Correction to the paper

"On relatively disjoint families of measures, with some applications to Banach space theory"
this volume pp. 13-36

by

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All references and notation are given in the above-mentioned paper, which appeared in Studia Mathematica.

As pointed out to us by I. Singer, our proof of Dieudonné's theorem, given preceding Section 2, is incorrect. We restate, then prove the result in question.

**Theorem.** Let \( (\mu_n) \) be a sequence of bounded finitely additive set functions (complex valued) defined on the discrete set \( A \).

(a) (Dieudonné [4]) If \( \sup_n |\mu_n(E)| < \infty \) for all \( E \in A \), then

\[
\sup_n ||\mu_n|| < \infty.
\]

(b) (Phillips [20]) If \( \lim_n \mu_n(E) = 0 \) for all \( E \in A \), then

\[
\lim_n \sum_{\omega \in \Omega} |\mu_n(\omega)| = 0.
\]

**Proof.** Let \( (\mu_n) \) satisfy the hypotheses of (a), put \( \mathcal{B} = \{\mu_n; \; n = 1, 2, \ldots\} \). Let us say that \( E \in A \) is a bad set if \( \sup_{\nu \in \mathcal{B}} |\nu|(E) = \infty \). We assume that \( A \) is a bad set and argue to a contradiction.

(i) Suppose there exists a bad set \( E \) which cannot be written as a disjoint union of two bad sets. Choose \( \nu_1 \in \mathcal{B} \) with \( |\nu_1|(E) > 1 + \sup_{\mu \in \mathcal{B}} |\mu|(E)| \).

Now choose \( F \in E \) with \( |\nu_1(F)| > |\nu_1|(E)/4 \). Then \( |\nu_1(E \setminus F)| > |\nu_1|(E)/8 \). Now if \( F \) is not a bad set, let \( E_1 = F \); otherwise let \( E_1 = E \setminus F \). Thus \( |\nu_1(E_1)| > |\nu_1|(E)/10 \).

Suppose \( E_1, \ldots, E_n \) and \( \nu_1, \ldots, \nu_n \in \mathcal{B} \) have been chosen with \( \bigcup_{i=1}^n E_i \subseteq E \).
and $E \sim \bigcup_{j=1}^{n} E_j$, a bad set. Then $\bigcup_{j=1}^{n} E_j$ is not a bad set, hence $\sup_{m \geq 0} |m| \left( \bigcup_{j=1}^{n} E_j \right) = \lambda < \infty$. Choose $m = \beta$ so that $|\eta_{\beta+1}(E)| > n + 1 + \delta(1 + \beta)$. Then $|\eta_{\beta+1}(E) \sim \bigcup_{j=1}^{n} E_j| \geq \frac{9}{10} |\eta_{\beta+1}(E)|$. Now choose $F' \subseteq E \sim \bigcup_{j=1}^{n} E_j$ with $|\eta_{\beta+1}(F')| \geq |\eta_{\beta+1}(E)|/5$. Let $E_{\beta+1} = E'$. If $E \sim \left( \bigcup_{j=1}^{n} E_j \cup F' \right)$ is a bad set; otherwise let $E_{\beta+1} = E \sim \left( \bigcup_{j=1}^{n} E_j \cup F' \right)$.

It follows that $|\eta_{\beta+1}(E_{\beta+1})| \geq |\eta_{\beta+1}(E)|/10$.

The sequence of subsets $E_1, E_2, \ldots$ of $E$ and functions $\eta_{\beta} \in \mathfrak{F}$ thus constructed are such that the $E_i$'s are disjoint and $|\eta_{\beta}(E_i)| \geq |\eta_{\beta}(E)|/10$ for all $i$.

By Lemma 1.1, we may choose $n_1 < n_2 < \ldots$ such that for all $i$, $|\eta_{n_i}(\bigcup_{j=1}^{n_i} E_j) < \frac{1}{2} |\eta_{n_i}(E_n)|$. (We apply 1.1 to the measures $|\eta_{n_i}|$ on $\mathfrak{S}_{n_i}$ and put $j = n_i |\eta_{n_i}(E_n)| < 10$ for all $n_i$, of course.) Then putting $G = \bigcup_{i=1}^{\infty} E_{n_i}$, $|\eta_{n_i}(G)| = |\eta_{n_i}(G)| + |\eta_{n_i}(G)|$, whence $|\eta_{n_i}(E_n)| \to 2$, a contradiction.

(ii) Now assume that every bad set can be written as a disjoint union of two bad sets. Then assuming $A$ is a bad set, there exists an infinite sequence of pairwise-disjoint bad sets, $E_1, E_2, \ldots$

We may then choose for all $i$, $\eta_{i+N}(E_i) \subseteq A_i$, $E_i \subseteq A_i$, and $\eta_{i+N}$ satisfying the relations: $n_i < n_{i+1}$, $N_i > N_{i+1}$, with $N_i$ infinite, $n_i \in N_i$, $F_i \subseteq E_i$, $|\eta_{n_i}(E_i)| \geq i + 4 \sum_{\beta=0}^{n_i} |\eta_{\beta}(F_i)|$, $|\eta_{n_i}(E_i)| / |\eta_{n_i}(E_i)| \leq 1$, and $|\eta_{n_i}(\bigcup E_i)| < 1$.

Indeed, let $N_i = N_i$, $n_i = n_i$, and choose $\eta_{i+j}$ and $E_i$ appropriately.

Having chosen everything up to the $i$th step, but not $N_{i+1}$, let $N_i = \bigcup_{j=1}^{M_i} M_j$ with $M_j \cap M_j = \emptyset$ and $M_j$ infinite for all $j$. Then $\sum_{j=1}^{M_i} |\eta_{n_i}(\bigcup E_j)| < \sum_{j=1}^{M_i} |\eta_{n_i}(E_j)| < 1$; hence for some $j$, $|\eta_{n_i}(\bigcup E_j)| < 1$; put $N_{i+1} = M_j$. Now let $n_{i+1}$ be the first element of $N_{i+1}$ greater than $n_i$, then choose $\eta_{n_{i+1}}$ and $F_{n_{i+1}} = E_{n_{i+1}}$ appropriately.

It follows that putting $F = \bigcup_{i=1}^{\infty} F_i$, then for each $i$, $|\eta_{n_i}(F)| \geq |\eta_{n_i}(F)| - \sum_{j=1}^{M_i} |\eta_{n_i}(F_j)| - |\eta_{n_i}(\bigcup E_j)| \geq \frac{1}{4} - |\eta_{n_i}(\bigcup E_j)| \geq \frac{1}{4} - 1$;

whence $|\eta_{n_i}(F)| \to 2$, a contradiction. This completes the proof of (a).

To prove (b), we have by (a) and the hypotheses of (b) that $\lim_{n \to \infty} |\eta_{n}(E)| < \infty$. Suppose that the conclusion of (b) were false. Then we could choose a $\delta > 0$, a subsequence $(\eta_n)$ of the $\eta_n$'s, and a sequence $(E_n)$ of disjoint subsets of $A$ such that for all $n$,

$$\sum_{i=1}^{n} |\eta_{n}(\bigcup E_i)| \geq 9 \delta,$$

$$\sum_{i=1}^{n} |\eta_{n}(\bigcup E_i)| \leq \delta,$$

$$E_n = A \sim \bigcup_{i=1}^{n} E_i,$$

and $|\eta_{n}(E_n)| \geq \frac{1}{4} \sum_{i=1}^{n} |\eta_{n}(\bigcup E_i)|$.

It would follow that for all $n$, $|\eta_{n}(\bigcup E_n)| \geq \delta$. Then by Lemma 1.1, there would exist an increasing sequence of indices $(n_i)$ such that for all $i$, $|\eta_{n_i}(\bigcup E_{n_i})| < \delta/2$.

Then putting $E = \bigcup_{i=1}^{n_i} E_{n_i}$,

$$|\eta_{n_i}(E)| < \delta/2$$

for all $i$, contradicting the hypotheses of (b).

Page 30 line 14 from the bottom; instead of “3.4” read “3.5”. Q.E.D.