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Correction to the paper "On relatively disjoint families of measures, with some applications to Banach space theory" this volume pp. 13-36

by

HASKELL P. ROSENTHAL (Berkeley, Calif.)

All references and notation are given in the above-mentioned paper, which appeared in Studia Mathematica.

As pointed out to us by I. Singer, our proof of Dieudonné's theorem, given preceding Section 2, is incorrect. We restate, then prove the result in question.

THEOREM. *Let* (μ_n) *be a sequence of bounded finitely additive set functions (complex valued) defined on the discrete set* Λ.

(a) (Dieudonné [4]) *If* $\sup_n |\mu_n(E)| < \infty$ *for all* $E \subset \Lambda$, *then*

$$\sup_n \|\mu_n\| < \infty .$$

(b) (Phillips [20]) *If* $\lim_n \mu_n(E) = 0$ *for all* $E \subset \Lambda$, *then*

$$\lim_n \sum_{j \in \Lambda} |\mu_n(\{j\})| = 0 .$$

Proof. Let (μ_n) satisfy the hypotheses of (a), put $\mathfrak{F} = \{\mu_n : n = 1, 2, \ldots\}$. Let us say that $E \subset \Lambda$ is a *bad* set if $\sup_{\mu \in \mathfrak{F}} |\mu|(E) = \infty$. We assume that Λ is a bad set and argue to a contradiction.

(i) Suppose there exists a bad set E which cannot be written as a disjoint union of two bad sets. Choose $\nu_1 \in \mathfrak{F}$ with $|\nu_1|(E) > 1 + 8 \sup_{\mu \in \mathfrak{F}} |\mu(E)|$. Now choose $F \subset E$ with $|\nu_1(F)| > |\nu_1|(E)/4$. Then $|\nu_1(E \sim F)| \geqslant |\nu_1(F)| - |\nu_1(E)| \geqslant |\nu_1|(E)/8$. Now if F is not a bad set, let $E_1 = F$; otherwise let $E_1 = E \sim F$. Thus $|\nu_1(E_1)| \geqslant |\nu_1|(E)/10$.

Suppose $E_1, \ldots, E_n$ and $\nu_1, \ldots, \nu_n \in \mathfrak{F}$ have been chosen with $\bigcup_{j=1}^{n} E_j \subset E$

and $E \sim \bigcup_{j=1}^{n} E_j$ a bad set. Then $\bigcup_{j=1}^{n} E_j$ is not a bad set, hence $\sup_{\mu\in\mathfrak{F}} |\mu|(\bigcup_{j=1}^{n} E_j) = \lambda < \infty$; also $\sup_{\mu\in\mathfrak{F}} |\mu(E \sim \bigcup_{j=1}^{n} E_j)| = \beta < \infty$. Choose $\nu_{n+1}\in\mathfrak{F}$ so that $|\nu_{n+1}|(E) > n+1+10(\lambda+\beta)$. Then $|\nu_{n+1}|(E \sim \bigcup_{j=1}^{n} E_j) \geqslant \frac{9}{10} |\nu_{n+1}|(E)$. Now choose $F' \subset E \sim \bigcup_{j=1}^{n} E_j$ with $|\nu_{n+1}(F')| \geqslant |\nu_{n+1}|(E)/5$. Let $E_{n+1} = F'$ if $E \sim \left(\bigcup_{j=1}^{n} E_j \cup F'\right)$ is a bad set; otherwise let $E_{n+1} = E \sim \left(\bigcup_{j=1}^{n} E_j \cup F'\right)$. It follows that $|\nu_{n+1}(E_{n+1})| \geqslant |\nu_{n+1}|(E)/10$.

The sequence of subsets $E_1, E_2, \ldots$ of E and functions $\nu_i\in\mathfrak{F}$ thus constructed are such that the E_i's are disjoint and $|\nu_i(E_i)| \geqslant |\nu_i|(E)/10 \geqslant i/10$ for all i.

By Lemma 1.1, we may choose $n_1 < n_2 < \ldots$ such that for all i, $|\nu_{n_i}|(\bigcup_{j\neq i} E_{n_j}) < \frac{1}{2}|\nu_{n_i}(E_{n_i})|$. (We apply 1.1 to the measures $|\nu_n|\,|_E/|\nu_n(E_n)|$ and put $\varepsilon = \frac{1}{2}$; $|\nu_n|(E)/\nu_n(E_n)| \leqslant 10$ for all n, of course.) Then putting $G = \bigcup_{j=1}^{\infty} \mathrm{E}_{n_j}$, $|\nu_{n_i}(G)| \geqslant |\nu_{n_i}|(E)/20$, whence $|\nu_{n_i}(G)| \to \infty$, a contradiction.

(ii) Now assume that every bad set can be written as a disjoint union of two bad sets. Then assuming Λ is a bad set, there exists an infinite sequence of pairwise-disjoint bad sets, $E_1, E_2, \ldots$

We may then choose for all $i\in N$ (the positive integers), by induction, $\nu_i\in\mathfrak{F}$, $N_i \subset N$, $F_i \subset \Lambda$, and $n_i\in N$ satisfying the relations: $n_i < n_{i+1}$, $N_i \supset N_{i+1}$ with N_i infinite, $n_i\in N_i$, $F_i \subset E_{n_i}$, $|\nu_i|(E_{n_i}) \geqslant i+4\sum_{j=1}^{i-1} \sup_{\mu\in\mathfrak{F}} |\mu(F_j)|$, $|\nu_i(F_i)| \geqslant |\nu_i|(E_{n_i})/4$, and $|\nu_i|(\bigcup_{j\in N_{i+1}} E_j) < 1$.

Indeed, let $N_1 = N$, $n_1 = 1$, and choose ν_1 and F_1 appropriately. Having chosen everything up to the ith step, but not N_{i+1}, let $N_i = \bigcup_{j=1}^{\infty} M_j$ with $M_j \cap M_{j'} = \emptyset$ and M_j infinite for all j and $j' \neq j$. Then $\sum_j |\nu_i|(\bigcup_{r\in M_j} E_r) \leqslant \|\nu_i\| < \infty$, hence for some j, $|\nu_i|(\bigcup_{r\in M_j} E_r) < 1$; put $N_{i+1} = M_j$. Now let n_{i+1} be the first element of N_{i+1} greater than n_i, then choose ν_{i+1} and $F_{i+1} \subset E_{n_{i+1}}$ appropriately.

It follows that putting $F = \bigcup_{i=1}^{\infty} F_i$, then for each i,

$$|\nu_i(F)| \geqslant |\nu_i(F_i)| - \sum_{j=1}^{i-1} |\nu_i(F_j)| - |\nu_i|\,(\bigcup_{j=i+1}^{\infty} F_j) \geqslant \frac{i}{4} - |\nu_i|(\bigcup_{j\in N_{i+1}} E_j) \geqslant \frac{i}{4} - 1;$$

whence $|\nu_i(F)| \to \infty$, a contradiction. This completes the proof of (a).

To prove (b), we have by (a) and the hypotheses of (b) that $\sup_n \|\mu_n\| < \infty$. Suppose that the conclusion of (b) were false. Then we could choose a $\delta > 0$, a subsequence (ν_n) of the μ_n's, and a sequence (E_n) of finite disjoint subsets of Λ such that for all n,

$$\sum \{|\nu_n(\{j\})|:\ j\in\Lambda\} \geqslant 6\delta,$$

$$\sum \{|\nu_n(\{j\})|:\ j\in \bigcup_{i=1}^{n-1} E_i\} < \delta,$$

$$E_n \subset \Lambda \sim \bigcup_{i=1}^{n-1} E_i,$$

and

$$|\nu_n(E_n)| \geqslant \tfrac{1}{5} \sum \{|\nu_n(\{j\})|:\ j\in\Lambda \sim \bigcup_{i=1}^{n-1} E_i\}.$$

It would follow that for all n, $|\nu_n(E_n)| \geqslant \delta$. Then by Lemma 1.1, there would exist an increasing sequence of indices (n_i) such that for all i,

$$|\nu_{n_i}|(\bigcup_{j\neq j} E_{n_j}) < \delta/2.$$

Then putting $E = \bigcup_{j=1}^{\infty} E_{n_j}$,

$$|\nu_{n_i}(E)| > \delta/2 \quad \text{for all} \quad i,$$

contradicting the hypotheses of (b). Q.E.D.

Page 30 line 14 from the bottom; instead of "3.4" read "3.5".