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Components and open mapping theorems

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Introduction. In a recent paper of De Wilde [8], strictly netted locally convex spaces are defined and some closed graph theorems are proved. It seems, as it is not the strict net itself in such a space which plays the essential role, but that it is only a tool for constructing a general structure on the space, a structure, which is independent of the actual construction and from which the closed graph theorems are a consequence. This is the background for this paper.

After the preliminaries are given in section 0 and 1, section 2 brings a general open mapping theorem for closed mappings from a pre-(F)-sequence into a not necessarily metrizable topological group, a generalization of Theorem 2 in [6], and it is from this that all other open mapping — and closed graph theorems presented here will be derived.

In section 3 come the main notions, those of a component and an overwhelming set of components for a locally convex space. The existence of such sets is responsible for the validity of the open mapping and closed graph theorems proved here. The notions are slightly different from those introduced by Słowikowski [7]; this enables us to include new cases, for example, it turns out (section 4) that the structure of strictly netted spaces and of Souslin spaces gives the possibility of constructing sets of components, which overwhelme.

In section 4 also examples from [7] are taken, and it is indicated that the class $\mathscr{D}_0$ considered by Raikov [3] is contained in the class of spaces having an overwhelming set of components.

I would take this opportunity to thank Dr. W. Słowikowski for his willingness to discuss the subject with me and for his encouraging attitude in general.

0. Notation and Terminology. All vector spaces in consideration are supposed to be over the complex numbers, and all locally convex spaces are supposed to be Hausdorff, unless something else is stated.

As it is customary, we shall write "the topological space X" instead of "the topological space (X, τ)", where no confusion about the topology τ will arise.

Besides the standard notation on set theory, topology, vector spaces, etc., we will use the following special notations:

N stands for the natural numbers.

C stands for the complex number.

N^∞ stands for the set of all sequences of natural numbers.

1. Preliminaries. Let us recall the following two definitions from [6]:

1.1 DEFINITION. A sequence $\{(V_n, \|\cdot\|_n) | n \in N\}$ of vector spaces, each V_n equipped with a seminorm $\|\cdot\|_n$, will be called a *pre-(F)-sequence*, if the following conditions are satisfied:

(i) For each $n \in N$, $V_{n+1} \subseteq V_n$ and $\|x\|_n \leqslant \|x\|_{n+1}$ for all $x \in V_{n+1}$.

(ii) The projective topology on $\bigcap_{n=1}^{\infty} V_n$ is Hausdorff.

If $\{(V_n, \|\cdot\|_n) | n \in N\}$ is a pre-(F)-sequence we can define the following positive functions on V_1:

$$\varrho_n(x) = \begin{cases} \dfrac{\|x\|_n}{1+\|x\|_n} & \text{for } x \in V_n, \\ 1 & \text{for } x \in V_1/V_n \end{cases}$$

and

$$\varrho(x) = \sum_{n=1}^{\infty} 2^{-n} \varrho_n(x) \quad \text{for all } x \in V_1.$$

Since all the ϱ_n's are subadditive, ϱ is subadditive and also $\varrho(x) = 0$ if and only if $x = 0$; hence ϱ defines a translation invariant metric on V_1, which turns the additive group $(V_1, +)$ into a topological group. In the following this group assigned to the pre-(F)-sequence $\{V_n\}$ will always be denoted by $[V_n]$.

1.2. DEFINITION. A pre-(F)-sequence $\{(V_n, \|\cdot\|_n) | n \in N\}$ is called an *(F)-sequence*, if the assigned group $[V_n]$ is complete.

Remark. Note that in case $[V_n]$ is complete, $\bigcap_{n=1}^{\infty} V_n$ is a Frechet space for the projective topology.

Let us end this section with the following:

1.3. DEFINITION. A pair (V, τ), where V is a vector space and τ a topology on V, will be called an *additive topological group with continuous scalar multiplication*, if

(i) $(V, +)$ is a topological group under τ.

(ii) For each $t \in C$ the map $x \to tx$ of V into itself is continuous for τ.

Note that the group assigned to a pre-(F)-sequence is an additive topological group with continuous scalar multiplication.

2. The Main Theorem. We are now able to prove the announced main theorem:

2.1. THEOREM. *Let $\{W_n | n \in N\}$ be an (F)-sequence and W an additive topological group with continuous scalar multiplication. Let $D_T \subseteq W_1$ be a subspace and $T: D_T \to W$ a linear map, such that:*

(i) *T is closed with respect to the topologies on $[W_n]$ and W.*

(ii) *For each $n \in N$, $T(D_T \cap W_n)$ is of second category in W.*

Under these circumstances T is open.

Proof. Let ϱ denote the metric on $[W_n]$ and let U_n denote the unit ball of W_n; for each $\varepsilon > 0$ we put

$$K_\varepsilon = \{x \in D_T | \varrho(x) \leqslant \varepsilon\}.$$

Let us first prove that for all $\varepsilon > 0$, $0 \in (T(K_\varepsilon))^{-\circ}$ (1).

Since $T(D_T \cap W_n)$ is of second category in W, we can conclude that $0 \in (T(2^{-n} U_n \cap D_T))^{-\circ}$ for all $n \in N$, hence if we make n so large that $2^{-n} U_n \cap D_T \subseteq K_\varepsilon$ we get the desired conclusion.

We will now prove that for every $\varepsilon > 0$

$$(*) \qquad (T(K_{\varepsilon/2}))^- \subseteq T(K_\varepsilon)$$

which together with the above will give us our result. Let us first determine n so large that $x \in 2^{-n+1} U_{n-1}$ implies $\varrho(x) \leqslant \varepsilon/2$.

Let $y \in (T(K_{\varepsilon/2}))^-$. Take $x_n \in K_{\varepsilon/2}$ such that

$$y - Tx_n \in (T(2^{-n} U_n \cap D_T))^-;$$

hence we take $x_{n+1} \in 2^{-n} U_n \cap D_T$ with

$$y - Tx_n - Tx_{n+1} \in (T(2^{-n-1} U_{n+1} \cap D_T))^-.$$

Continuing in that way we reach a sequence $\{x_k | k \geqslant n\} \subseteq D_T$ with the properties:

$$(1) \qquad x_k \in 2^{-k+1} U_{k-1} \quad \text{for } k > n,$$

$$(2) \qquad y - \sum_{k=n}^{m} Tx_k \in (T(2^{-m} U_m \cap D_T))^-.$$

(1) gives that $\sum_{k=n}^{\infty} x_k$ is convergent in $[W_n]$; put $x = \sum_{k=n}^{\infty} x_k$.

Since

$$\sum_{k=n+1}^{m} x_k \in \sum_{k=n+1}^{m} 2^{-k+1} U_{k-1} \subseteq 2^{-n+1} U_{n-1}$$

we get $\varrho(x - x_n) \leqslant \varepsilon/2$ and hence $\varrho(x) \leqslant \varepsilon$.

(1) $-^\circ$ denotes the subsequent formation of closure and interior.

Let now U be a symmetric neighbourhood of 0 in $[W_n]$ and V a symmetric neighbourhood of 0 in W; from (2) we get:

$$y - \sum_{k=n}^{m} Tx_k \in T(2^{-m} U_m \cap D_T) + V, \quad \forall m \geqslant n. \tag{3}$$

Determine now m so large that $2^{-m} U_m \subseteq U$ and $x - \sum_{k=n}^{m} x_k \in U$, then from (3)

$$x \in \Big(x - \sum_{k=n}^{m} x_k\Big) + T^{-1}\Big(\sum_{k=n}^{m} Tx_k\Big) \subseteq U + T^{-1}\big(y + T(U \cap D_T) + V\big). \tag{4}$$

From (4) it follows that to each U and V we can find element $a(U, V) \in D_T$, $b(U, V) \in U$, $c(U, V) \in U \cap D_T$ and $d(U, V) \in V$ such that

$$x = a(U, V) + b(U, V), \qquad T\big(a(U, V)\big) = y + T\big(c(U, V)\big) + d(U, V). \tag{5}$$

Since

$$\lim_{(U,V)} b(U, V) = \lim_{(U,V)} c(U, V) = \lim_{(U,V)} d(U, V) = 0$$

we get

$$\lim_{(U,V)} \big(a(U, V) - c(U, V)\big) = x,$$

$$\lim_{(U,V)} T\big(a(U, V) - c(U, V)\big) = y,$$

hence by the closedness of T, $x \in D_T$ and $Tx = y$. Q.E.D.

Remark. In case W is the group assigned to some (F)-sequence the above proof can be much simplified, see [6], Theorem 2.

3. Locally convex spaces with an overwhelming set of components. In this section we will define a wide class of locally convex spaces in which we can apply Theorem 2.1.

Let in the following (E, τ) denote a locally convex space.

3.1. DEFINITION. A pre-(F)-sequence $\{(V_n, \|\cdot\|_n| \; n \in N\}$ is called a *component* for (E, τ), if the following conditions hold:

(i) V_1 is a subspace of E.

(ii) The injection $[V_n] \to (E, \tau)$ is continuous.

(iii) The injection in (ii) can be extended to a continuous map from the completion $\widetilde{[V_n]}$ into (E, τ).

A component for (E, τ) is called a *complete component*, if it is an (F)-sequence.

3.2. DEFINITION. A set Σ of components for (E, τ) is called *overwhelming*, if to any pre-(F)-sequence $\{(V_p, \|\cdot\|_p)| \; p \in N\}$ and to any linear map T from a subspace D_T of E onto a second category subset of $[V_p]$, there is $\{W_n | \; n \in N\} \in \Sigma$ such that $T(D_T \cap W_n)$ is of second category in $[V_p]$ for every $n \in N$.

A set Σ of components is called *strongly overwhelming*, if it has the above property with the pre-(F)-sequence $\{V_p | \; p \in N\}$ interchanged by an arbitrary additive topological group with continuous scalar multiplication.

3.3. LEMMA. *If (E, τ) has a component, then it has also a complete component.*

Proof. Let $\{V_n | \; n \in N\}$ be a component for (E, τ); by assumption the injection $T: [V_n] \to (E, \tau)$ can be extended to a continuous map $\tilde{T}: \widetilde{[V_n]} \to (E, \tau)$. Using now Proposition 2 of [6], we can find an (F)-sequence $\{W_n | \; n \in N\}$ such that $[W_n] = \widetilde{[V_n]}$.

Since $[W_n]$ is an additive topological group with continuous scalar-multiplication we get by the linearity of T and the continuity of $\tilde{T}$ that $\tilde{T}$ is a linear map of W_1 into E, and hence $L = \tilde{T}^{-1}(0)$ is a $[W_n]$-closed subspace of W_1.

From Proposition 4 in [6] it now follows that there is an (F)-sequence $\{H_n | \; n \in N\}$, such that $[H_n]$ is isomorphic to $[W_n]/L$.

Identifying now the H_n's with subspaces of E and carrying the respective seminorms over to E we get the desired result.

The following proposition is now trivial to prove:

3.4. PROPOSITION. *Let (E, τ) be a locally convex space having a (strongly) overwhelming set of components, then (E, τ) also has a (strongly) overwhelming set of complete components.*

Having Lemma 3.3 and Proposition 3.4 in mind, one may ask why a component is not once for all defined to be complete instead of having condition (iii) of Definition 3.1. The reason for this is that in concrete cases, where one perhaps is not so much interested in the components themselves, but merely the properties they induce on the space under consideration, it is often easy to find components, which need not be complete, while complete components are not straightforward at hand, so that one has to go through the procedure in 3.3 in order to find them.

In section 4 we will give some important examples of locally convex spaces with overwhelming sets of components.

If E is a locally convex space with a component $\{V_n | \; n \in N\}$ and M is a linear subspace of E, then we will denote the subgroup $[V_n] \cap M$ of $[V_n]$ by $[V_n \cap M]$.

The following theorems are all corollaries of the main Theorem 2.1:

3.5. THEOREM. *Let E be a locally convex space, V an additive topological group with continuous scalar multiplication and D_T a subspace of E. Further* *let $T\colon D_T \to V$ be a linear map, mapping D_T onto a second category subspac of V. If one of the two conditions*

(i) *E has an overwhelming set of components, V is metrizable and T is sequentially closed,*

(ii) *E has a strongly overwhelming set of components and T is closed,*

is satisfied, then there is a complete component $\{W_n \mid n \in N\}$ for E such that $T|[W_n \cap D_T]$ is an open map.

Proof. Direct application of Theorem 2.1.

3.6. COROLLARY. *Let E be a locally convex space with a strongly overwhelming set of components, F a locally convex space and D_T a subspace of E. If $T\colon D_T \to F$ is a closed linear map, then either $T(D_T)$ is of first category in F or $T(D_T) = F$ and T is open.*

The conclusion is still true, if E has an overwhelming set of components, F is a Fréchet space and T is sequentially closed.

From the foregoing we now get the following closed graph theorem, which at the same time is a kind of localization theorem for locally convex spaces with a (strongly) overwhelming set of components:

3.7. THEOREM. *Let E be a Baire space and F a locally convex space having a strongly overwhelming set of components. If $T\colon E \to F$ is a closed linear map, then there is a complete component $\{W_n \mid n \in N\}$ for F such that $T(E) \subseteq \bigcap_{n=1}^{\infty} W_n$ and T is continuous from E into $\bigcap_{n=1}^{\infty} W_n$ (hence also continuous from E into F).*

The same is true for a sequentially closed map from a Fréchet space into a locally convex space with an overwhelming set of components.

Proof. From Theorem 3.5 we can conclude that there is a complete component $\{W_n \mid n \in N\}$ such that $T(E) \subseteq W_1$ and T is continuous from E into $[W_n]$, and using [6], Proposition 1.D we get that actually $T(E) \subseteq \bigcap_{n=1}^{\infty} W_n$. Q. E. D.

Let us end this section by defining an inductive limit topology on a locally convex space with an overwhelming set of components, which seems to play an important role in the theory:

Let (E, τ) be a locally convex space with an overwhelming set Σ of complete components, say $\Sigma = \{\{W_n^a\}_{n=1}^{\infty} \mid a \in A\}$. Define for $a \in A$, $F_a = \bigcap_{n=1}^{\infty} W_n^a$ and equip it with the projective topology; from Theorem 3.7 it easily follows that $E = \bigcup_{a \in A} F_a$. We can order A partially by putting $a \leqslant \beta$ if $F_a \subseteq F_\beta$, and since every F_a can be mapped continuously into (E, τ), we see that if $a \leqslant \beta$, then the imbedding of F_a into F_β is closed and hence continuous.

Let now $\{a_1, a_2, \ldots, a_n\} \subseteq A$ and define $F = \operatorname{span}(\bigcup_{i=1}^{n} F_{a_i})$; the topology on F induced by the quotient topology on

$$\prod_{i=1}^{n} F_{a_i}/L, \qquad L = \{(x_{a_i}) \in \prod_{i=1}^{n} F_{a_i} \mid \quad \sum_{i=1}^{n} x_{a_i} = 0\}$$

turns F into a Fréchet space, which can be mapped continuously into (E, τ), hence using Theorem 3.7 we find an $a \in A$ with $F \subseteq F_a$. This proves that the ordering on A is filtered upwards.

The inductive limit topology on E with respect to the family $\{F_a \mid a \in A\}$ exists, since τ is Hausdorff; let us denote it by τ_Σ; from Theorem 3.7 we get that any continuous operator from a Fréchet space into (E, τ) is also continuous for τ_Σ.

Let us call a topology τ_σ on E *maximal with respect to* Σ, if Σ is an overwhelming set of components for (E, τ_σ) and τ_σ is the finest locally convex topology on E with this property. It is not difficult to see that such a topology exists and that it is weaker than τ_Σ. One may pose the following:

3.8. QUESTION. *Is $\tau_\Sigma = \tau_\sigma$?*

Probably the answer is negative in general, but in many concrete cases it is positive. If for example (E, τ) is an ultrabornological space, then $\tau = \tau_\Sigma$ and hence maximal; this shows that the inductive topologies of the families which have an overwhelming set of components in the sense of [7] are all maximal.

Let us finally prove the following proposition concerning 3.8:

3.9. PROPOSITION. *If there is a co-complete locally convex topology ξ on E, such that Σ is an overwhelming set of components for (E, ξ), then the answer to 3.8 is positive and Σ is an overwhelming set of components for the family $\{F_a \mid a \in A\}$ in the sense of [7].*

Remark. Co-completeness of (E, ξ) means that every convex, balanced, closed and bounded subset of (E, ξ) spans a Banach space (see [1], p. 382).

Proof of 3.9. Let $\{W_n \mid n \in N\} \in \Sigma$ and let (x_m) be a Cauchy sequence in $[W_n]$; denote the unit ball in W_n by U_n. To each $n \in N$ we can find a number m_n such that

$$n(x_m - x_{m_n}) \in n^{-1} U_n \quad \text{for all } m \geqslant m_n \tag{1}$$

and this implies that there is a bounded, convex balanced and closed set B in (E, ξ) with:

$$\bigcup_{n=1}^{\infty} \{n(x_m - x_{m_n}) \mid m \geqslant m_n\} \subseteq B;$$

hence for all $p, q \geqslant m_n$:

(2) $$x_p - x_q = (x_p - x_{m_n}) + (x_{m_n} - x_q) \in 2n^{-1}B.$$

Since (E, ξ) is co-complete, there is an $\alpha \in A$ such that B is a bounded subset of F_α. From (2) it then follows that (x_n) is convergent in F_α. The rest of the proof is now trivial. Q.E.D.

4. Permanence properties and examples. It is seen immediately that the class of locally convex spaces having a (strongly) overwhelming set of components is closed under the following operations: Countable inductive limits, continuous images by linear maps, going to separated quotients and going to closed subspaces. Hence for example any countable inductive limit of Fréchet spaces has a strongly overwhelming set of components.

The type of spaces we are going to discuss in the following examples often occur in analysis.

Example 1. *σ^2-spaces* (cf. [7]). Let E be a vector space. By a *σ^2-family* in E we understand a family $\{E_{k_1,\ldots,k_n} \mid n \in N,\ k_1, k_2, \ldots, k_n \in N\}$ of subspaces, each $E_{k_1,\ldots,k_n}$ equipped with a seminorm $\|\cdot\|_{k_1,\ldots,k_n}$ such that

(i) For each $\boldsymbol{k} = (k_n) \in N^\infty$, $\{E_{k_1,\ldots,k_n} \mid n \in N\}$ is a pre-(F)-sequence.

(ii) For each $\boldsymbol{k} \in N^\infty$, $E_{\boldsymbol{k}} = \bigcap_{n=1}^{\infty} E_{k_1,\ldots,k_n}$ is a Fréchet space under the projective topology.

(iii) If $\boldsymbol{k} = (k_n) \in N^\infty$, $\boldsymbol{k}' = (k_n') \in N^\infty$ and $k_n \leqslant k_n'$ all $n \in N$, then

$$E_{k_1,\ldots,k_n} \subseteq E_{k_1',\ldots,k_n'}$$

and

$$\|x\|_{k_1',\ldots,k_n'} \leqslant \|x\|_{k_1,\ldots,k_n} \quad \text{for all } x \in E_{k_1,\ldots,k_n}.$$

(iv) $\mathrm{E} = \bigcup_{k_1=1}^{\infty} E_{k_1},\ E_{k_1,\ldots,k_{n-1}} = \bigcup_{k_n=1}^{\infty} E_{k_1,\ldots,k_n}$.

Definition. A locally convex space (E, τ) is called a *σ^2-space* if there is a σ^2-family $\{E_{k_1,\ldots,k_n} \mid n \in N,\ k_1, \ldots, k_n \in N\}$ in E such that (E, τ) is the inductive limit of the family $\{E_{\boldsymbol{k}} \mid \boldsymbol{k} \in N^\infty\}$.

It is not difficult to see that if E is a σ^2-space with respect to the σ^2-family $\{E_{k_1,\ldots,k_n} \mid n \in N,\ k_1, \ldots, k_n \in N\}$, then the set

$$\Sigma = \{\{E_{k_1,\ldots,k_n}\}_{n=1}^{\infty} \mid \boldsymbol{k} \in N^\infty\}$$

is a strongly overwhelming set of components for E; it can even be proved that Σ consists of complete components (this is heavily dependent on conditions (ii) and (iii) above).

The class of σ^2-spaces is a subclass of the class of spaces occurring in the next example.

Example 2. *Strictly netted spaces* (cf. [8]).

Definition. A locally convex space E is called *strictly netted* if there is a family $\{e_{k_1,k_2,\ldots,k_n} \mid n \in N,\ k_1, k_2, \ldots, k_n \in N\}$ of convex and balanced subsets of E, satisfying the following conditions:

(i) $\bigcup_{k_1=1}^{\infty} e_{k_1}$ is absorbing in E.

(ii) For fixed $k_1, \ldots, k_{n-1}$, $\bigcup_{k_n=1}^{\infty} e_{k_1,\ldots,k_n}$ is absorbing in $e_{k_1,\ldots,k_{n-1}}$.

(iii) If $\boldsymbol{k} = (k_n) \in N^\infty$ and $\{x_n\}_{n=0}^{\infty} \subseteq E$ such that $x_n - x_{n-1} \in e_{k_1,k_2,\ldots,k_n}$ for all $n \in N$, then $\{x_n\}$ is convergent in E.

The family $\{e_{k_1,\ldots,k_n}\}$ is called a *strict net* in E.

It is readily seen that if E is a strictly netted space, then it is no restriction to assume about the strict net $\{e_{k_1,\ldots,k_n}\}$ in E that $e_{k_1,\ldots,k_n} \subseteq e_{k_1,\ldots,k_{n-1}}$ for all $n \in N$, so let us always do that from now on.

Let now E be a strictly netted locally convex space with strict net $\{e_{k_1,\ldots,k_n}\}$; for each $(k_1, \ldots, k_n)$ we define the subspace

$$E_{k_1,k_2,\ldots,k_n} = \operatorname{span}(e_{k_1,\ldots,k_n})$$

and we turn $E_{k_1,\ldots,k_n}$ into a seminormed space by introducing the Minkowski functional for $e_{k_1,\ldots,k_n}$ on it.

It follows from the conditions above that

$$E = \bigcup_{k_1=1}^{\infty} E_{k_1} \qquad E_{k_1,\ldots,k_{n-1}} = \bigcup_{k_n=1}^{\infty} E_{k_1,\ldots,k_n}.$$

Proposition. *For every $\boldsymbol{k} = (k_n) \in N^\infty$, $\{E_{k_1,\ldots,k_n} \mid n \in N\}$ is a component for E.*

Proof. Let $\boldsymbol{k} = (k_n) \in N^\infty$ be given. From (iii) above, it follows that to any 0-neighbourhood U in E there is an $n \in N$ with $e_{k_1,\ldots,k_n} \subseteq U$; this proves

1° that the projective topology on $E_{\boldsymbol{k}} = \bigcap_{n=1}^{\infty} E_{k_1,\ldots,k_n}$ is Hausdorff, hence that $\{E_{k_1,\ldots,k_n} \mid n \in N\}$ is a pre-(F)-sequence;

2° that the injection of $[E_{k_1,\ldots,k_n}]$ into E is continuous.

If $\{x_p \mid p \in N\}$ is a Cauchy sequence in $[E_{k_1,\ldots,k_n}]$, we can find a subsequence $\{x_{p_n} \mid n = 0, 1, \ldots\}$ with

$$x_{p_n} - x_{p_{n-1}} \in e_{k_1,\ldots,k_n} \quad \text{for all } n \in N$$

and hence $\{x_{p_n}\}$ and therefore also $\{x_p\}$ is convergent in E. Q.E.D.

It is now trivial to prove:

Proposition. *The set*

$$\Sigma = \{\{E_{k_1,\ldots,k_n}\}_{n=1}^{\infty} \mid \boldsymbol{k} \in N^\infty\}$$

is a strongly overwhelming set of components for E.

Example 3. *Souslin spaces* (cf. [2] and [4]). We shall here prove:

PROPOSITION. *If E is a co-complete locally convex Souslin space, then E has an overwhelming set of components.*

Proof. Since E is Souslin, we can find a continuous map f of N^∞ onto E. Let us for $n \in N$, $k_1, \ldots, k_n \in N$ define

$$N^\infty(k_1, \ldots, k_n) = \{\boldsymbol{m} = (m_j) \in N^\infty \mid m_j = k_j,\ j = 1, 2, \ldots, n\}.$$

It is easy to see that for $\boldsymbol{k} = (k_n) \in N^\infty$, the set

$$\{N^\infty(k_1, \ldots, k_n) \mid n \in N\}$$

forms a neighbourhood basis at $\boldsymbol{k}$ for the product topology on N^∞.

If $\boldsymbol{k} = (k_n) \in N^\infty$ it follows from the continuity of f that

(1) $$\big(x_n \in f(N^\infty(k_1, \ldots, k_n)) \text{ for all } n\big) \Rightarrow \big(x_n \to f(\boldsymbol{k}),\ n \to \infty\big).$$

(2) $$\{f(\boldsymbol{k})\} = \bigcap_{n=1}^{\infty} f\big(N^\infty(k_1, k_2, \ldots, k_n)\big).$$

Also we have

$$E = \bigcup_{k_1=1}^{\infty} f\big(N^\infty(k_1)\big),$$

(3) $$f\big(N^\infty(k_1, k_2, \ldots, k_{n-1})\big) = \bigcup_{k_n=1}^{\infty} f\big(N^\infty(k_1, k_2, \ldots, k_n)\big).$$

Define now

$$U_{k_1,\ldots,k_n} = \text{convex balanced hull of } f\big(N^\infty(k_1, k_2, \ldots, k_n)\big)$$

and

$$E_{k_1,k_2,\ldots,k_n} = \bigcup_{m=1}^{\infty} m\, U_{k_1,\ldots,k_n}.$$

The gauge function for $U_{k_1,k_2,\ldots,k_n}$ turns $E_{k_1,\ldots,k_n}$ into a seminormed space.

We shall now show that for fixed $\boldsymbol{k} = (k_n) \in N^\infty$, $\{E_{k_1,k_2,\ldots,k_n} \mid n \in N\}$, is a component for E.

Let U be a convex and balanced 0-neighbourhood in E; from (1) it follows that if $x_n \in n^{-1} f\big(N^\infty(k_1, k_2, \ldots, k_n)\big)$ all $n \in N$, then $x_n \to 0$ for $n \to \infty$; hence we can find an $n \in N$ such that

$$n^{-1} f\big(N^\infty(k_1, k_2, \ldots, k_n)\big) \subseteq U,$$

and then also

$$n^{-1} U_{k_1,k_2,\ldots,k_n} \subseteq U.$$

This proves that $\{E_{k_1,\ldots,k_n} \mid n \in N\}$ is a pre-(F)-sequence, and that the imbedding of $[E_{k_1,\ldots,k_n}]$ into E is continuous.

Let now $\{x_p\}$ be a Cauchy sequence in $[E_{k_1,\ldots,k_n}]$ and take a subsequence such that

$$x_{p_n} - x_{p_{n-1}} \in n^{-1} 2^{-n} U_{k_1,\ldots,k_n}.$$

Then $2^n(x_{p_n} - x_{p_{n-1}}) \to 0$ for $n \to \infty$, let B be a bounded convex and balanced closed subset of E with

$$2^n(x_{p_n} - x_{p_{n-1}}) \in B \quad \text{all } n \in N.$$

Now

$$x_{p_n} - x_{p_m} = \sum_{i=m+1}^{n} (x_{p_i} - x_{p_{i-1}}) \in \Big(\sum_{i=m+1}^{n} 2^{-i}\Big) B$$

and from the co-completeness of E it follows that $\{x_{p_n}\}$ and hence $\{x_p\}$ is convergent in E.

From (3) it is now easy to see that the set

$$\Sigma = \big\{\{E_{k_1,k_2,\ldots,k_n}\}_{n=1}^{\infty} \mid \boldsymbol{k} \in N^\infty\big\}$$

is an overwhelming set of components for E. Q.E.D.

From the above proposition we see that there is an overlapping between the graph theorems proved here and the graph theorems proved by Schwartz [4] and Martineau [2] for Souslin spaces, but it is only an overlapping; there is no hope of proving the Borelgraph theorem of Schwartz within the theory developed here, it requires a completely different technique.

Example 4. The class $\mathscr{D}_0$ considered by Raikov [3].

We will not work out in detail here that any locally convex space of type $\mathscr{D}_0$ has an overwhelming set of components, but just mention that this follows from conditions (α), (β) and (γ) on p. 293 in [3] together with Lemma 1, p. 469 in [5].

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