STUDIA MATHEMATICA, T. XXXVII. (1971)

On certain hypersingular integrals

by

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1. Introduction. Let $f(x)$ be a function of a single real variable. We say that $f(x)$ has at a point x_0, $r-1$ *generalized derivatives* iff

$$f(x_0+t) = P_{r-1}(x_0, t)+o(t^{r-1}), \tag{1.1}$$

where $P_{r-1}(x_0, t)$ is a polynomial in t of degree $r-1$.

If f has at x_0, $r-1$ generalized derivatives we define $\delta_r(x_0, t)$ by

$$\begin{aligned}\{f(x_0+t)+(-1)^{r+1}f(x_0-t)\} \\ = \{P_{r-1}(x_0, t)+(-1)^{r+1}P_{r-1}(x_0, -t)\}+\frac{\delta_r(x_0, t)}{r!}t^r.\end{aligned} \tag{1.2}$$

The hypersingular transform of f, of order r, is then defined as

$$\lim_{\varepsilon\to 0}\left\{-\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{\delta_r(x_0,t)}{t}\,dt\right\}. \tag{1.3}$$

Of course, the definition is constructed so as to give for a good function $f(x)$, the Hilbert transform of its r-th derivative.

The early results of Calderón and Zygmund on singular integrals in n dimensions, [1]-[3], have been generalized by Muckenhoupt in [4]. He replaced the kernel $\mathfrak{N}(t)/|t|^n$ by $\mathfrak{N}(t)/|t|^{n+i\gamma}$, γ real.

The starting point for this generalization is that the Fourier transform of the kernel will be positive homogeneous of degree $i\gamma$ and so will be bounded. An important application of this generalization is the construction of an analytic family of linear operators having as intermediate values the fractional integral operators, and using appropriate interpolation theorems this enables one to prove the boundedness of these operators on various classes of functions. We refer the reader to [4] and [7] for a discussion of this method.

* Research supported in part by NSF grant number GP-8289, and by the Multinational Project of Mathematics of the OAS.

Our aim here is to extend Muckenhoupt's results in the one dimensional case to hypersingular integrals. The hypersingular transform of order r and type γ is defined as

$$\lim_{\varepsilon\to 0}\left\{-\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{\delta_r(x_0,t)}{t^{1+i\gamma}}\,dt\right\}. \tag{1.4}$$

The main result will be the following theorem: If f has $r-1$ generalized derivatives in a set E of positive measure, then a necessary and sufficient condition for the existence almost everywhere in E of the hypersingular integral of order r and type γ, is that the indefinite integral of f has $r+1$ generalized derivatives almost everywhere in E. This theorem in the case $\gamma=0$ was proved by Weiss and Zymgund in [8].

A general remark on complex homogeneity is in order. The introduction of a factor $t^{-i\gamma}$ to the kernel should in general improve things. This is so because near $t=0$ the factor $t^{-i\gamma}$ oscillates rapidly and so improves the cancellation near the singularity, while for large values of t this factor has no effect. (This enabled Muckenhoupt in [4] to take the integral in summation method rather than taking principal value.) Thus the fact that a condition which is sufficient for the case $\gamma=0$ is sufficient also when $\gamma\neq 0$ is to be expected. The main point is therefore that the same condition is necessary when $\gamma\neq 0$ as when $\gamma=0$.

This work presents the main results of my thesis, done at the University of Chicago. I wish to thank my advisor Prof. A. Zygmund for suggesting the problem and for helpful discussions. During the early part of the work on this problem, I received help and encouragement from the late Mary C. Weiss. I wish to record my debt to her.

2. The kernel $1/t\,|t|^{i\gamma}$.

THEOREM 1. *If*

$$K_{\varepsilon,N}(t)=\begin{cases}1/t\,|t|^{i\gamma}, & \varepsilon<|t|<N,\\ 0 & \textit{otherwise},\end{cases}$$

then for every $x\neq 0$,

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\hat{K}_{\varepsilon,N}(x)=-C_\gamma\, i\,|x|^{i\gamma}\ \operatorname{sign}x,$$

where

$$C_\gamma=2\int_0^\infty\frac{\sin t}{t^{1+i\gamma}}\,dt. \tag{2.1}$$

Further, for all $\varepsilon, N, x\neq 0$, *we have* $|\hat{K}_{\varepsilon,N}(x)|\leqslant C$, C *a constant which is independent of* ε, N, x.

Proof. We have

$$\hat{K}_{\varepsilon,N}(x)=\int_{-N}^{-\varepsilon}\frac{e^{-ixt}}{t(-t)^{i\gamma}}\,dt+\int_{\varepsilon}^{N}\frac{e^{-ixt}}{t^{1+i\gamma}}\,dt$$
$$=-2i\int_{\varepsilon}^{N}\frac{\sin xt}{t^{1+i\gamma}}\,dt=-2i\,|x|^{i\gamma}\operatorname{sign}x\int_{\varepsilon|x|}^{N|x|}\frac{\sin t}{t^{1+i\gamma}}\,dt.$$

If necessary, write

$$\int_{\varepsilon|x|}^{N|x|}\frac{\sin t}{t^{1+i\gamma}}\,dt=\int_{\varepsilon|x|}^{1}\frac{\sin t}{t^{1+i\gamma}}\,dt+\int_{1}^{N|x|}\frac{\sin t}{t^{1+i\gamma}}\,dt.$$

The first integral converges to

$$\int_0^1\frac{\sin t}{t^{1+i\gamma}}\,dt\qquad\text{as }\ \varepsilon\to 0$$

and never exceeds 1 in absolute value. Using integration by parts, one proves the uniform boundedness of the second integral as well as its convergence to

$$\int_1^\infty\frac{\sin t}{t^{1+i\gamma}}\,dt\qquad\text{as }\ N\to\infty.$$

We therefore have

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\hat{K}_{\varepsilon,N}(x)=-iC_\gamma|x|^{i\gamma}\operatorname{sign}x,$$

and the theorem has been proved.

The existence of the transform

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{f(x-t)}{t\,|t|^{i\gamma}}\,dt$$

for $f\in L^p(R)$, $1\leqslant p<\infty$, as well as class preservation for $1<p<\infty$, follow from Muckenhoupt's work. We therefore consider only the periodic case.

THEOREM 2. *Let f be integrable over* $(-\pi,\pi)$, *and periodic* (2π). *Then:*

(a) $$\textit{p.v.}\int_{-\infty}^{\infty}f(x-t)\frac{1}{t\,|t|^{i\gamma}}\,dt\quad \textit{exists a.e.}$$

(b) *We can find a function $p(t)$ defined in $(-\pi,\pi)$ so that*

$$p.v.\int_{-\infty}^{\infty} f(x-t)\frac{1}{t|t|^{i\gamma}}\,dt = p.v.\int_{-\pi}^{\pi} f(x-t)p(t)\,dt,$$

$$p(t) = \frac{1}{t|t|^{i\gamma}} + q(t), \qquad q(t) \text{ continuous in } (-\pi,\pi).$$

(c) *Moreover,*

$$\hat{p}(n) = -\frac{C_\gamma}{2\pi} i|n|^{i\gamma}\operatorname{sign} n. \tag{2.2}$$

Proof.

$$\text{p.v.}\int_{-(2N+1)\pi}^{(2N+1)\pi} f(x-t)\frac{1}{t|t|^{i\gamma}}\,dt$$

$$= \text{p.v.}\int_{-\pi}^{\pi} f(x-t)\left\{\frac{1}{t|t|^{i\gamma}} + \sum_{k=1}^{N}\frac{1}{(2k\pi+t)^{1+i\gamma}} - \frac{1}{(2k\pi-t)^{1+i\gamma}}\right\}dt,$$

$$\sum_{k=1}^{N}\frac{1}{(2k\pi+t)^{1+i\gamma}} - \frac{1}{(2k\pi-t)^{1+i\gamma}} = \sum_{k=1}^{N}\frac{(2k\pi-t)^{1+i\gamma}-(2k\pi+t)^{1+i\gamma}}{(4k^2\pi^2-t^2)^{1+i\gamma}},$$

$$|(2k\pi-t)^{1+i\gamma}-(2k\pi+t)^{1+i\gamma}| = |1+i\gamma|\left|\int_{2k\pi-t}^{2k\pi+t} u^{i\gamma}\,du\right| \leqslant |1+i\gamma|\,|2t|.$$

Thus the series

$$\sum_{k=1}^{\infty}\frac{1}{(2k\pi+t)^{1+i\gamma}} - \frac{1}{(2k\pi-t)^{1+i\gamma}}$$

converges abolutely and uniformly in $|t|\leqslant\pi$. Let its sum be $q(t)$. Since the integrals

$$\int_{\pm(2N+1)\pi}^{\pm(2N+3)\pi} |f(x-t)|\frac{1}{|t|}\,dt$$

tend to 0 as $N\to\infty$, we have the convergence of

$$\text{p.v.}\int_{-\infty}^{\infty} f(x-t)\frac{1}{t|t|^{i\gamma}}\,dt,$$

and further

$$\text{p.v.}\int_{-\infty}^{\infty} f(x-t)\frac{1}{t|t|^{i\gamma}}\,dt = \text{p.v.}\int_{-\pi}^{\pi} f(x-t)p(t)\,dt,$$

where

$$p(t) = \frac{1}{t|t|^{i\gamma}} + q(t),$$

$q(t)$ continuous in $(-\pi,\pi)$.

Taking in particular $f(t) = e^{int}$, $x = 0$, we get $\hat{k}(n) = 2\pi\hat{p}(n)$ and so

$$\hat{p}(n) = -\frac{C_\gamma}{2\pi} i|n|^{i\gamma}\operatorname{sign} n.$$

This concludes the proof of the theorem.

Given now a trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx = \sum_{n=0}^{\infty} A_n(x),$$

we call

$$\sum_{n=1}^{\infty}(a_n\sin nx - b_n\cos nx) = \sum_{n=1}^{\infty} B_n(x)$$

its *conjugate series*. Taking a trigonometric polynomial $\sum_{n=0}^{N} A_n(x)$, its transform by convolution with $p(t)$ will be, from the expression for $\hat{p}(n)$: $C_\gamma\sum_{n=1}^{N} B_n(x)n^{i\gamma}$.

This leads us to the following definition:

Definition 3. Let $\sum_{0}^{\infty} A_n(x)$ be a trigonometric series. Define

$$M_\gamma\Big(\sum_{n=0}^{\infty} A_n(x)\Big) = C_\gamma\sum_{n=1}^{\infty} B_n(x)n^{i\gamma}.$$

Call the new series the *γ-conjugate* of $\sum_{n=0}^{\infty} A_n(x)$.

Let $f\in L$. Clearly, the n-th partial sum of the transformed series will be given by

$$\frac{C_\gamma}{\pi}\int_{-\pi}^{\pi} f(x-t)\Big(\sum_{k=1}^{n} k^{i\gamma}\sin kt\Big)\,dt.$$

Therefore

$$\frac{C_\gamma}{\pi}\sum_{k=1}^{n} k^{i\gamma}\sin kt$$

can serve as the γ-conjugate Dirichlet kernel. This function, however, is less manageable than its counterpart on the line. We will later prove that

$$\frac{C_\gamma}{\pi}\int_0^\omega u_{i\gamma}\sin ut\,du$$

is this kernel. For the proof of this we shall need only part (a) of the following lemma. The other estimates will be needed later when we shall deal with the C-means of the kernel.

LEMMA 4. *Let $0 \leqslant j \leqslant \beta$ be integers. Write*

$$A_j^\beta(x) = x^{-\beta}\int_0^x (x-u)^{\beta-j}u^{j+i\gamma}\sin u\,du. \tag{2.3}$$

Then, as $x\to\infty$:

(a) $A_\beta^\beta(x) = O(1)$.

(b) *For $1\leqslant\beta$ we have*

$$A_0^\beta(x) = \int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^\infty u^{i\gamma-1}\cos u\,du + O\left(\frac{1}{x}\right).$$

(c) *For $1\leqslant j\leqslant\beta-1$ we have $A_j^\beta(x) = O(1/x)$.*

Proof. (a) If $\beta=0$, then

$$A_0^0(x) = \int_0^x u^{i\gamma}\sin u\,du$$
$$= \int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^x u^{i\gamma-1}\cos u\,du - x^{i\gamma}\cos x = O(1).$$

If now $\beta\geqslant 1$, then

$$A_\beta^\beta(x) = x^{-\beta}\int_0^x u^{\beta+i\gamma}\sin u\,du = -x^{i\gamma}\cos x + (\beta+i\gamma)x^{-\beta}\int_0^x u^{\beta-1+i\gamma}\cos u\,du$$
$$= -x^{i\gamma}\cos x + (\beta+i\gamma)x^{i\gamma}\int_0^1 u^{\beta-1+i\gamma}\cos(ux)\,du = O(1).$$

(b) $A_0^\beta(x)$

$$= x^{-\beta}\int_0^x (x-u)^\beta u^{i\gamma}\sin u\,du = \int_0^x u^{i\gamma}\sin u\,du$$

$$+\sum_{k=1}^{\beta}\binom{\beta}{k}(-1)^k x^{-k}\Big[-x^{k+i\gamma}\cos x + (k+i\gamma)\int_0^x u^{k-1+i\gamma}\cos u\,du\Big]$$

$$= \int_0^{\pi/2} u^{i\gamma}\sin u\,du + \int_{\pi/2}^x u^{i\gamma}\sin u\,du$$
$$+\sum_{k=1}^{\beta}\binom{\beta}{k}(-1)^k(k+i\gamma)x^{-k}\int_0^x u^{k-1+i\gamma}\cos u\,du - x^{i\gamma}\cos x\sum_{k=1}^{\beta}\binom{\beta}{k}(-1)^k$$

$$= \int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^x u^{i\gamma-1}\cos u\,du -$$
$$-x^{i\gamma}\cos x + \sum_{k=1}^{\beta}(-1)^k\binom{\beta}{k}(k+i\gamma)x^{-k}\int_0^x u^{k-1+i\gamma}\cos u + x^{i\gamma}\cos x.$$

All that remains is to show for $k\geqslant 1$,

$$x^{-k}\int_0^x u^{k-1+i\gamma}\cos u\,du = O\left(\frac{1}{x}\right) \quad \text{as } x\to\infty.$$

For $k=1$ this follows from $\int_0^x u^{i\gamma}\cos u\,du = O(1)$.

For $2\leqslant k$ we have

$$x^{-k}\int_0^x u^{k-1+i\gamma}\cos u\,du = x^{-k}\Big[x^{k-1+i\gamma}\sin x - (k-1+i\gamma)\int_0^x u^{k-2+i\gamma}\sin u\,du\Big]$$
$$= O\left(\frac{1}{x}\right) + \frac{k-1+i\gamma}{x}x^{i\gamma}\int_0^1 u^{k-2+i\gamma}\sin ux\,du = O\left(\frac{1}{x}\right).$$

Since, finally,

$$\int_x^\infty u^{i\gamma-1}\cos u\,du = O\left(\frac{1}{x}\right),$$

we have

$$A_0^\beta(x) = \int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^\infty u^{i\gamma-1}\cos u\,du + O\left(\frac{1}{x}\right).$$

(c) We shall prove $A_j^\beta(x) = O(1/x)$ for $1 \leqslant j \leqslant \beta-1$ simultaneously on β, by induction on j:

$$A_j^\beta(x) = x^{-\beta}\int_0^x (x-u)^{\beta-j} u^{j+i\gamma}\sin u\,du$$
$$= -x^{-\beta}\int_0^x (x-u)^{\beta-(j-1)}u^{j-1+i\gamma}\sin u\,du +$$
$$+x^{-(\beta-1)}\int_0^x (x-u)^{\beta-1-(j-1)}u^{j-1+i\gamma}\sin u\,du,$$

and so

(2.4) $$A_j^\beta(x) = A_{j-1}^{\beta-1}(x) - A_{j-1}^\beta(x).$$

We have to prove therefore only $A_1^\beta(x) = O(1/x)$. Using (2.4), however, with $j=1$ and part (b) of the lemma we get

$$A_1^\beta(x) = A_0^{\beta-1}(x) - A_0^\beta(x) = O\left(\frac{1}{x}\right).$$

This completes the proof of the theorem.

THEOREM 5.

$$\frac{2}{\pi}\int_0^\infty \sin kt\int_0^\omega u^{i\gamma}\sin ut\,du\,dt = \begin{cases} 0, & 0<\omega<k,\\ \tfrac{1}{2}k^{i\gamma}, & \omega = k,\\ k^{i\gamma}, & k<\omega.\end{cases}$$

Proof. Let us first show the existence of the integral

$$\int_0^\omega u^{i\gamma}\sin ut\,du = \frac{1}{t^{1+i\gamma}}\int_0^{\omega t} u^{i\gamma}\sin u\,du = \frac{1}{t^{1+i\gamma}}A_0^0(\omega t).$$

It is therefore sufficient to show

$$\int_0^T \sin kt\int_0^{\omega t} u^{i\gamma}\sin u\,du\,dt = O(1) \quad \text{as space } T\to\infty;$$

$$\int_0^T \sin kt\int_0^{\omega t} u^{i\gamma}\sin u\,du\,dt$$
$$= -\frac{1}{k}\cos kT\int_0^{\omega T} u^{i\gamma}\sin u\,du + \int_0^T \frac{1}{k}\cos kt\,\omega(\omega t)^{i\gamma}\sin \omega t\,dt$$
$$= O(1) + \tfrac{1}{2}\omega^{1+i\gamma}\left[\int_0^T t^{i\gamma}\sin(\omega+k)t\,dt + \int_0^T t^{i\gamma}\sin(\omega-k)t\,dt\right] = O(1)$$

by part (a) of Lemma 4.

We have therefore shown the existence of the integral.

Calculate it now. Note that for $\gamma = 0$ the theorem reduces to

$$\frac{2}{\pi}\int_0^\infty \sin kt\,\frac{1-\cos\omega t}{t}\,dt = \begin{cases} 0, & 0<\omega<k,\\ \tfrac{1}{2}, & \omega = k,\\ 1, & \omega>k,\end{cases}$$

as is well known.

For $\gamma \neq 0$ we use the result for $\gamma = 0$ in the following way:

$$\frac{2}{\pi}\int_0^\infty \sin kt\int_0^\omega u^{i\gamma}\sin ut\,du\,dt$$
$$= \omega^{i\gamma}\frac{2}{\pi}\int_0^\infty \sin kt\int_0^\omega \sin ut\,du\,dt - \frac{2}{\pi}\int_0^\infty \sin kt\int_0^\omega (\omega^{i\gamma}-u^{i\gamma})\sin ut\,du\,dt$$
$$= \omega^{i\gamma}\begin{Bmatrix} 0 & 0<\omega<k\\ \tfrac{1}{2} & \omega = k\\ 1 & \omega>k\end{Bmatrix} - \frac{2}{\pi}\int_0^\infty \sin kt\int_0^\omega\int_u^\omega i\gamma\lambda^{i\gamma-1}d\lambda\,\sin ut\,du\,dt.$$

Assume now that we can interchange the order of integration in the last integral. We get

$$i\gamma\int_0^\omega \lambda^{i\gamma-1}\frac{2}{\pi}\int_0^\infty \sin kt\,\frac{1-\cos\lambda t}{t}\,dt\,d\lambda = i\gamma\int_0^\omega \lambda^{i\gamma-1}\cdot\begin{Bmatrix} 0 & (0<\lambda<k)\\ \tfrac{1}{2} & (\lambda = k)\\ 1 & (\lambda>k)\end{Bmatrix}d\lambda.$$

We therefore have

$$\frac{2}{\pi}\int_0^\infty \sin kt\int_0^\omega u^{i\gamma}\sin ut\,du\,dt = \begin{Bmatrix} 0 & (\omega<k)\\ \tfrac{1}{2}k^{i\gamma} & (\omega = k)\\ \omega^{i\gamma} & (\omega>k)\end{Bmatrix} - \begin{Bmatrix} 0 & (\omega<k)\\ 0 & (\omega = k)\\ \omega^{i\gamma}-k^{i\gamma} & (\omega>k)\end{Bmatrix}$$
$$= \begin{cases} 0, & \omega<k,\\ \tfrac{1}{2}k^{i\gamma}, & \omega = k,\\ k^{i\gamma}, & \omega>k.\end{cases}$$

To complete the proof, now we have to show that we can interchange the order of integration.

For every T we have

$$\int_0^T \sin kt\int_0^\omega\left(\int_u^\omega \lambda^{i\gamma-1}d\lambda\right)\sin ut\,du\,dt = \int_0^\omega \lambda^{i\gamma-1}\int_0^T \sin kt\int_0^\lambda \sin ut\,du\,dt\,d\lambda$$

$$= \int_0^{\omega} \lambda^{i\gamma-1} \int_0^{T} \sin kt \, \frac{1-\cos \lambda t}{t}\, dt\, d\lambda$$

$$= \int_0^{\omega} \lambda^{i\gamma-1} \Big[\int_0^{kT} \frac{\sin t}{t}\, dt - \frac{1}{2} \int_0^{(k+\lambda)T} \frac{\sin t}{t}\, dt - \frac{1}{2} \int_0^{(k-\lambda)T} \frac{\sin t}{t}\, dt \Big]\, d\lambda.$$

Thus for every λ the integrand has a limit as $T \to \infty$. The integrand is obviously bounded as $T \to \infty$, uniformly in $0 < \varepsilon \leqslant \lambda \leqslant \omega$. To show uniform boundedness near $\lambda = 0$, note that the expression inside the brackets vanishes for $\lambda = 0$, and using the mean-value theorem it is equal to

$$-\frac{\lambda}{2} \Big\{ T \frac{\sin(k+\lambda_0)T}{(k+\lambda_0)T} - T \frac{\sin(k-\lambda_0)T}{(k-\lambda_0)T} \Big\}$$

with $0 < \lambda_0 < \lambda < \varepsilon < 1$ and so the integrand is uniformly bounded. This concludes the proof of the theorem.

DEFINITION 6. Let $\sum_0^{\infty} A_k(x)$ be a trigonometric series. Write

$$S^*_{\gamma,\omega}(x) = C_\gamma \sum_{0<k\leqslant\omega}^{*} k^{i\gamma} B_k(x), \tag{2.5}$$

where the star indicates that if ω is an integer, then the term corresponding to $k = \omega$ is multiplied by $\frac{1}{2}$.

DEFINITION 7.

$$W_\omega(t) = \frac{C_\gamma}{\pi} \int_0^{\omega} u^{i\gamma} \sin ut\, du.$$

THEOREM 8. *If $f \in L(-\pi, \pi)$, then*

$$S^*_{\gamma,\omega}(x) = \int_{-\infty}^{\infty} f(x-t)\, W_\omega(t)\, dt.$$

Proof. (a) If f is a trigonometric polynomial, we may assume that it consists of a single term:

$$f(x) = a_k \cos kx + b_k \sin kx.$$

We have then

$$\int_{-\infty}^{\infty} f(x-t)\, W_\omega(t)\, dt = \frac{C_\gamma}{\pi} B_k(x) \int_{-\infty}^{\infty} \sin kt \int_0^{\omega} u^{i\gamma} \sin ut\, du\, dt$$

$$= C_\gamma B_k(x) k^{i\gamma} \begin{cases} 0, & \omega < k, \\ \frac{1}{2}, & \omega = k, \\ 1, & \omega > k. \end{cases}$$

(b) $f \in L(-\pi, \pi)$.

$$W_\omega(t) = \frac{C_\gamma}{\pi} \int_0^{\omega} u^{i\gamma} \sin ut\, du$$

is of bounded variation over any finite interval $[-T, T]$. Thus

$$\int_{-T}^{T} f(x-t)\, W_\omega(t)\, dt = \sum_{k=1}^{\infty} \int_{-T}^{T} A_k(x-t)\, W_\omega(t)\, dt$$

$$= S^*_{\gamma,\omega}(x) - \sum_{\omega<k} \int_{T<|t|} A_k(x-t)\, W_\omega(t)\, dt - \sum_{k\leqslant\omega}^{*} \int_{T<|t|} A_k(x-t)\, W_\omega(t)\, dt.$$

The last sum tends to 0 as $T \to \infty$ and so all we have to show is that

$$\sum_{\omega<k} \int_{T<|t|} A_k(x-t)\, W_\omega(t)\, dt = o(1) \quad \text{as } T \to \infty.$$

Since $W_\omega(t)$ is an odd function of t, we have

$$\sum = \frac{2C_\gamma}{\pi} \sum_{\omega<k} B_k(x) \int_T^{\infty} \sin kt \int_0^{\omega} u^{i\gamma} \sin ut\, du\, dt.$$

Write

$$G_k(T) = \int_T^{\infty} \sin kt \int_0^{\omega} u^{i\gamma} \sin ut\, du\, dt. \tag{2.6}$$

We have

$$G_k(T) = \omega^{i\gamma} \int_T^{\infty} \sin kt \, \frac{1-\cos \omega t}{t}\, dt - i\gamma \int_0^{\omega} \lambda^{i\gamma-1} \int_T^{\infty} \sin kt \, \frac{1-\cos\lambda t}{t}\, dt\, d\lambda;$$

$$\int_T^{\infty} \sin kt \, \frac{1-\cos\lambda t}{t}\, dt = \int_{kT}^{\infty} \frac{\sin t}{t}\, dt - \frac{1}{2} \int_{(k+\lambda)T}^{\infty} \frac{\sin t}{t}\, dt - \frac{1}{2} \int_{(k-\lambda)T}^{\infty} \frac{\sin t}{t}\, dt$$

$$= \frac{\cos kT}{kT} - \int_{kT}^{\infty} \frac{\cos t}{t^2}\, dt - \frac{1}{2} \frac{\cos(k+\lambda)T}{(k+\lambda)T} + \frac{1}{2} \int_{(k-\lambda)T}^{\infty} \frac{\cos t}{t^2}\, dt -$$

$$- \frac{1}{2} \frac{\cos(k-\lambda)T}{(k-\lambda)T} + \frac{1}{2} \int_{(k-\lambda)T}^{\infty} \frac{\cos t}{t^2}\, dt$$

$$= \frac{2\cos kT - \cos(k+\lambda)T - \cos(k-\lambda)T}{2kT} + \frac{\lambda \cos(k+\lambda)T}{2k(k+\lambda)T} - \frac{\lambda\cos(k-\lambda)T}{2k(k-\lambda)T} +$$

$$+ \frac{1}{2} \int_{(k+\lambda)T}^{\infty} \frac{\cos t}{t^2}\, dt + \frac{1}{2} \int_{(k-\lambda)T}^{\infty} \frac{\cos t}{t^2}\, dt - \int_{kT}^{\infty} \frac{\cos t}{t^2}$$

$$= \frac{\cos kT[1-\cos\lambda T]}{kT} + \lambda\left\{\frac{\cos(k+\lambda)T}{2k(k+\lambda)T} + \frac{\cos(k-\lambda)T}{2k(k-\lambda)T}\right\} +$$
$$+\lambda\left\{-\frac{1}{2}\frac{\cos(k+\lambda_0)T}{(k+\lambda_0)^2T} + \frac{1}{2}\frac{\cos(k-\lambda_0)T}{(k-\lambda_0)^2T}\right\},$$

where $0<\lambda_0<\lambda\leqslant\omega<k$.

Thus

$$G_k(T) = \omega^{i\gamma}\frac{\cos kT(1-\cos\omega T)}{kT} + O\left(\frac{1}{Tk^2}\right) -$$
$$-i\gamma\frac{\cos kT}{kT}\int\limits_0^{\omega}\lambda^{i\gamma-1}(1-\cos\lambda T)\,d\lambda + O\left(\frac{1}{Tk^2}\right),$$

$$\int\limits_0^{\infty}\lambda^{i\gamma-1}(1-\cos\lambda T)\,d\lambda = T^{-i\gamma}\int\limits_0^{\omega T}\lambda^{i\gamma-1}(1-\cos\lambda)\,d\lambda = O(1)$$

as $T\to\infty$, and so

$$\sum_{\omega<k} a_k\sin kx\,G_k(T) = O\left(\frac{1}{T}\right)\sum_{\omega<k}\frac{a_k}{k}\sin kx\cos kT + O\left(\frac{1}{T}\sum_{\omega<k}|a_k|\cdot\frac{1}{k^2}\right).$$

The last term is clearly $O(1/T)$ as $T\to\infty$. Denote now by Φ the integral of the even part of

$$f(x) - \sum_{k<\omega}A_k(x), \qquad \Phi(x) = \sum_{\omega<k}\frac{a_k}{k}\sin kx.$$

We have

$$\frac{1}{2}[\Phi(x+T)+\Phi(x-T)] = \sum_{\omega<k}\frac{a_k}{k}\sin kx\cos kT,$$

and so the last expression is bounded uniformly in T.

This shows

$$\sum_{\omega<k} a_k\sin kx\,G_k(T) = O\left(\frac{1}{T}\right).$$

Similarly one shows

$$\sum_{\omega<k} b_k\cos kx\,G_k(T) = O\left(\frac{1}{T}\right),$$

and the proof is complete.

3. Hypersingular integrals.

DEFINITION 1.

$$\tilde{f}(x) = \text{p.v.}\int\limits_{-\infty}^{\infty} f(x-t)\frac{1}{t|t|^{i\gamma}}\,dt. \tag{3.1}$$

If f has at x, $r-1$ generalized derivatives, define also

$$\tilde{f}_r(x) = -\text{p.v.}\int\limits_0^{\infty}\delta_r(x,t)\frac{1}{t^{1+i\gamma}}\,dt. \tag{3.2}$$

THEOREM 1. *If $f\in C^r$, f periodic (2π), then*

$$\tilde{f}_r(x) = \frac{r!}{(1+i\gamma)\ldots(r+i\gamma)}\tilde{f}^{(r)}(x).$$

Proof. Integration by parts.

THEOREM 2. *Let $E\subset(-\pi,\pi)$, $|E|>0$. If f has $r-1$ generalized derivatives in E, and if the indefinite integral Φ of f has $r+1$ generalized derivatives in E, then $\tilde{f}_r(x)$ exist a.e. in E.*

Proof. We include the proof for completeness. It is analogous to the proof of the corresponding theorem in [8].

Let $P\subset E$ be a closed set so that $|E-P|<\varepsilon$ and so that if we write $\chi(x)$ = distance (x,P), we have a decomposition $\Phi = G+H$, where $G\in C^{r+1}$, and $|H(x)|\leqslant C\chi^{r+1}(x)$ for all x except possibly those belonging to one of finitely many intervals contiguous to P (see [10], XI, § 4).

Since we have $\Phi' = f$ a.e., $H = \Phi - G$, H is differentiable a.e. Let $H' = h$, $G' = g$. We have

$$f = g+h, \tag{3.3}$$

$g\in C^r$ and so $\tilde{g}_r(x)$ exists a.e. It therefore suffices to show the existence a.e. in P of $\tilde{h}_r$.

Since $|H(x)|\leqslant C\chi^{r+1}(x)$, it follows that if x is a point of density of P we have

$$H(x+t) = o(t^{r+1}). \tag{3.4}$$

So H has $r+1$ generalized derivatives at points of density of P all of them 0. Since f, g have $r-1$ generalized derivatives at all points of P so does h, and from (3.4) follows further that at points of density of P,

$$h(x) = h_1(x) = \ldots = h_{r-1}(x) = 0. \tag{3.5}$$

By a well-known theorem of Marcinkiewicz, we have for almost every $x\in P$

$$\int\limits_{-\pi}^{\pi}\frac{\chi^{r+1}(t)}{|x-t|^{r+2}}\,dt < \infty. \tag{3.6}$$

Let x be a point of density of P where (3.6) holds. To show the existence of $\tilde{h}_r(x)$, we have to show only the existence of

$$\int\limits_0^{\eta}\frac{h(x-t)}{t^{r+1+i\gamma}}\,dt \quad \text{for an} \quad \eta>0. \tag{3.7}$$

Take η so that $(x-\eta, x)$ does not intersect any exceptional interval. It will also be convenient to take η so that $x-\eta \in P$.

For $0 < t < \eta$ we have $|H(x-t)| \leqslant C\chi^{r+1}(t)$.

Integrate (3.7) by parts,

$$\left|\int_0^\eta \frac{h(x-t)}{t^{r+1+i\gamma}}\,dt\right| = |r+1+i\gamma|\left|\int_0^\eta \frac{H(x-t)}{t^{r+2+i\gamma}}\,dt\right|$$

$$\leqslant C|r+1+i\gamma| \int_{-\pi}^{\pi} \frac{\chi^{r+1}(x-t)}{|t|^{r+2}}\,dt < \infty,$$

and the theorem has been proved.

Our next objective is the following theorem: Let $S[f]$ be the Fourier series of f, and $S^{(r)}[f]$ the r-th termwise derivative of $S[f]$. Then, if $\tilde{f}_r(x)$ exists, $M_\gamma(S^{(r)}[f])(x)$ converges $(C, r+2)$ to

$$\frac{\Gamma(r+1+i\gamma)}{\Gamma(1+i\gamma)\Gamma(r+1)}\,\frac{C_\gamma}{\pi}\left(\int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^{\infty} u^{i\gamma-1}\cos u\,du\right)\tilde{f}_r(x). \tag{3.8}$$

The proof shall require some intermediate results.

Let us write

$$\sigma^\beta_{\gamma,\omega}(x) = C_\gamma \sum_{k\leqslant\omega} k^{i\gamma} B_k(x)\left(1-\frac{k}{\omega}\right)^\beta. \tag{3.9}$$

THEOREM 3.

$$\sigma^\beta_{\gamma,\omega}(x) = \int_{-\infty}^{\infty} (x-t)\, W^\beta_\omega(t)\,dt,$$

where

$$W^\beta_\omega(t) = \frac{C_\gamma}{\pi}\,\omega^{-\beta}\int_0^\omega (\omega-u)^\beta u^{i\gamma}\sin ut\,du.$$

Proof.

$$\sigma^\beta_{\gamma,\omega}(x) = C_\gamma\,\omega^{-\beta}\sum_{k\leqslant\omega} k^{i\gamma}B_k(x)(\omega-k)^\beta$$

$$= \omega^{-\beta}\int_0^\omega (\omega-u)^\beta\,dS^*_{\gamma,u}(x) = -\omega^{-\beta}\int_0^\omega S^*_{\gamma,u}(x)\,d(\omega-u)^\beta$$

$$= -\omega^{-\beta}\int_0^\omega\left(\int_{-\infty}^{\infty} f(x-t)\,W_u(t)\,dt\right)d(\omega-u)^\beta$$

$$= -\omega^{-\beta}\int_{-\infty}^{\infty} f(x-t)\left(\int^{\omega} W_u(t)\,d(\omega-u)^\beta\right)dt$$

$$= \int_{-\infty}^{\infty} f(x-t)\left[\omega^{-\beta}\int_0^\omega (\omega-u)^\beta\,dW_u(t)\right]dt$$

$$= \int_{-\infty}^{\infty} f(x-t)\left[\frac{C_\gamma}{\pi}\,\omega^{-\beta}\int_0^\omega (\omega-u)^\beta u^{i\gamma}\sin ut\,du\right]dt.$$

We shall now establish some estimates for the kernels $W^\beta_\omega(t)$.

LEMMA 4.

$$\left|\frac{d^k}{dt^k}\,W^\beta_\omega(t)\right| \leqslant \frac{1}{\pi}\,\omega^{k+1}.$$

Proof.

$$\left|\frac{d^k}{dt^k}\,W^\beta_\omega(t)\right| \leqslant \left|\frac{C_\gamma}{\pi}\int_0^\omega\left(1-\frac{u}{\omega}\right)^\beta u^{k+i\gamma}(\sin ut)^{(k)}\,du\right|$$

$$\leqslant \left|\frac{C_\gamma}{\pi}\int_0^\omega u^k\,du\right| \leqslant \frac{|C_\gamma|}{\pi}\,\omega^{k+1}.$$

LEMMA 5. *For* $0\leqslant k\leqslant\omega$ *we have as* $t\to\infty$, $\dfrac{d^k}{dt^k}W^\beta_\omega(t) = O\left(\dfrac{1}{t^{k+1}}\right)$ *the* 0 *uniform in* ω, *for* $0<\omega_0\leqslant\omega$.

Proof.

$$\frac{d^k}{dt^k}\,W^\beta_\omega(t) = \frac{C_\gamma}{\pi}\,\frac{1}{t^{k+1+i\gamma}}\int_0^{\omega t}\left(1-\frac{u}{\omega t}\right)^\beta u^{k+i\gamma}\,\frac{d^k}{du^k}(\sin u)\,du$$

$$= \frac{C_\gamma}{\pi}\,\frac{1}{t^{k+1+i\gamma}}\,\frac{1}{(\omega t)^\beta}\int_0^{\omega t}\sin u\sum_{j=0}^{k} C(k,j,\beta,\gamma)(\omega t-u)^{\beta-j}u^{j+i\gamma}\,du.$$

Since $j\leqslant k\leqslant\beta$ we can use Lemma II 4, and get as $\omega t\to\infty$

$$\sum_{j=0}^{k} C(k,j,\beta,\gamma)\,\frac{1}{(\omega t)^\beta}\int_0^{\omega t}(\omega t-u)^{\beta-j}u^{j+i\gamma}\sin u\,du = O(1)$$

and the proof is complete.

LEMMA 6. *If we write*

$$W^\beta_\omega(t) = \frac{A}{t^{1+i\gamma}} + H^\beta_\omega(t),$$

with

$$A = \frac{C_\gamma}{\pi}\left(\int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^{\infty} u^{i\gamma-1}\cos u\,du\right)$$

we have for $2 \leqslant \beta$, $1 \leqslant \omega t$

$$\left|\frac{d^{\beta-2}}{dt^{\beta-2}}H_\omega^\beta(t)\right| \leqslant \frac{C}{\omega t^\beta}, \tag{3.10}$$

$$\left|\frac{d}{dt}\left\{t^{\beta-2}\frac{d^{\beta-2}}{dt^{\beta-2}}H_\omega^\beta(t)\right\}\right| \leqslant \frac{C}{\omega t^3}. \tag{3.11}$$

Proof.

$$\frac{d^k}{dt^k}W_\omega^\beta(t) = \frac{C_\gamma}{\pi}\frac{(-1)^k}{t^{k+1+i\gamma}}\sum_{j=0}^{k}\binom{k}{j}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}\frac{\Gamma(k+1+i\gamma)}{\Gamma(j+1+i\gamma)}\times$$

$$\times(-1)^j\frac{1}{(\omega t)^\beta}\int_0^{\omega t}(\omega t-u)^{\beta-j}u^{j+i\gamma}\sin u\,du.$$

We are considering only the cases $k = \beta-1$, $k = \beta-2$, and so we have $j \leqslant k \leqslant \beta-1$.

Using the appropriate estimates of Lemma II. 4 we get

$$\frac{d^k}{dt^k}W_\omega^\beta(t) = \frac{C(k,\gamma)}{t^{k+1+i\gamma}} + \frac{1}{t^{k+1+i\gamma}}O\left(\frac{1}{\omega t}\right) = \frac{C(k,\gamma)}{t^{k+1+i\gamma}} + O\left(\frac{1}{\omega t^{k+2}}\right)$$

with

$$C(k,\gamma) = \frac{C_\gamma}{\pi}(-1)^k\frac{\Gamma(k+1+i\gamma)}{\Gamma(1+i\gamma)}\left(\int_0^{\pi/2}u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^{\infty}u^{i\gamma-1}\cos u\,du\right).$$

We have

$$\frac{C(k+1,\gamma)}{C(k,\gamma)} = -(k+1+i\gamma)$$

and so

$$\frac{d}{dt}\left(\frac{C(k,\gamma)}{t^{k+1+i\gamma}}\right) = \frac{C(k+1,\gamma)}{t^{k+2+i\gamma}}.$$

Thus

$$\frac{d^k}{dt^k}\left(W_\omega^\beta(t) - \frac{A}{t^{1+i\gamma}}\right) = O\left(\frac{1}{\omega t^{k+2}}\right)$$

with

$$A = C(0,\gamma) = \frac{C_\gamma}{\pi}\left(\int_0^{\pi}(u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^{\infty}u^{i\gamma-1}\cos u\,du\right).$$

Substituting $k = \beta-2$ we get the first estimate. To get estimate (3.11):

$$\frac{d}{dt}\left(t^{\beta-2}\frac{d^{\beta-2}}{dt^{\beta-2}}H_\omega^\beta(t)\right) = (\beta-2)t^{\beta-3}\frac{d^{\beta-2}}{dt^{\beta-2}}H_\omega^\beta(t) + t^{\beta-2}\frac{d^{\beta-1}}{dt^{\beta-1}}H_\omega^\beta(t)$$

$$= t^{\beta-3}O\left(\frac{1}{\omega t^\beta}\right) + t^{\beta-2}O\left(\frac{1}{\omega t^{\beta+1}}\right) = O\left(\frac{1}{\omega t^3}\right),$$

and Lemma 6 is proved.

LEMMA 7. (a) *If* $r \leqslant \beta+1$ *is odd, we have*

$$\int_0^\infty \{W_\omega^\beta(t)\}^{(r)}dt = \int_0^\infty t^2\{W_\omega^\beta(t)\}^{(r)}dt = \ldots = \int_0^\infty t^{r-1}\{W_\omega^\beta(t)\}^{(r)}dt = 0. \tag{3.12}$$

(b) *If* $r \leqslant \beta+1$ *is even, we have*

$$\int_0^\infty t\{W_\omega^\beta(t)\}^{(r)}dt = \ldots = \int_0^\infty t^{r-1}\{W_\omega^\beta(t)\}^{(r)}dt = 0. \tag{3.13}$$

Proof. $W_\omega^\beta(t)$ is an odd function of t. It therefore vanishes, with all its derivatives of even order at $t = 0$. Moreover, for $0 \leqslant k \leqslant \beta$ we have $\{W_\omega^\beta(t)\}^{(k)} = O(1/t^{k+1})$, and for all k we have

$$|\{W_\omega^\beta(t)\}^{(k)}| \leqslant \frac{1}{\pi}\omega^{k+1}.$$

Using these, both (3.12) and (3.13) follow by integration by parts.

THEOREM 8. *If* $\tilde f^{(r)}(x)$ *exists, then* $M_\gamma(S^{(r)}[f])$ *is* $(C, r+2)$ *summable to*

$$\frac{\Gamma(r+1+i\gamma)}{\Gamma(1+i\gamma)\Gamma(r+1)}\frac{C_\gamma}{\pi}\left(\int_0^{\pi/2}u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^{\infty}u^{i\gamma-1}\cos u\,du\right)\tilde f_r(x). \tag{3.14}$$

Proof. The proof is along the lines of that of Lemma 5 in [8]:

$$\sigma_{\gamma,\omega}^{r+2}(x) = \int_{-\infty}^{\infty} f(t)\,W_\omega^{r+2}(x-t)\,dt,$$

and so the $r+2$ means of $S^{(r)}[f]$, which are also $\frac{d^r}{dx^r}\sigma_{\gamma,\omega}^{r+2}(x)$ are given by

$$\int_{-\infty}^{\infty} f(t)\frac{d^r}{dx^r}W_\omega^{r+2}(x-t)\,dt = \int_0^\infty [f(x-t)+(-1)^{r+1}f(x+t)]\{W_\omega^{r+2}(t)\}^{(r)}dt$$

$$= (-1)^{r+1}\int_0^\infty [f(x+t)+(-1)^{r+1}f(x-t)]\{W_\omega^{r+2}(t)\}^{(r)}dt$$

$$= (-1)^{r+1}\int_0^\infty \Big[P_{r-1}(x,t)+(-1)^{r+1}P_{r-1}(x,-t)+\frac{1}{r!}t^r\delta_r(x,t)\Big]\{W_\omega^{r+2}(t)\}^{(r)}dt$$

$$= (-1)^{r+1}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\{W_\omega^{r+2}(t)\}^{(r)}dt.$$

The last equality follows from (3.12) if r is odd, from (3.13) if r is even. Using now the decomposition of $W_\omega^{r+2}(t)$, given by Lemma 6, we have

$$(-1)^{r+1}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\frac{d^r}{dt^r}\big(W_\omega^{r+2}(t)\big)dt$$

$$= (-1)^{r+1}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\frac{d^r}{dt^r}\frac{A}{t^{1+i\gamma}}dt+(-1)^{r+1}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\frac{d^r}{dt^r}H_\omega^{r+2}(t)dt$$

$$= -A\frac{\Gamma(r+1+i\gamma)}{\Gamma(1+i\gamma)}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\frac{1}{t^{r+1+i\gamma}}dt+$$

$$+(-1)^{r+1}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\frac{d^r}{dt^r}H_\omega^{r+2}(t)dt$$

$$= A\frac{\Gamma(r+1+i\gamma)}{\Gamma(1+i\gamma)\Gamma(r+1)}\tilde{f}_r(x)+(-1)^{r+1}\int_0^\infty \delta_r(x,t)\frac{t^r}{r!}\frac{d^r}{dt^r}H_\omega^{r+2}(t)dt.$$

And to complete the proof we have to show that the last integral tends to 0 as $\omega\to\infty$.

Estimate (3.10), which is certainly valid for $1\leqslant t, 1\leqslant\omega$, gives

$$\Big|\int_1^\infty \delta_r(x,t)t^r\frac{d^r}{dt^r}H_\omega^{r+2}(t)dt\Big|\leqslant\frac{C}{\omega}\int_1^\infty\frac{|\delta_r(x,t)|}{t^2}dt = O\Big(\frac{1}{\omega}\Big).$$

Thus we have to deal only with the integral between 0 and 1. Break it into 2 parts, between 0 and $1/\omega$, and between $1/\omega$ and 1. Write

$$\int_0^t \frac{\delta_r(x,s)}{s^{1+i\gamma}}ds = F(t).$$

We have

$$\Delta(t) = \int_0^t \delta_r(x,s)ds = \int_0^t s^{1+i\gamma}\frac{\delta_r(x,s)}{s^{1+i\gamma}}ds$$

$$= s^{1+i\gamma}F(s)\Big]_0^t-(1+i\gamma)\int_0^t F(s)s^{i\gamma}ds.$$

Since $F(s) = o(1)$ as $s\to 0$, we have $\Delta(t) = o(t)$.

$$\int_0^{1/\omega}\delta_r(x,t)t^r\frac{d^r}{dt^r}H_\omega^{r+2}(t)dt$$

$$= \int_0^{1/\omega}\delta_r(x,t)t^r\frac{d^r}{dt^r}W_\omega^{r+2}(t)dt-\int_0^{1/\omega}\delta_r(x,t)t^r\frac{d^r}{dt^r}\frac{A}{t^{1+i\gamma}}dt.$$

The 2nd integral tends to 0 as $\omega\to\infty$, and

$$\int_0^{1/\omega}\delta_r(x,t)t^r\frac{d^r}{dt^r}W_\omega^{r+2}(t)d_t$$

$$= \Delta(t)t^r\frac{d^r}{dt^r}W_\omega^{r+2}(t)\Big]_0^{\frac{1}{\omega}}-\int_0^{1/\omega}\Delta(t)\frac{d}{dt}\Big\{t^r\frac{d^r}{dt^r}W_\omega^{r+2}(t)\Big\}dt$$

$$= o\Big(\frac{1}{\omega}\Big)\frac{1}{\omega^r}\omega^{r+1}-\int_0^{1/\omega}\Delta(t)rt^{r-1}\frac{d^r}{dt^r}W_\omega^{r+2}(t)dt-\int_0^{1/\omega}\Delta(t)t^r\frac{d^{r+1}}{dt^{r+1}}W_\omega^{r+2}(t)dt$$

$$= o(1)-\int_0^{1/\omega}o(t^r)\omega^{r+1}dt-\int_0^{1/\omega}o(t^{r+1})\omega^{r+2}dt = o(1).$$

Finally,

$$\int_{1/\omega}^1\delta_r(x,t)t^r\frac{d^r}{dt^r}H_\omega^{r+2}(t)dt$$

$$= \Delta(t)t^r\frac{d^r}{dt^r}H_\omega^{r+2}(t)\Big]_{\frac{1}{\omega}}^1-\int_{1/\omega}^1\Delta(t)\frac{d}{dt}\Big\{t^r\frac{d^r}{dt^r}H_\omega^{r+2}(t)\Big\}d^t$$

$$= o(t)O\Big(\frac{1}{\omega t^2}\Big)\Big]_{\frac{1}{\omega}}^1-\int_{1/\omega}^1 o(t)O\Big(\frac{1}{\omega t^3}\Big)dt = o(1)$$

and the proof is complete.

We can now prove, quite easily, the following identity:

$$2\int_0^\infty\frac{\sin u}{u^{1+i\gamma}}du\Big(\int_0^{\pi/2}u^{i\gamma}\sin u\,du+i\gamma\int_{\pi/2}^\infty u^{i\gamma-1}\cos u\,du\Big) = \pi. \tag{3.15}$$

Take in theorem 8, $f(x) = \sin x$, $r = 1$,

$$\tilde{f}_1(x) = -\int_0^\infty \frac{\delta_1(x,t)}{t^{1+i\gamma}}\,dt = -\int_0^\infty \frac{f'(x+t)-f'(x-t)}{t^{1+i\gamma}}\,dt\,\frac{1}{1+i\gamma}$$
$$= \frac{1}{1+i\gamma}\int_0^\infty \frac{2\sin x\sin t}{t^{1+i\gamma}} = \frac{1}{1+i\gamma}C_\gamma \sin x.$$

By theorem 8, however, $M_\gamma(S^{(1)}[f])$ is $(C, 3)$ summable to

$$\tilde{f}_1(x)\frac{\Gamma(2+i\gamma)}{\Gamma(1+i\gamma)}\frac{1}{\pi}\left(\int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^\infty u^{i\gamma-1}\cos u\,du\right).$$

However, $M_\gamma(S^{(1)}[f]) = C_\gamma \sin x$. Therefore

$$C_\gamma \sin x = \frac{\sin x}{1+i\gamma}C_\gamma\frac{\Gamma(2+i\gamma)}{\Gamma(1+i\gamma)}\frac{C_\gamma}{\pi}\left(\int_0^{\pi/2} u^{i\gamma}\sin u\,du + i\gamma\int_{\pi/2}^\infty u^{i\gamma-1}\cos u\,du\right)$$

and hence (3.15).

Substituting (3.15) in (3.14) we get: if $\tilde{f}_r(x)$ exists,

$$(3.16)\qquad (C, r+2)\, M_\gamma(S^{(r)}[f]) = \frac{\Gamma(r+1+i\gamma)}{\Gamma(r+1)\Gamma(1+i\gamma)}\tilde{f}_r(x).$$

THEOREM 9. *If a trigonometric series* $\sum_0^\infty A_k(x)$ *is* (C, β) *summable in a set* E *of positive measure,* $0 \leqslant \beta$ *an integer, then* $M_\gamma\left(\sum A_k(x)\right)$ *is* (C, β) *summable almost everywhere in* E.

Proof. This is immediate in view of the main theorem in [9], and Lemma 3 in [8]. See also [6], where a more general situation is discussed.

THEOREM 10. *If* $\sum_1^\infty A_l(x)$ *is* (C, β) *summable to* $s(x)$ *in a set* E, $|E| > 0$ *then if* $k \geqslant \beta+2$, *and if* $F(x)$ *is the sum of the k-th termwise integral of* $\sum A_l(x)$, *then* $F_k(x)$ *exists and equals* $s(x)$ *almost everywhere in* E.

Proof. Since $\sum A_l(x)$ is (C, β) summable in E, $\sum B_l(x)$ is (C, β) summable a.e. in E (see [5]). We next apply Theorem XI. 2. 22 in [10]. For a statement of the theorem see also Lemma 4 in [8].

THEOREM 11. *Let f have* $r-1$ *generalized derivatives in a set* E, $|E| > 0$. *Assume also the existence of* $\tilde{f}_r(x)$ *at each point of* E. *Then let* Φ *be the indefinite integral of f. Claim:* Φ *has* $r+1$ *generalized derivatives in* E.

Proof. The proof of this theorem in the case $\gamma = 0$, given in section 4 of [8], goes over without change.

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Reçu par la Rédaction le 21. 3. 1970