admissible then there exists $M \in \mathscr{M}_S$ such that*

(5) $$M|F \preccurlyeq_F \psi.$$

(b) *If $Q \in \mathscr{P}$ and Q edges S then there exists $M \in \Delta(S)$ such that $M \preccurlyeq Q$.*

(c) *If $\emptyset \neq B \subset D$ and B satisfies (3) then there exists $M \in \Delta(S)$ such that $\sup M(B) = \sup -S(-B)$.*

(d) *If $d \in D$ then there exists $M \in \Delta(S)$ such that $M(d) = -S(-d)$.*

Proofs. (a) We write $\mathscr{Y} = \{P\colon P \in \mathscr{P}, P \preccurlyeq S*\psi\}$. If $\mathscr{T}$ is a $(\preccurlyeq)$-chain in $\mathscr{Y}$ and, for $g \in E$, we write $T(g) = \inf\{P(g)\colon P \in \mathscr{T}\}$ then $T \in \mathscr{Y}$. The result now follows from Zorn's Lemma and Lemma 8(c) ($\Rightarrow$).

(b) The nonempty intersection of a decreasing chain of faces is a face and so, by an argument similar to that in (a), $\mathscr{Y} = \{P\colon P \in \mathscr{P}, P \preccurlyeq Q, P \text{ edges } S\}$ has a minimal element M. If $d \in D$ then, from Lemma 10(b) and Lemma 14(c), $M_d \preccurlyeq M$ and $M_d \in \mathscr{Y}$ hence $M \preccurlyeq M_d$ and so $M(d) \leqslant M_d(d) \leqslant -M(-d)$. From Theorem 12, $M \in \mathscr{M}$ hence $M \in \Delta(S)$.

(c) If $M \in \Delta(S)$ and $b \in B$ then $M(b) \geqslant -S(-b)$ so $\sup M(B) \geqslant \sup -S(-B)$. If $\sup -S(-B) = \infty$ then the result follows from (b) with $Q = S$. If, on the other hand, $\sup -S(-B) < \infty$ then, from (b) with $Q = S*\psi$ and Lemma 14(a), there exists $M \in \Delta(S)$ such that $M \preccurlyeq S*\psi$. Then, for all $b \in B$, $M(b) \leqslant S*\psi(b) \leqslant \beta = \sup -S(-B)$ hence $\sup M(B) \leqslant \sup -S(-B)$.

(d) is a special case of (c).

16. Theorem. *We write E^* for the algebraic dual of E and, if $S\in\mathscr{S}$ $E^*_S = E^*\cap\mathscr{S}_S$.*

(a) *The linear functionals on E are the $(\prec)$-minimal sublinear functionals on E.*

(b) *If $S\in\mathscr{S}$ and $g\in E$ then there exists $L\in E^*_S$ such that $L(g) = S(g)$.*

(c) *Any sublinear functional on E is the upper envelope of the linear functionals on E that it dominates.*

Proofs. (a) follows from Theorem 12. (b) follows from Theorem 15(d) applied to Example 2. (c) is immediate from (b).

17. Theorem. *We suppose that $S\in\mathscr{P}$ and $Q\in\mathscr{S}_S$. Then the following conditions are equivalent:*

(a) *If $P\in\mathscr{S}_S$ and $P|D\prec_D Q|D$ then $P = Q$.*

(b) *$Q\in E^*_S$ and if $P\in E^*_S$ and $P|D\prec_D Q|D$ then $P = Q$.*

(c) *$Q\in\mathscr{M}\cap E^*$.*

Proof. We first observe from Lemma 8(c) that if $P\in\mathscr{S}_S$ then $P|D\prec_D Q|D\Leftrightarrow P\prec S*(Q|D)$. If (a) is true then, from Theorem 16(a), $Q = S*(Q|D)\in E^*$ hence (b) is true. If (b) is true then, from Theorem 16(c), $Q = S*(Q|D)$ hence, from Lemma 9(b), $Q\in\mathscr{P}$ and so, from Theorem 12, (c) is true. If (c) is true then, from Lemma 9(b), $S*(Q|D) = Q\in E^*$ and so, from Theorem 16(a), (a) is true.

2. THE UNIQUENESS OF CERTAIN SUBLINEAR FUNCTIONALS

18. Notation. We write $\mathscr{N} = \{S\colon S\in\mathscr{P}, \mathscr{M}_S$ contains exactly one element$\}$.

19. Theorem. *If $S\in\mathscr{P}$ the following conditions are equivalent:*

(a) *$S\in\mathscr{N}$.*

(b) *There exists $M\in\mathscr{M}_S$ such that $M|-D = S|-D$.*

(c) *$S|-D$ is linear on $-D$.*

(d) *For all $g\in -D$, $S(g)\leqslant\inf\{-S(-d)\colon d\vdash g\}$.*

(e) *If $Q\in\mathscr{P}_S$ then $Q|-D = S|-D$.*

(f) *If $M\in\mathscr{M}_S$ then $M|-D = S|-D$.*

Proofs. If (a) is true, $\mathscr{M}_S = \{M\}$ and $d\in D$ then, from Theorem 15(d), there exists $N\in\mathscr{M}_S$ such that $N(-d) = S(-d)$ hence, since $N = M$, $M(-d) = S(-d)$. We have proved that (a) $\Rightarrow$ (b). It is immediate from Theorem 12 that (b) $\Rightarrow$ (c). If $g\in -D$ and $d\vdash g$ then $-d\in -D$ hence, if (c) is true, $S(g)+S(-d) = S(g-d)\leqslant 0$ and so (d) is true. If (d) is true and $Q\in\mathscr{P}_S$ then, for all $g\in -D$,

$$Q(g)\leqslant S(g)\leqslant\inf\{-S(-d)\colon d\vdash g\}\leqslant\inf\{Q(d)\colon d\vdash g\} = Q(g)$$

and so (e) is true. It is trivial that (e) $\Rightarrow$ (f). If, finally, (f) is true and $M, N\in\mathscr{M}_S$ then $M|-D = N|-D$. From Theorem 12 and Lemma 6, $M = N$ and so (a) is true.

20. Theorem. *We suppose that $S\in\mathscr{P}$ and $Q\in\mathscr{S}_S$. We write $\psi = Q|D$. Then $S*\psi|-D$ is linear on $-D\Leftrightarrow$*

$$\text{for all } g, h\in -D,\ S*\psi(g)+S*\psi(h)\leqslant S(g+h).$$

Proof. $(\Rightarrow)$ $S*\psi(g)+S*\psi(h) = S*\psi(g+h)\leqslant S(g+h)$ from Lemma 8(a). $(\Leftarrow)$ if $g, h\in -D$ and $d\in D$ then, since $h-d\in -D$,

$$\begin{aligned} S\big((g+h)-d\big)+Q(d) &= S\big(g+(h-d)\big)+Q(d)\\ &\geqslant S*\psi(g)+S*\psi(h-d)+Q(d)\\ \text{(using Lemma 8(b))}\quad &\geqslant S*\psi(g)+S*\psi(h-d)+S*\psi(d)\\ \text{(using Lemma 8(a))}\quad &\geqslant S*\psi(g)+S*\psi(h). \end{aligned}$$

Taking the inf over d yields

$$S*\psi(g+h)\geqslant S*\psi(g)+S*\psi(h)$$

as required.

3. A GENERAL SHILOV THEOREM

21. Theorem. *We suppose that $S\in\mathscr{P}$, $\mathscr{A}\subset\mathscr{S}_S$ and that either*

(a) *$\mathscr{A}$ is closed in $\mathscr{S}_S$ in the (possibly non-Hausdorff) topology (t) of pointwise convergence on $D-D$ or*

(b) *$\mathscr{A}$ is closed in $\mathscr{S}_S$ in the topology (p), induced from the product topology of R^E and $D-D$ is dense in E in the topology given by the seminorm $g\to S(g)\vee S(-g)$.*

Then $\mathscr{A}\supset\Delta(S)\Leftrightarrow$ for all $d\in D$, $\inf\{P(d)\colon P\in\mathscr{A}\}\leqslant -S(-d)$.

Proof. The density condition in (b) implies that $(\mathscr{S}_S, t)$ is Hausdorff hence, since $(t)\subset(p)$ and $(\mathscr{S}_S, p)$ is compact, $(t) = (p)$. So (b) is a special case of (a). We shall establish (a).

$(\Rightarrow)$ is immediate from Theorem 15(d).

$(\Leftarrow)$ We suppose that $M\in\Delta(S)\setminus\mathscr{A}$. Then there exist $f_1, \ldots, f_m\in D-D$ such that if $P\in\mathscr{A}$ then there exists i $(=1, \ldots, m)$ such that $|P(f_i)-M(f_i)|\geqslant 1$. Now $P(f_i)\geqslant -P(-f_i)$ and (since $f_i\in D-D$ and $M\in\mathscr{M}$) $M(f_i) = -M(-f_i)$. Consequently, if $\{g_1, \ldots, g_n\} = \{\pm f_1, \ldots, \pm f_m\}$ and, for $i = 1, \ldots, n$,

$$\mathscr{A}_i = \{P\colon P\in\mathscr{S}_S, P(g_i)\geqslant M(g_i)+1\}$$

then

$$\mathscr{A}\subset\mathscr{A}_1\cup\ldots\cup\mathscr{A}_n.$$

We write

$$\mathscr{A}' = \bigcup \{\lambda_1\mathscr{A}_1 + \ldots + \lambda_n\mathscr{A}_n : \lambda_1, \ldots, \lambda_n \geqslant 0, \lambda_1 + \ldots + \lambda_n = 1\}.$$

$\mathscr{S}_M$ is a face of $\mathscr{S}_S$ and, for each $i = 1, \ldots, n$, $\mathscr{A}_i \cap \mathscr{S}_M = \emptyset$ hence $\mathscr{A}' \cap \mathscr{S}_M = \emptyset$, i.e.,

for all $P \in \mathscr{A}'$ there exists $h \in E$ such that $M(h) < P(h)$.

By the usual "continuous image" argument, $\mathscr{A}'$ is compact in R^E hence there exist $\delta > 0, h_1, \ldots, h_k \in E$ such that

for all $P \in \mathscr{A}'$ there exists j $(= 1, \ldots, k)$ such that $M(h_j) + \delta < P(h_j)$.

The sets

$$A = \{(P(h_1), \ldots, P(h_k)) : P \in \mathscr{A}'\}$$

and

$$B = \{x : x \in R^k, x_j \leqslant M(h_j) + \delta \text{ for all } j = 1, \ldots, k\}$$

are convex and disjoint in R^k, hence there exists a nonzero linear functional φ on R^k such that $\sup\varphi(B) \leqslant \inf\varphi(A)$. It is immediate that φ is of the form $x \to \lambda_1 x_1 + \ldots + \lambda_k x_k$, where $\lambda_1, \ldots, \lambda_k \geqslant 0$. We write $h = \lambda_1 h_1 + \ldots + \lambda_k h_k$. If $P \in \mathscr{A}$ we choose i $(= 1, \ldots, n)$ such that $P \in \mathscr{A}_i$. From Theorem 16(b), there exists $L \in E_P^*$ such that $L(g_i) = P(g_i)$. Then $L \in \mathscr{A}_i \subset \mathscr{A}'$ hence

$$\begin{aligned} (6) \qquad P(h) \geqslant L(h) &= \lambda_1 L(h_1) + \ldots + \lambda_k L(h_k) \\ &\geqslant \lambda_1(M(h_1) + \delta) + \ldots + \lambda_k(M(h_k) + \delta) \\ &\geqslant M(h) + (\lambda_1 + \ldots + \lambda_k)\delta, \end{aligned}$$

from which $M(h) < \inf\{P(h) : P \in \mathscr{A}\}$. Since $M \in \mathscr{P}$, there exists $d \vdash h$ (hence $d \in D$) such that

$$M(d) < \inf\{P(h) : P \in \mathscr{A}\} \leqslant \inf\{P(d) : P \in \mathscr{A}\}.$$

The proof is completed by the observation that $M(d) \geqslant -S(-d)$.

22. Remarks. In the above proof we can use a separation theorem in R^E rather than in R^k. This needs an extra application of the axiom of choice. See also Remark 29. The appeal to Theorem 16(b) (and the axiom of choice) can also be avoided. What we need for (6) is $Q \in \mathscr{S}_P$ such that $Q(g_i) = P(g_i)$ and $Q(h_j) = -Q(-h_j)$ for all $j = 1, \ldots, k$. Such a Q can be constructed explicitly by using the reducing operation of Lemma 10(b) a finite number of times, taking first $d = -g_i$ and then $d = h_1, \ldots, h_k$ in sequence (imagining for this construction that we are in the case of Example 2).

4. A THEOREM ON SEQUENCES

23. Notation. If $S \in \mathscr{P}$ we say that S *linearizes* $\vdash$ if $\mathscr{M}_S \subset E^*$.

24. Theorem. *If $S \in \mathscr{P}$, S linearizes $\vdash$ and $g_1, g_2, \ldots \in E$ are such that, for all $n \geqslant 2$, $S(g_{n-1} - g_n) \leqslant 0$ and, for all $M \in \Delta(S)$, $\sup_n M(g_n) \geqslant \alpha$ then, for all $P \in \mathscr{P}_S$, $\sup_n P(g_n) \geqslant \alpha$.*

Proof. We suppose that $P \in \mathscr{P}_S$. From Theorem 15(a), there exists $Q \in \mathscr{M}_P$ $(\subset \mathscr{M}_S \subset E_S^*)$. Let $\varepsilon > 0$. We choose $d_1 \vdash g_1$ such that

$$Q(d_1) \leqslant Q(g_1) + \varepsilon/2$$

and, for $n = 2, 3, \ldots$, d_n inductively such that

$$d_n \vdash (d_{n-1} - g_{n-1} + g_n) \quad \text{and} \quad Q(d_n) \leqslant Q(d_{n-1} - g_{n-1} + g_n) + \varepsilon/2^n.$$

From these relationships

$$(7) \qquad S(g_1 - d_1) \leqslant 0,$$

$$(8) \qquad \text{for all } n \geqslant 2, S(d_{n-1} - g_{n-1} + g_n - d_n) \leqslant 0,$$

and

$$(9) \qquad \text{for all } n \geqslant 2 \quad Q(d_n) - Q(g_n) \geqslant Q(d_{n-1}) - Q(g_{n-1}) + \varepsilon/2^n.$$

From (7) and (8) and induction, for each $n \geqslant 1$, $S(g_n - d_n) \leqslant 0$ hence, for all $M \in \Delta(S)$, $M(d_n) \geqslant M(g_n)$, from which

$$(10) \qquad \text{for all } M \in \Delta(S), \sup_n M(d_n) \geqslant \sup_n M(g_n) \geqslant \alpha.$$

From (8), for each $n \geqslant 2$, $S(d_{n-1} - d_n) \leqslant S(g_{n-1} - g_n) \leqslant 0$ hence, from (10) and Theorem 15(c) with $B = \{d_n : n \geqslant 1\}$,

$$(11) \qquad \sup_n(-S(-d_n)) \geqslant \alpha.$$

From (9) and induction, for each $n \geqslant 1$,

$$Q(d_n) - Q(g_n) \leqslant \varepsilon\left(\frac{1}{2} + \ldots + \frac{1}{2^n}\right) \leqslant \varepsilon$$

and so, from (11),

$$\sup_n P(g_n) \geqslant \sup_n Q(g_n) \geqslant \sup_n Q(d_n) - \varepsilon \geqslant \sup_n - S(-d_n) - \varepsilon \geqslant \alpha - \varepsilon.$$

The result follows.

25. Remark. The above proof is an adaptation of that of [5], Theorem 31, p. 233, arranged so as to avoid the lattice operations.

5. RESULTS ON SUBSPACES AND LINEARITY

26. LEMMA. *We suppose that $S \in \mathscr{P}$, F $(\subset D)$ is a subspace of E and $\psi \in F^*_{S|F}$.* (We write "ex" for "extreme points of".)

(a) *ψ is $S-F$-admissible.*

(b) *(1) $\Leftrightarrow S*\psi|F = \psi$.*

(c) *(2) $\Leftrightarrow Q \prec S$ and $Q|F = \psi$.*

(d) *(5) $\Leftrightarrow M|F = \psi$.*

(e) *There exists $M \in \mathscr{M}_S$ such that $M|F = \psi$.*

(f) *$\psi \in \mathrm{ex} F^*_{S|F} \Leftrightarrow S*\psi$ edges S.*

(g) *If $\psi \in \mathrm{ex} F^*_{S|F}$ then there exists $M \in \Delta(S)$ such that $M|F = \psi$.*

Proofs. (a) is immediate, (b), (c) and (d) follow from Theorem 16 (a) applied to F and ψ. (e) follows from (d) and Theorem 15 (a). (f) ($\Rightarrow$) follows from (b) and Lemma 8 (c) ($\Leftarrow$). (f) ($\Leftarrow$) follows from (b), Lemma 9 (a) and Lemma 8 (b). (g) follows from (f), (c), Theorem 15 (b) and Lemma 8 (c) ($\Rightarrow$).

27. LEMMA. *We suppose that $S \in \mathscr{P}$.*

(a) *If S linearizes $\vdash$ then $\Delta(S) = \mathscr{P} \cap \mathrm{ex} E^*_S$.*

(b) *If $D-D$ is dense in E in the topology given by the seminorm $g \to S(g) \vee S(-g)$ then S linearizes $\vdash$.*

Proofs. It is easily seen that $\mathrm{ex} E^*_S = E^* \cap \mathrm{ex} \mathscr{S}_S$ (cf. the proof of Lemma 26 (f) ($\Rightarrow$)) hence

$$\mathscr{P} \cap \mathrm{ex} E^*_S = \mathscr{P} \cap E^* \cap \mathrm{ex} \mathscr{S}_S = E^* \cap (\mathscr{M} \cap \mathrm{ex} \mathscr{S}_S).$$

Further, $\Delta(S) = E^* \cap \Delta(S)$. However, if $L \in E^*$ then, from Theorem 16 (a), $\mathscr{S}_L = \{L\}$ hence L edges $S \Leftrightarrow L \in \mathrm{ex} \mathscr{S}_S$. The result follows.

(b) is immediate from Theorem 12 and the fact that if $Q \in \mathscr{S}_S$ then Q is continuous in the seminorm topology.

6. APPLICATIONS: THE HAHN–BANACH THEOREM

28. THEOREM. *We suppose that $S \in \mathscr{S}$.*

(a) *If $g \in E$ then there exists $L \in \mathrm{ex} E^*_S$ such that $L(g) = S(g)$* (cf. Theorem 16 (b).)

(b) *If F is a subspace of E and $\psi \in F^*_{S|F}$ then there exists $L \in E^*_S$ such that $L|F = \psi$.*

(c) *If F is a subspace of E and $\psi \in \mathrm{ex} F^*_{S|F}$ then there exists $L \in \mathrm{ex} E^*_S$ such that $L|F = \psi$.*

(d) *If $\emptyset \neq A$ $(\subset E)$ is convex then there exists $L \in E^*_S$ such that $\inf L(A) = \inf S(A)$.*

(e) *If $\emptyset \neq B$ $(\subset E)$ is convex, $\inf S(B) > -\infty$ and $\leqslant$ is a preorder on E compatible with the linear space structure then there exists $(\leqslant)$-positive $L \in E^*_S$ such that $\inf L(B) = \inf S(B) \Leftrightarrow$*

$$f \in B \text{ and } g \geqslant f \text{ imply that } S(g) \geqslant \inf S(B).$$

Proofs. We suppose throughout that the notation is as in Example 2 and we use Theorem 16 (a) and Lemma 27. (a) follows from Theorem 15 (d).

(b) follows from Lemma 26 (e). (c) follows from Lemma 26 (g).

(d) If $\inf S(A) = -\infty$ then the result is immediate from (a) or (b). If $\inf S(A) = a > -\infty$ then the result follows from Theorem 15 (a) with $F = \{\lambda g\colon \lambda \leqslant 0, g \in A\}$ and, for $f \in F$, $\psi(f) = \inf\{\lambda a\colon \lambda \leqslant 0, f \in \lambda A\}$.

(e) ($\Rightarrow$) is trivial.

(e) ($\Leftarrow$) We write $A = \{g\colon g \in E, \text{ there exists } f \in B \text{ such that } g \geqslant f\}$. A is convex hence, from (d), there exists $L \in E^*_S$ such that $\inf L(A) = \inf S(A)$. Clearly $\inf S(B) \geqslant \inf L(B)$ and, since $A \supset B$, $\inf L(B) \geqslant \inf L(A)$. Thus the hypotheses imply that

$$\inf S(B) = \inf L(B) = \inf L(A) = \inf S(A).$$

If $g \in E$ and $g \geqslant 0$ then $g + B \subset A$ hence

$$L(g) + \inf L(B) = \inf L(g+B) \geqslant \inf L(A) = \inf L(B)$$

and, since $\inf L(B) > -\infty$, $L(g) \geqslant 0$. We have proved that L is $(\leqslant)$-positive.

29. REMARKS. In the above theorem, (a) is equivalent to the Krein–Milman theorem via the bipolar theorem. (If K' is a compact convex set in a Hausdorff locally convex space E' and E is the dual of E', for each $g \in E$ we write $S(g) = \sup\langle K', g\rangle$. E^*_S can be identified with K'.) (b) is the usual Hahn–Banach theorem. (c) can also be proved by copying the usual proof of the Hahn–Banach theorem and preserving the extreme property of the extension at each stage (cf. [2], Lemma 11, p. 171).

If $X \neq \emptyset$, E is the vector lattice of all bounded real functions on X and, for $g \in E$, $S(g) = \sup g(X)$ then, from (d), *if $\emptyset \neq A$ $(\subset E)$ is convex then there exists a positive linear functional L on E such that $L(1) = 1$ and* $\inf L(A) = \inf S(A)$. This is a result used by Kelley in [4]. (e) strengthens [7], Corollary 7 (in which it was assumed that $\inf S(B) > 0$) which, in turn, strengthened a result of Sikorski in [4] (in which S was a norm such that $0 \leqslant f \leqslant g$ implies that $S(f) \leqslant S(g)$ and B was a set of $(\leqslant)$-positive elements).

From any of the above results we can deduce: *if $\emptyset \neq A$ $(\subset R^k)$ is convex then there exist $\lambda_1, \ldots, \lambda_k \geqslant 0$, $\lambda_1 + \ldots + \lambda_k = 1$ such that*

$$\inf\{\lambda_1 a_1 + \ldots + \lambda_k a_k\colon a \in A\} = \inf\{a_1 \vee \ldots \vee a_k\colon a \in A\}.$$

This generalizes the result we deduced from a separation theorem in the proof of Theorem 21.

7. APPLICATIONS: CONES OF CONTINUOUS FUNCTIONS

In this section we use the notation of Example 3. If $x \in X$ and $g \in E$ we write $\varepsilon_x(g)$ for $g(x)$. We use the words "directed" and "envelope" with respect to the usual ordering on E. (We do not need to assume that C is closed under $\wedge$.)

30. THEOREM.

(a) *There is a subset ∂X of X such that*

$$\Delta(S) = \{0\} \cup \{\varepsilon_x: x \in \partial X\}.$$

(b) *If $\mathscr{B}$ is a closed subset of $\mathscr{S}_S$ such that*

$$\text{for all } d \in C, \ \inf\{P(d): P \in \mathscr{B}\} \geqslant -1 \text{ implies that } \inf d(X) \geqslant -1$$

then $\mathscr{B} \supset \{\varepsilon_x: x \in \partial X\}$.

(c) *If $A \subset X$ and*

$$\text{for all } d \in C, \ \inf d(A) \geqslant -1 \text{ implies that } \inf d(X) \geqslant -1$$

then $\bar{A} \supset \partial X$.

(d) *If the extended real valued function f on X is the upper envelope of an upwards-directed subset B of C and either $\inf f(X) < 0$ or there exists $b \in B$ such that $\inf b(X) > 0$ then there exists $x \in \partial X$ such that $f(x) = \inf f(X)$.*

(e) *If $d \in C$ and $\inf d(X) \neq 0$ then there exists $x \in \partial X$ such that $d(x) = \inf d(X)$.*

(f) *If $x \in X$ there exists $M \in \mathscr{M}_S$ such that $M|C \prec_C \varepsilon_x|C$ and, for all $f \in C \cap -C$, $M(f) = f(x)$.*

(g) *If $M \in \mathscr{M}_S$ then there exists a regular Borel measure μ on X such that $\mu(X) \leqslant 1$, for all $g \in E$, $M(g) = \int g\,d\mu$ and $\mu(Y) = 0$ if Y is any compact G_δ in X disjoint from ∂X.*

Proofs. If $\mathscr{B}$ is as in (b) we write $\mathscr{A} = \mathscr{B} \cup \{0\}$. Then, for all $d \in C$,

$$\inf\{P(d): P \in \mathscr{A}\} = 0 \wedge \inf\{P(d): P \in \mathscr{B}\} \leqslant 0 \wedge \inf d(X) = -S(-d).$$

From Theorem 21(b), $\mathscr{A} \supset \Delta(S)$. Taking $\mathscr{B} = \{\varepsilon_x: x \in X\}$

$$\Delta(S) \subset \{0\} \cup \{\varepsilon_x: x \in X\}.$$

It is easily seen that $0 \in \Delta(S)$ and so (a) follows. If, again, $\mathscr{B}$ is as in (b),

$$\{0\} \cup \{\varepsilon_x: x \in \partial X\} \subset \mathscr{B} \cup \{0\}$$

and (b) follows. (c) follows from (b) by writing $\mathscr{B} = \{\varepsilon_x: x \in \bar{A}\}$.

(d) We first observe that $\sup_{b \in B} \inf b(X) \leqslant \inf f(X)$, so that conditions ensure that $\sup_{b \in B} \inf b(X) \neq 0$. The result follows from (a) and Theorem 15(c). (e) is a special case of (d).

The first observation of (f) follows from Theorem 15(a) with $F = C$ and $\psi = \varepsilon_x|F$. The second observation follows from Theorem 16(a), with E replaced by $C \cap -C$.

(g) If $M \in \mathscr{M}_S$ then, from Lemma 27, $M \in E_S^*$ from which M is a positive linear functional on E and $M(1) \leqslant 1$. From the Riesz representation theorem, there exists a regular Borel measure μ on X with $\mu(X) \leqslant 1$ such that, for all $g \in E$, $M(g) = \int g\,d\mu$. From Theorem 24, if $g_1, g_2, \ldots \in E$, $g_1 \leqslant g_2 \leqslant \ldots$ and, for all $x \in \partial X$, $\sup_n g_n(x) \geqslant 0$ then $\sup_n M(g_n) \geqslant 0$. The proof is completed as in [6], p. 28, or [5], Theorem 32, p. 233.

31. THEOREM. *We suppose that $P \in \mathscr{S}_S$. If $g \in E$ we write*

$$\tilde{g}(P) = \inf\{P(d): d \vdash g\}.$$

Then (a)–(f) *are equivalent.*

(a) *There exists a unique $M \in \mathscr{M}_S$ such that $M|C \prec_C P|C$.*

(b) *There exists $M \in \mathscr{M}_S$, $M|C \prec_C P|C$ such that, for all $g \in -C$ $M(g) = \tilde{g}(P)$.*

(c) *If $g, h \in -C$, $\widetilde{g+h}(P) = \tilde{g}(P) + \tilde{h}(P)$.*

(d) *If $M \in \mathscr{M}_S$ and $M|C \prec_C P|C$ then, for all $g \in -C$, $M(g) = \tilde{g}(P)$.*

(e) *If $g, h \in -C$ then $\tilde{g}(P) + \tilde{h}(P) \leqslant S(g+h)$.*

(f) *If $g \in -C$ and $d \vdash g$ then $\tilde{g}(P) \leqslant -(\widetilde{-d})(P)$.*

Proof. If $f \in C$ then $\big(S(g-f)\cdot 1 + f\big) \vdash g$ hence

$$\tilde{g}(P) \leqslant P\big(S(g-f)\cdot 1 + f\big) \leqslant S(g-f) + P(f)$$

and, taking the inf over f, $\tilde{g}(P) \leqslant \big(S^*(P|C)\big)(g)$. On the other hand, if $d \vdash g$ then

$$P(d) \geqslant S(g-d) + P(d) \geqslant \big(S^*(P|C)\big)(g)$$

hence $\tilde{g}(P) \geqslant \big(S^*(P|C)\big)(g)$. We have proved that

$$\tilde{g}(P) = \big(S^*(P|C)\big)(g).$$

Further, from Lemma 8(c), if $M \in \mathscr{M}_S$, $M|C \prec_C P|C \Leftrightarrow M \prec S^*(P|C)$. Hence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follows from Theorem 19(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (f) applied to $S^*(P|C)$. (c) $\Leftrightarrow$ (e) follows from Theorem 20. If (e) is true then (f) follows by putting $h = -d$. If (f) is true and $g, h \in -C$ we write $k = S(g+h)\cdot 1$. Then $k \in C$ and, from (f) with $d = k - h$,

$$\tilde{g}(P) + \tilde{h}(P) \leqslant \tilde{g}(P) + \widetilde{-d}(P) + \tilde{k}(P) \leqslant S(k) = S(g+h),$$

hence (e) is true.

32. REMARKS. If $P, Q \in \mathscr{S}_S$ we say that "P balays Q" if $P|C \prec_C Q|C$. This then extends the notion of balayage introduced in [5], Definition 41, p. 239, for elements of E_S^*. From Theorem 17(b) $\Leftrightarrow$ (c) and Lemma 27(b),

$\mathscr{M}_S$ coincides with the family of "maximal measures on X of mass $\leqslant 1$". It follows that ∂X coincides with $\partial_C X$ as defined in [5], Definition 42, p. 240. (If $x \in \partial_C X$ then $\varepsilon_x \in \mathscr{M}_S \subset \mathscr{P}$ and, arguing as in [3], p. 441, $\varepsilon_x \in \operatorname{ex} E_S^*$ hence, from Lemma 27(a), $\varepsilon_x \in \Delta(s)$. If, conversely, $\varepsilon_x \in \Delta(S) \subset \mathscr{M}_S$ then $x \in \partial_C X$.) Theorem 30(c) is then [5], Theorem 48(b), p. 241, and Theorem 30(b) permits generalizations to (for instance) Shilov sets of measures of mass $\leqslant 1$. Theorem 30(e) strengthens [5], Theorem 48(a), p. 241, in that it replaces "$<$" by "$\neq$". Substituting $P = \varepsilon_x$ in Theorem 31 gives a variety of conditions for ε_x to have a unique maximal balayage — we observe that $\tilde{g}(\varepsilon_x) = \hat{g}(x)$ as defined in [5], Definition 44, p. 240. These results do not appear in [5].

33. THEOREM. *We suppose that $-1 \in C$. We write C^u for the family of extended real functions on X that are the upper envelopes of an upwards-directed subset of C and C^{ul} for the family of extended real functions on X that are bounded below and the lower envelope of a subset of C^u.*

(a) *In Theorem* 30(g) *we now have $\mu(X) = 1$.*

(b) *If $f \in C^u$ then there exists $x \in \partial X$ such that $f(x) = \inf f(X)$.*

(c) *If $f \in C^{ul}$ then $\inf f(\partial X) = \inf f(X)$.*

(d) *If $f \in C^{ul}$, $g \in -C^{ul}$ and, for all $x \in \partial X$, $f(x) \geqslant g(x)$ then, for all $x \in X$, $f(x) \geqslant g(x)$.*

(e) *If $f, g \in C^{ul} \cap -C^{ul}$ and $f|\partial X = g|\partial X$ then $f = g$.*

Proofs. (a) is immediate. (b) follows from applying Theorem 30(d) to f minus a sufficiently large positive constant. (c), (d) and (e) follow in sequence from (b).

34. THEOREM. *We suppose that F is a subspace of E such that $1 \in F \subset C$ and each $d \in C$ is the lower envelope of a subset of F.*

(a) *For all $x \in X$, $S^*(\varepsilon_x|C) = S^*(\varepsilon_x|F)$.*

(b) *$\partial X = \{x\colon x \in X,\ \varepsilon_x|F \in \operatorname{ex} F^*_{S|F}\}$ and $\operatorname{ex} F^*_{S|F} = \{\varepsilon_x|F\colon x \in \partial X\} \cup \{0\}$.*

(c) *If $\mathscr{B}$ is a closed subset of $\mathscr{S}_S$ such that for all $f \in F$ $\inf\{P(f)\colon P \in \mathscr{B}\} \geqslant -1$ implies that $\inf f(X) \geqslant -1$ then $\mathscr{B} \supset \{\varepsilon_x\colon x \in \partial X\}$.*

(d) *If $P \in \mathscr{S}_S$ and $x \in X$ then $P|C \prec_C \varepsilon_x|C \Leftrightarrow P|F = \varepsilon_x|F$.*

Proofs. (a) is immediate from the definitions.

(b) If $x \in \partial X$ then $\varepsilon_x \in \mathscr{P}_S$ and so, from (a) and Lemma 9(b),

$$S^*(\varepsilon_x|F) = S^*(\varepsilon_x|C) = \varepsilon_x$$

which edges S. Hence, from Lemma 26(f), $\varepsilon_x|F \in \operatorname{ex} F^*_{S|F}$. If, conversely, $\psi \in \operatorname{ex} F^*_{S|F} \setminus \{0\}$ then, from Lemma 26(g), there exists $y \in \partial X$ such that $\varepsilon_y|F = \psi$. The result follows since F separates the points of X.

(c) is immediate.

(d) ($\Rightarrow$) follows from the argument used in the proof of the second assertion of Theorem 30(f). (d) ($\Leftarrow$) follows from (a) and Lemma 8(c).

35. REMARK. If F is a subspace of E such that $F \ni 1$ and F separates points we write $C = \{f_1 \wedge \ldots \wedge f_n\colon n \geqslant 1, f_1, \ldots, f_n \in F\}$. The Stone–Weierstrass theorem then shows that all the results of this section are valid. From Theorem 34(b), ∂X is the Choquet boundary of F as defined in [6], p. 38. ($F^*_{S|F} = \{\lambda\psi\colon 0 \leqslant \lambda \leqslant 1, \psi \in K(F)\}$ in the notation of [6]. This little problem can be avoided by defining $S(g) = \sup g(X)$ in which case $F^*_{S|F} = K(F)$ and $\Delta(S) = \{\varepsilon_x\colon x \in \partial X\}$.) We can now deduce the real versions of [6], Proposition 6.3, p. 40, [6], Proposition 6.4, p. 40, and [6], p. 43, from Theorem 33(b), Theorem 34(c), and Theorem 30(g) and Theorem 33(a), respectively. The remaining parts of Theorems 30, 31, 33 and 34 give further results, including various conditions equivalent to the statement "there exists a unique maximal measure M on X such that $M|F = \varepsilon_x|F$".

8. APPLICATIONS: COMPACT CONVEX SETS

We suppose in this section that the notation is as in Example 4. We write $F = \{f\colon$ there exists $\lambda \in R$ and $x' \in V'$ such that, for all $x \in X$, $f(x) = \langle x, x'\rangle + \lambda\}$. Since F separates the points of X, the map $x \to \varepsilon_x|F$ of X into $F^*_{S|F}$ is injective; it is clearly affine. A simple application of the bipolar theorem shows that $F^*_{S|F} = \{\lambda(\varepsilon_x|F)\colon 0 \leqslant \lambda \leqslant 1, x \in X\}$. It follows from [5], Theorem 7(a), p. 222, that Theorem 34 is applicable. Then Theorem 36(a) follows from Theorem 34(b) and Theorem 36(b) follows from Theorem 34(c).

From [5], Theorem 7(b), (c), p. 222, any extended real-valued concave lower semicontinuous function on X that is bounded below is in C^u and any bounded affine semicontinuous function on X is in $C^{ul} \cap -C^{ul}$. Then Theorem 36(c), (d) follow from Theorem 33(b), (e). Finally, Theorem 36(e) follows from Theorem 30(g) and Theorem 33(a).

36. THEOREM.

(a) $\partial X = \{\varepsilon_x\colon x \in \operatorname{ex} X\}$.

(b) *If $\mathscr{B}$ is closed in $\mathscr{S}_S$ and, for all $x' \in V'$,*

$$\inf\{P(\langle\cdot, v'\rangle|X)\colon P \in \mathscr{B}\} \leqslant \inf\langle X, v'\rangle$$

then $\mathscr{B} \supset \{\varepsilon_x\colon x \in \operatorname{ex} X\}$.

(c) *If f is an extended real valued concave lower semicontinuous function on X that is bounded below then there exists $x \in \operatorname{ex} X$ such that $f(x) = \inf f(X)$.*

(d) *If f and g are bounded affine semicontinuous functions on X and $f|\operatorname{ex} X = g|\operatorname{ex} X$ then $f = g$.*

(e) *If $x \in X$ then x is the barycenter of a maximal measure μ on X. If μ is a maximal measure on X then $\mu(X) = 1$ and $\mu(Y) = 0$ whenever Y is a compact G_δ in X such that $Y \cap \operatorname{ex} X = \emptyset$.*

If $g \in E$ *and* $x \in X$ *we write* $\hat{g}(x) = \inf\{d(x): d \in C, d \geqslant g\}$.

37. THEOREM. *If* $x \in X$ *then conditions* (a)–(g) *are equivalent.*

(a) *x is the barycenter of a unique maximal measure on X.*

(b) *x is the barycenter of a maximal measure μ on X such that, for all* $g \in -C$, $\int g\,d\mu = \hat{g}(x)$.

(c) *If* $g, h \in -C$, $\hat{g}(x) + \hat{h}(x) = \widehat{g+h}(x)$.

(d) *If x is the barycenter of a maximal measure μ on X then, for all* $g \in -C$, $\int g\,d\mu = \hat{g}(x)$.

(e) *If* $g, h \in -C$ *then* $\hat{g}(x) + \hat{h}(x) \leqslant S(g+h)$.

(f) *If* $g \in -C$, $d \in C$ *and* $g \leqslant d$ *then* $\hat{g}(x) \leqslant -(\widehat{-d})(x)$.

(g) *If, for* $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$, $a_i \geqslant 0$, $\beta_j \geqslant 0$, $y_i \in X$, $z_j \in X$, $\sum_i a_i = 1$, $\sum_j \beta_j = 1$, $\sum_i a_i y_i = x = \sum_j \beta_j z_j$ *then there exist* $\gamma_{ij} \geqslant 0$, $t_{ij} \in X$ *such that, for each* i, $\sum_j \gamma_{ij} = a_i$ *and* $\sum_j \gamma_{ij} t_{ij} = a_i y_i$ *and, for each* j, $\sum_i \gamma_{ij} = \beta_j$ *and* $\sum_i \gamma_{ij} t_{ij} = \beta_j z_j$.

Proof. We apply Theorem 34(d) and then the equivalence of (a)–(f) follows from the corresponding statements in Theorem 31, with $P = \varepsilon_x$.

(a) $\Rightarrow$ (g). If a_i, β_j, y_i, z_j are as in (g) then, from Theorem 36(e), there exists maximal measures μ_i and ν_j with barycenters y_i and z_j, respectively. Then $\sum_i a_i \mu_i$ and $\sum_j \beta_j \nu_j$ are both maximal measures with barycenter x. By (a), $\sum_i a_i \mu_i = \sum_j \beta_j \nu_j$. From the decomposition property, there exist $\gamma_{ij} \geqslant 0$ and probability measures ϱ_{ij} such that, for each i, $\sum_j \gamma_{ij} \varrho_{ij} = a_i \mu_i$ and, for each j, $\sum_i \gamma_{ij} \varrho_{ij} = \beta_j \nu_j$. The required result follows with t_{ij} the barycenter of ϱ_{ij}.

(g) $\Rightarrow$ (e). We suppose that $g, h \in -C$ and $\sum a_i y_i = x = \sum \beta_j z_j$, where a_i, β_j, y_i, z_j are as in (g). Since g and h are convex,

$$\sum_i a_i g(y_i) + \sum_j \beta_j h(z_j) \leqslant \sum_{ij} \gamma_{ij} [g(t_{ij}) + h(t_{ij})] \leqslant S(g+h).$$

Hence, from [6], Lemma 9.6, p. 66, $\hat{g}(x) + \hat{h}(x) \leqslant S(g+h)$, as required.

38. REMARKS. Theorem 36(b) implies Milman's theorem [3], Lemma V. 8.5, p. 440. Theorem 36(c), (d) are due to Bauer [1]. Of course, we can generalize (d) to "if $f, g \in C^{ul} \cap -C^{ul} \ldots$". All the functions in $C^{ul} \cap -C^{ul}$ are affine — it might be interesting to find exactly which affine functions are in $C^{ul} \cap -C^{ul}$. Theorem 36(e) is the Choquet–Bishop–deLeeuw theorem [6], p. 24.

We can consider the ordering defined on $V \times R$ by the cone

$$Y = \{(\lambda x, \lambda): x \in X, \lambda \geqslant 0\}.$$

Then Theorem 37(g) says simply that if, for all i, j, $a_i, b_j \in V \times R$, $a_i, b_j \geqslant 0$ and $\sum_i a_i = (x, 1) = \sum_j b_j$ then there exist $c_{ij} \in Y$ such that, for each i, $\sum_j c_{ij} = a_i$ and for each j, $\sum_i c_{ij} = b_j$. We can think of this as a "local decomposition property" at $(x, 1)$ (cf. [5], Theorem 29, p. 231). Consequently

(12) each $x \in X$ is the barycenter of a unique maximal measure on X

if, and only if, the ordering on $V \times R$ satisfies the decomposition property. This is, in turn, equivalent to the statement that Y induces a lattice ordering on $Y - Y$, i.e., that X is a simplex. This is part of the Choquet–Meyer theorem ([6], p. 66, (5) $\Leftrightarrow$ (1)). [6], p. 66, (5) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) is immediate from Theorem 37(a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). Finally, (12) is equivalent to

(13) for all $g \in -C$, $\hat{g}$ is affine

(see [6], p. 66, (2)). (It is immediate from Theorem 37(a) $\Rightarrow$ (f) that (12) $\Rightarrow$ (13). It follows from Lemma 9(b), Theorem 36(d) and Theorem 37(c) $\Rightarrow$ (a) that (13) $\Rightarrow$ (12).) See Theorem 41 for a different approach.

9. APPLICATIONS: CONES OF CONTINUOUS AFFINE FUNCTIONS

In the context of Section 7 (and Section 8), $\mathrm{ex}\,E_S^* = \{0\} \cup \{\varepsilon_x: x \in X\}$ which is closed in R^E. In this section we discuss a generalization of the results of Section 7 in which $\mathrm{ex}\,E_S^*$ is not necessarily closed in R^E.

We suppose that the notation is as in Example 5. If $x \in X$ and $g \in E$ we write $\varepsilon_x(g)$ for $g(x)$. We use the words "directed" and "envelope" with respect to the usual ordering on E. Then (cf. the introductory remarks in Section 8)

$$E_S^* = \{\lambda \varepsilon_x: 0 \leqslant \lambda \leqslant 1, x \in X\} \quad \text{and} \quad \mathrm{ex}\,E_S^* = \{0\} \cup \{\varepsilon_x: x \in \mathrm{ex}\,X\}.$$

We note from Theorem 16(c) that $P \in \mathscr{S}_S \Leftrightarrow P$ is of the form $g \to \sup \lambda_a g(x_a)$ where $0 \leqslant \lambda_a \leqslant 1$ and $x_a \in X$.

39. THEOREM.

(a) *There exists* $\partial X \subset \mathrm{ex}\,X$ *such that* $\Delta(S) = \{0\} \cup \{\varepsilon_x: x \in \partial X\}$.

(b) *Theorem* 30(c), (d), (e), (f) *are true as stated.*

(c) *If* $M \in \mathscr{M}_S$ *there exists* λ, $0 \leqslant \lambda \leqslant 1$ *and* $y \in Y$ *such that* $M = \lambda \varepsilon_y$ *and if* $g_1, g_2, \ldots \in E$, $g_1 \leqslant g_2 \leqslant \ldots$ *and, for all* $x \in \partial X$, $\sup_n g_n(x) \geqslant 0$ *then* $\sup_n g_n(y) \geqslant 0$.

(d) *Theorem* 31 *is true as stated.*

Proofs. (a) follows from the remarks above and Lemma 27 (b), (c) and (d) are proved by analogy with the proofs we already have.

40. REMARK. It is well known that if X is compact Hausdorff, $\mathscr{C}(X)$ can be identified with the set of continuous affine functions on a certain compact simplex with closed extreme points. Thus Theorem 39 in fact represents a considerable generalization of Theorem 30 and Theorem 31.

We can also make the appropriate modifications and obtain the analogy of Theorem 33 and Theorem 34.

10. A DIFFERENT APPROACH TO THE UNIQUENESS PROBLEM

In this section we return to the general considerations of Section 1 and Section 2. We suppose that S, F are as in Lemma 8 and that $\mathscr{K}$ is a compact convex subset of R^F such that each $\psi \in \mathscr{K}$ is $S-F$-admissible. Further, we suppose that $\mathscr{P}_S$ is convex.

41. Theorem. *For all $\psi \in \mathscr{K}$, $S*\psi \in \mathscr{N} \Leftrightarrow$ for all $\psi \in \operatorname{ex}\mathscr{K}$, $S*\psi \in \mathscr{N}$ and, for all $g \in -D$, the map $\psi \to S*\psi(g)$ is affine on $\mathscr{K}$.*

Proof. ($\Rightarrow$) We suppose that $g \in -D$, $\psi, \psi' \in \mathscr{K}$ and $0 \leqslant a \leqslant 1$. Then, from Lemma 9(a),

$$S*\big(a\psi+(1-a)\psi'\big) > a(S*\psi)+(1-a)(S*\psi').$$

By hypothesis, the left hand element is in $\mathscr{N}$ and the right hand one is in $\mathscr{P}$. Hence, from Theorem 19(a) $\Rightarrow$ (e)

$$S*\big(a\psi+(1-a)\psi'\big)|-D = \big(a(S*\psi)+(1-a)(S*\psi')\big)|-D$$

and this gives the required result.

($\Leftarrow$) We suppose that $g, h \in -D$. From Theorem 19(a) $\Rightarrow$ (c)

$$\text{for all } \psi \in \operatorname{ex}\mathscr{K},\ S*\psi(g)+S*\psi(h) = S*\psi(g+h).$$

Now the functions $\psi \to S*\psi(g)+S*\psi(h)$ and $\psi \to S*\psi(g+h)$ are both bounded, affine and upper semicontinuous on $\mathscr{K}$, hence from Theorem 36(d), they coincide on $\mathscr{K}$. The result follows from Theorem 19(c) $\Rightarrow$ (a).

42. Remark. We mention two applications of Theorem 41.

In the context of Theorem 28: if $S \in \mathscr{S}$ and F is a subspace of E then for all $\psi \in F^*_{S|F}$ there is a unique $L \in E^*_S$ such that $L|F = \psi \Leftrightarrow$ for all $\psi \in \operatorname{ex}F^*_{S|F}$ there is a unique $L \in E^*_S$ such that $L|F = \psi$ and, for all $g \in E$, the map $\psi \to S*\psi(g)$ is affine on $F^*_{S|F}$.

In the context of Theorem 34 (and Remark 35), for all $\psi \in F^*_{S|F}$ there is a unique $M \in \mathscr{M}_S$ such that $M|F = L|F \Leftrightarrow$ for all $x \in \partial X$ there is a unique $M \in \mathscr{M}_S$ such that $M|F = \varepsilon_x|F$ and, for all $g \in -C$, the map $\psi \to \inf\{\psi(f)\colon f \in F, f \geqslant g\}$ is affine on $F^*_{S|F}$. In the context of Section 8 this collapses to the statement (12) $\Leftrightarrow$ (13).

11. THE CHOQUET–BISHOP–deLEEUW THEOREM

One might consider Theorem 43 as the linear space part of the Choquet–Bishop–deLeeuw theorem. The measure theoretic result can be deduced from Theorem 43(e) as in [6], p. 28, or [5], Theorem 32, p. 233.

43. Theorem. *Let X be a compact convex subset of a real Hausdorff locally convex space and $x \in X$. We write $E = \mathscr{C}(X)$ and $C = \{d\colon d \in E$, d is concave$\}$. If $g \in E$ we write $S_x(g) = \inf\{d(x)\colon d \in C, d \geqslant g\}$. We write $\prec$ for the pointwise ordering on R^E and $\mathscr{P}_x = \{P\colon P \in R^E, P$ is sublinear on E, $P \prec S_x$ and, for all $g \in E, P(g) = \inf\{P(d)\colon d \in C, d \geqslant g\}\}$.*

(a) *$\mathscr{P}_x \neq \emptyset$.*

(b) *There is a $(\prec)$-minimal element M of $\mathscr{P}_x$.*

(c) *M is a positive linear functional on E, $M(1) = 1$ and x is the barycenter of M.*

(d) *If $d_1, d_2, \ldots \in C$ and $d_1 \leqslant d_2 \leqslant \ldots$ then there exists $x \in \operatorname{ex}X$ such that $\lim_n d_n(x) = \lim_n \inf d_n(X)$.*

(e) *If $g_1, g_2, \ldots \in E$, $g_1 \leqslant g_2 \leqslant \ldots$ and, for all $x \in \operatorname{ex}X$, $\lim_n g_n(x) \geqslant 0$ then $\lim_n M(g_n) \geqslant 0$.*

Proofs. (a) is immediate since $S_x \in \mathscr{P}_x$ and (b) follows from Zorn's Lemma.

(c) If $d \in C$ and $g \in E$ we write $N(g) = \inf\{M(g-\lambda d)-M(-\lambda d)\colon \lambda \geqslant 0\}$. Then $N \in \mathscr{P}_x$ and $N \prec M$. Since M is minimal, $M(d) \leqslant N(d) \leqslant -M(-d)$ and so M is linear on $C-C$. M is $\|\cdot\|$-continuous on E and, from the Stone–Weierstrass Theorem, $C-C$ is $\|\cdot\|$-dense in E hence M is linear on E. If $g \in E$, $g \leqslant 0$ then $M(g) \leqslant S_x(g) \leqslant O(x) = 0$ hence M is positive. If $d \in C$ then $M(d) \leqslant S_x(d) = d(x)$, i.e., M is a balayage of ε_x. This implies that $M(1) = 1$ and x is the barycenter of M.

(d) For all $n \geqslant 1$ there exists $x_n \in X$ such that $d_n(x_n) \leqslant \inf d_n(X)+\dfrac{1}{n}$. We let y be a cluster point of $\{x_n\}_{n\geqslant 1}$. If $n \geqslant m$ then $d_m(x_n) \leqslant d_n(x_n) \leqslant \inf d_n(X)+\dfrac{1}{n}$ hence, letting $n \to \infty$, $d_m(y) \leqslant \lim_n \inf d_n(X)$ and so $\lim_m d_m(y) \leqslant \lim_n \inf d_n(X)$. On the other hand, for all $x \in X$, $\lim_m d_m(x) \geqslant \lim_n \inf d_n(X)$. It follows that

$$\{y\colon y \in X, \lim_m d_m(y) = \lim_n \inf d_n(X)\}$$

is a closed face of X and, by the usual argument, contains an extreme point of X.

(e) Let $\varepsilon > 0$. We choose $d_1, d_2, \ldots \in C$ such that

$$d_1 \geqslant g_1 \quad \text{and} \quad M(d_1) \leqslant M(g_1)+\varepsilon/2$$

and, for $n = 2, 3, \ldots$

$$d_n \geqslant d_{n-1} - g_{n-1} + g_n \quad \text{and} \quad M(d_n) \leqslant M(d_{n-1} - g_{n-1} + g_n) + \varepsilon/2^n,$$

from which it follows that $d_1 \leqslant d_2 \leqslant \ldots$ and, for $n = 1, 2, \ldots$

$$(14) \qquad d_n \geqslant g_n \quad \text{and} \quad M(d_n) - M(g_n) \leqslant \varepsilon\left(\frac{1}{2} + \ldots + \frac{1}{2^n}\right) \leqslant \varepsilon.$$

For all $x \in \operatorname{ex} X$, $\lim_n d_n(x) \geqslant \lim_n g_n(x) \geqslant 0$ hence, from (d), $\lim_n \inf d_n(X) \geqslant 0$ and so $\lim_n M(d_n) \geqslant 0$. Combining this with (14), $\lim_n M(g_n) \geqslant -\varepsilon$ and the required result follows since ε is arbitrary.

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On the function g_λ^* and the heat equation

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INTRODUCTION AND NOTATIONS

In the present paper, a function analogous to the g_λ^* function of Littlewood, Paley, Zygmund, Stein (see [13] and [10]) is introduced for functions $u(x, t, y)$ which are solutions of the boundary problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial y^2}, \qquad y > 0$$

and

$$\lim_{y\to 0} u(x, t, y) = f(x, t).$$

The definition of g_λ^* is given in section 2, (2.1), and its properties concerning the preservation of L^p classes are discussed in theorems (2.2), (2.3), and (2.4). The method used here is an adaptation to the parabolic case of the one found in C. L. Fefferman's doctoral dissertation [2]. In section 3, theorem (3.1), the function g_λ^* is applied to obtain a characterization of the $\mathscr{L}_a^p$ spaces introduced by B. F. Jones in [4] and [5]. This characterization is suggested by those given by Hirschman [3] and Stein [9]. Also, a generalization of the g-function of Littlewood–Paley involving fractional derivatives is considered (theorem (2.25)). For an analogue in the case of analytic and harmonic functions, see [3] and [8].

We shall denote by E_{n+1} the set of all $(n+1)$-tuples $(x_1, \ldots, x_n, t) = (x, t)$ of real numbers, with the explicit intention of distinguishing the last variable. E_{n+2}^+ denotes the set of all $(n+2)$-tuples $(x_1, \ldots, x_n, t, y)$ of real numbers with $y > 0$. By $|x|$ we denote the absolute value of $(x_1, \ldots, x_n)$, which is given by $\left(\sum_1^n x_i^2\right)^{1/2}$. The complement of a set A is denoted by A' and its Lebesgue measure by $|A|$. The definition of Fourier