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Inequalities for Fourier transforms*

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1. Introduction. The primary purpose of this paper is the derivation of necessary and sufficient conditions in order that the Fourier transformation carry one given Orlicz space into another (Theorem (2.16)). The sufficiency follows from an argument due to R. O'Neil [21] on the interpolation of operations of types $(1, \infty)$ and $(2, 2)$. Necessity is based on examples exploiting the lacunarity argument in Example 27, Chapter V of Zygmund's book [35].

The following special case exhibits the general form of the characteristic condition. Let A, B denote convex functions defined on $[0, \infty)$, with $A(x)/x$ and $B(x)/x$ increasing strictly from zero to positive infinity with x. Then there exists $k > 0$ such that

$$\int_{-\infty}^{\infty} B\big(k|\hat{f}(x)|\big)\,dx \leqslant 1$$

whenever

$$\int_{-\infty}^{\infty} A\big(|f(x)|\big)\,dx \leqslant 1$$

if and only if for some constant $c > 0$

$$(*) \qquad B^{-1}(tx) \geqslant cxA^{-1}(t/x) \quad \text{for } x > 0,\ 0 < t \leqslant 1.$$

Here A^{-1}, B^{-1} denote the functions inverse to A, B respectively, and $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt$. The Fourier transform of f is defined since $(*)$ implies (by setting $x = 1/t$) that f is the sum of an integrable function and a square-integrable function (see (5.8)).

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In terms of the Young's complement $\bar{A}$ of A (see section 2 for definitions and notation) condition (*) reads: for some constant $k > 0$, and for $x > 0$, $t \geqslant 1$

$$B(kx) \leqslant 1/t^2 \bar{A}(1/tx).$$

That is, $B(kx)$ is dominated by the largest among those functions C for which

$$C(x) \leqslant 1/\bar{A}(1/x) \text{ and } C(x)/x^2 \text{ is non-decreasing.}$$

This paper is divided into six sections. Preliminary matters are discussed for reference in section 2. The reader familiar with Orlicz spaces might start at (2.13). In section 3, the interpolation of operations of types $(1, \infty)$ and $(2, 2)$ is treated; section 4 consists of further remarks on such operations. Section 5 is devoted to a necessary condition for the Fourier transformation to be of type (A, B). Section 6 contains notes and remarks not already given in the text, dealing with references [1]-[4], [7], [8], [13]-[16], [23]-[26], [28]-[32].

We are indebted to Professor Zygmund for suggesting this problem and for valuable comments, and to Professor R. O'Neil for pointing out that his theorem could be modified. The extension to arbitrary abelian groups was aided by J. B. Dowling, who suggested the basic "lacunary" set, and by Phillip Griffith, who explained to us the structure of torsion groups.

2. Preliminaries. (a) *Young's Functions* (see [19], [22], [35], etc.). The letters A, B, C are reserved in this paper for functions $A(x)$, defined for $x \geqslant 0$, such that

(i) $0 \leqslant A(x) \leqslant +\infty$, $A(0) = 0$;

(ii) $A(x)/x$ increases (in the wide sense);

(iii) A is left-continuous $(A(x) = A(x-0))$;

(iv) A is non-trivial $(0 \not\equiv A(x) \not\equiv +\infty$ for $x > 0)$.

By the phrase "$A(x)/x$ increases" we refer to these conditions. Let

$$A_0(x) = \int_0^x \frac{A(t)}{t}\, dt. \tag{2.1}$$

Then A_0 is convex, increasing, zero at zero, and non-trivial (A_0 is positive and finite in the same set as A). Moreover,

$$A_0(x) \leqslant A(x) \leqslant A_0(2x). \tag{*}$$

Functions which have the properties of A_0 are called *Young's Functions*.

Inequalities of the Hausdorff-Young type have been studied, when $A(x)/x$ increases, by Mulholland [20]. Inequalities (*) show that such inequalities (for example, the special case in the introduction) are equivalent to ones with convex functions. However, it is often convenient to use A instead of its *regularization* A_0. For example, $\min(A(x), B(x))$ satisfies (i)-(iv) but is not in general convex when A, B are convex.

b) *Orlicz spaces*. Let (X, μ) be a positive measure space, and let $A(x)/x$ be increasing. The *Orlicz space* $L_A(X, \mu)$ is defined to be the space (of equivalence classes modulo equality almost everywhere with respect to μ) of measurable functions f such that

$$\int_X A(\varepsilon|f(x)|)\, d\mu(x) < \infty$$

for some $\varepsilon > 0$ (depending on f). It is easy to see that L_A is a linear space. It is a Banach space with respect to the norm

$$\|f\|_A = \inf\{k > 0 \colon \int_X A_0(|f(x)|/k)\, d\mu(x) \leqslant 1\},$$

where A_0 is A if A is a Young's function, and is given by (2.1) otherwise. These spaces are studied in detail in [18], [19], [22]. Note that L_A and L_{A_0} consist of the same functions.

Inclusion relations among Orlicz spaces are characterized by inequalities on the functions A determining them, and by properties of the measure space (X, μ). In any case,

(2.2) If there exists $k > 0$ such that $A(x) \leqslant B(kx)$, $x > 0$ then $L_A(X, \mu) \supseteq L_B(X, \mu)$.

We use the following converse cases:

(i) If μ is non-atomic and $\mu(X) = +\infty$, the converse of (2.2) holds.

(ii) If μ is non-atomic and $\mu(X) < \infty$, the condition of (2.2), holding for x sufficiently large, is necessary and sufficient for the inclusion to hold.

(iii) If μ is purely atomic, with $\mu(\{t\}) = a > 0$ for each $t \in X$, the validity of the inequality in (2.2) for x sufficiently small is necessary and sufficient.

This leads to the following definition.

(2.3) Definition. We say that B *dominates* A and write $A \leqslant B$ if the inequality in (2.2) holds. $A \leqslant B$ for large values, $A \leqslant B$ for small values are defined similarly. For example, $A \leqslant B$ for large values if there exist positive constants k, x_0 such that $A(x) \leqslant B(kx)$ for $x > x_0$. As part of the preceding definition, we say that A is *equivalent to* B (for all values, large values, etc.) if $A \preccurlyeq B$ and $B \preccurlyeq A$ (in the appropriate intervals). This is denoted by $A \sim B$. Thus $A \sim B$ means $L_A = L_B$; the norms are equivalent. For example,

$$xe^x \sim e^x \quad \text{for large values},$$

since for $x \geqslant 1$,

$$e^x \leqslant xe^x \leqslant e^{2x}.$$

Finally we note that $L_A \cap L_B = L_{\max(A,B)}$, $L_A + L_B = L_{\min(A,B)}$ and the norms are equivalent (see [17]).

(c) *Operations of strong and weak types* (A, B); *rearrangements*. A linear mapping $T\colon L_A(X, \mu) \to L_B(Y, \nu)$ is continuous if and only if there is a constant $K > 0$ such that

$$\int_Y B\big(|Tf(y)|/K\big)\,d\nu(y) \leqslant 1 \tag{2.4a}$$

whenever

$$\int_X A\big(|f(x)|\big)\,d\mu(x) \leqslant 1 \quad (\text{that is, } \|f\|_A \leqslant 1). \tag{2.4b}$$

The best constant K is called the *norm* of T. Even if T is not linear (2.4) may hold; this is expressed by saying that T is of (*strong*) *type* (A, B). Notation: When A and B are powers, we use the exponents. Thus if $A(x) = x^2$, L_A is L^2, a map of type (A, A) is of type $(2, 2)$, etc. It is natural to call the function A given by

$$A(x) = 0 \text{ for } 0 \leqslant x \leqslant 1, \qquad A(x) = +\infty \text{ for } x > 1,$$

a "power"; we sometimes write x^∞. That $L_{x^\infty} = L^\infty$ follows from the definition of Orlicz spaces.

To define weak type (A, B), we first recall the notation $m(f, y)$ for the *distribution function* of a measurable function f on (X, μ):

$$m(f, y) = \mu(\{x\colon |f(x)| > y\}), \qquad y > 0. \tag{2.5}$$

If T is a function from $L_A(X, \mu)$ to $L_B(Y, \nu)$ for which inequalities (2.4) hold, then for each $y > 0$

$$m(Tf, y) \leqslant 1/B(y/K). \tag{2.6}$$

In case (2.6) (but not necessarily (2.4a)) holds whenever (2.4b) holds, we say that T is of *weak type* (A, B), with norm $\leqslant K$.

In addition to the distribution function, we shall use the *rearrangement* of f, denoted f^*, and defined to be the distribution function of $m(f, y)$ (see [12]). Since also $m(f^*, y) = m(f, y)$ (this follows using the right-continuity of $m(f, y)$), f^* and f are "equi-distributed".

If $A(x) = \int_0^x a(t)\,dt$, $a \geqslant 0$ and increasing (i.e., non-decreasing) — which means that A is a Young's function —, then

$$\int_X A\big(|f(x)|\big)\,d\mu(x) = \int_0^\infty a(y)\,m(f, y)\,dy = \int_0^\infty A\big(f^*(t)\big)\,dt. \tag{2.7}$$

(d) *Orlicz spaces related to* L_A; *operations on Young's functions*. Let $A(x)/x$ increase. The *Young's complement* of A, denoted $\bar{A}$, is given by

$$\bar{A}(x) = \sup_{y \geqslant 0} \big(xy - A(y)\big). \tag{2.8}$$

$\bar{A}$ is a Young's function, $\bar{\bar{A}} \sim A$, and $A \preccurlyeq B$ if and only if $\bar{B} \preccurlyeq \bar{A}$. The inequality

$$xy \leqslant A(y) + \bar{A}(x) \tag{2.9}$$

is called *Young's inequality*.

By (2.9),

$$\int_X |f(x)|\,|g(x)|\,d\mu(x) \leqslant 2\|f\|_A \|g\|_{\bar{A}} \quad (\textit{Hölder's inequality}). \tag{2.10}$$

Also (Luxemburg [19], Theorem 3 in section 2 of Chapter II and after),

$$N(f) \equiv \sup_{\|g\|_{\bar{A}} \leqslant 1} \Big| \int_X f(x)g(x)\,d\mu(x) \Big|$$

is a norm on L_A, equivalent to $\|f\|_A$. In case $A(2x) \leqslant CA(x)$ for some constant C, $L_{\bar{A}}$ is the space of continuous linear functionals on L_A (see [19]).

The *inverse* of A is defined on $[0, \infty]$ by

$$A^{-1}(y) = \inf\{x\colon A(x) > y\} \qquad (\inf \emptyset = +\infty),$$

$A^{-1}(y)$ is positive and finite for $y > 0$, $A^{-1}(\infty) = \infty$, A^{-1} is non-decreasing and right continuous, and $A^{-1}(y)/y$ is decreasing,

$$A\big(A^{-1}(x)\big) \leqslant x \leqslant A^{-1}\big(A(x)\big), \qquad x \geqslant 0, \tag{2.11}$$

$$A(x) = \sup\{y\colon A^{-1}(y) < x\} \qquad (\sup \emptyset = 0),$$

$A(x) = x^\infty$ $(= 0$ for $0 \leqslant x \leqslant 1$, $= +\infty$ for $x > 1)$ is an illuminating example.

If p and p' are conjugate exponents, the relation $x^{1/p} \cdot x^{1/p'} = x$ is the prototype of

$$x \leqslant A^{-1}(x)\bar{A}^{-1}(x) \leqslant 2x. \tag{2.12}$$

Also, $A^{-1}(x) \leqslant B^{-1}(x)$ (for all, large, small values respectively) if and only if $B(x) \leqslant A(x)$ (for all, large, small values respectively). The inverse of $kB(Kx)$ is

$$\frac{1}{K} B^{-1}\left(\frac{x}{k}\right).$$

For our purposes, there are special operations to mention. Let $A(x)/x$ increase, and put

$$RAR(x) = \begin{cases} 0, & x = 0, \\ \sup_{y<x} 1/A(1/y), & x > 0, \end{cases} \tag{2.13}$$

where $1/0 = +\infty$ and $1/\infty = 0$. Then RAR is left-continuous, non-trivial, $RAR(x)/x$ increases. The effect of this operation is to reverse the behavior at zero with that at infinity. It is easy to check that $RAR \leqslant RBR$ if and only if $B \leqslant A$ (with "for large values" and "for small values" interchanged). Since $(RAR)^{-1}(x) = 1/A^{-1}(1/x)$, we use (2.12) et seq. and the remarks after (2.8) to see that

$$\overline{RAR} \sim R\bar{A}R, \quad A \leqslant B \text{ if and only if } R\bar{A}R \leqslant R\bar{B}R. \tag{2.14}$$

We later consider functions A such that $A(x)/x^2$ is monotone. Then $\bar{A}(x)/x^2$ and $R\bar{A}R(x)/x^2$ are monotone in the opposite sense, and $RAR(x)/x^2$ in the same sense. If only $A(x) \geqslant mx^2$ near zero, we define

$$A_1(x) = x^2 \inf_{y \leqslant x} A(y)/y^2.$$

Then $A_1(x) \leqslant A(x)$, $A_1(x)/x^2$ decreases, $A_1(x)/x$ increases. A_1 is largest with respect to the first two of these properties; the second and third imply that $A_1(x)/x$ is continuous; A_1 is non-trivial.

(2.15a) Definition. If $A(x) \geqslant mx^2$ near 0, let

$$\hat{A}(x) = R\bar{A}_1 R(x).$$

$\hat{A}$ is maximal in the ordering $\leqslant$ (for large values or for all values) defined by (2.3), with respect to the properties

(i) $C(x)/x^2$ increases,

(ii) $C \leqslant R\bar{A}R$,

provided $A(x) \geqslant mx^2$ near 0. But (2.15a) depends on the behavior of $A(x)$ for small x, and for some spaces L_A this does not matter. In this case let C denote $R\bar{A}R$ and set

$$\hat{A}(x) = \begin{cases} x^2 \inf_{x \leqslant y \leqslant C^{-1}(1)} C(y)/y^2 & (C = R\bar{A}R), \\ +\infty & (x > C^{-1}(1)). \end{cases} \tag{2.15b}$$

It may be shown that the equivalence class (for small values) of $\hat{A}$ depends only on that of A (for large values), and that $\hat{A}$ is maximal with respect to properties (i) and (ii) above, in the sense of the ordering $\leqslant$ (for small values).

We may now state the main result.

(2.16) Theorem. *Let G be a locally compact abelian group, and denote its dual $\hat{G}$. Then if $A(x)/x$ increases, $L_A(G)^\wedge \subseteq L_B(\hat{G})$ if and only if $A(x) \geqslant mx^2$ near 0 (no restriction if G is compact) and*

$$L_{\hat{A}}(\hat{G}) \subseteq L_B(\hat{G}).$$

To prove this, we show that if $A(x) \geqslant mx^2$ near 0, then the Fourier transformation is of type $(A_1, R\bar{A}_1R)$ (Theorem (3.1)). Since

$$L_A(G) \subseteq L_{A_1}(G) \xrightarrow{\wedge} L_{R\bar{A}_1R}(\hat{G}) = L_{\hat{A}}(\hat{G}) \subseteq L_B(\hat{G})$$

and the inclusions are continuous, $L_A(G)^\wedge \subseteq L_B(\hat{G})$. The closed graph theorem then gives the continuity of the map when the inclusion holds. To prove necessity, we first make a number of reductions, and then derive an inequality which (in the non-compact case) forces B to be trivial unless $A(x) \geqslant mx^2$ near 0.

3. Interpolation of operations of types $(1, \infty)$ and $(2, 2)$.

The results of this section give the following theorem.

(3.1) Theorem. *Let G be a locally compact abelian group, and let Γ denote its dual group. Suppose that $A(x)/x$ increases. Then the Fourier transformation is a continuous mapping of $L_A(G)$ into $L_{\hat{A}}(\Gamma)$ ($\hat{A}$ is defined by (2.15)), if $\hat{A}$ is non-trivial.*

This follows from (3.10), applied to $A_1 \equiv R\hat{\bar{A}}R$ ($\sim A$ if $A(x)/x^2$ decreases), for $A_1 \leqslant A$ and $A_1(x)/x^2$ decreases.

In [20], H. P. Mulholland essentially stated (3.1) for the circle and the integers in order to prove special cases. Now (3.1) is a *convergence theorem*: one only asserts that $\int A(f) \leqslant 1$ implies $\int B(\varepsilon\hat{f}) \leqslant 1$ for some $\varepsilon > 0$. It is also of interest to know when, for example, $\int B(\varepsilon\hat{f}) \leqslant \int A(f)$. Mulholland considers this question and obtains a number of results; the choice of material for this section and the next is based on his paper.

Theorem (3.2) the basic one for this section, is essentially the Marcinkiewicz interpolation argument, combined with the idea of Cotlar [5] in which the availability of *strong* type (2, 2) is used to advantage. The extension to certain Orlicz spaces appears in the thesis of O'Neil [21].

(3.2) THEOREM. *Let* $A(x) = \int_0^x a(t)\,dt$, *where* $a(t)$ *is* (i) *non-negative, continuous, and strictly increasing from* 0 *to* $+\infty$ *with* x, *and* (ii) $a(t)/t$ *is non-increasing. Denote by* $\bar{a}(t)$ *the inverse function to* $a(t)$ *and set*

$$A^*(x) = \int_0^x dt/\bar{a}(1/t). \tag{3.3}$$

Then if T *is a sublinear operation simultaneously of types* $(1, \infty)$ *and* $(2, 2)$ *of norm* $\leqslant 1$ *in each case, it follows that*

$$\int_0^\infty A^*\big(\tfrac{1}{2}(Tf)^*(x)\big)\,dx \leqslant \int_0^\infty A\big(f^*(x)\big)\,dx \tag{3.4}$$

whenever the right-hand side is $\leqslant 1$.

Proof. Let b denote a non-negative function such that $a(y) \equiv b(y)/y$ is non-decreasing, and set $\alpha = a(0+)$. Suppose $\int_0^\infty A(f^*(x))\,dx \leqslant 1$. Then by Fubini's Theorem, the inequality $m(f+g, 2y) \leqslant m(f, y) + m(g, y)$ and the sublinearity of T,

$$\begin{aligned}\int_0^\infty m(Tf, 2y)\,b(y)\,dy &= \int_0^\infty ym(Tf, 2y)\int_0^y da(z)\,dy + \alpha\int_0^\infty ym(Tf, 2y)\,dy\\ &= \int_0^\infty\int_z^\infty ym(Tf, 2y)\,dy\,da(z) + K\\ &\leqslant \int_0^\infty\int_z^\infty ym(Tf^u, y)\,dy\,da(z) + \\ &\quad+\int_0^\infty\int_z^\infty ym(Tf_u, y)\,dy\,da(z) + K\\ &= I+J+K,\end{aligned}$$

where $f_u = \operatorname{sgn} f\cdot\min(u, |f|)$ and $f^u = f - f_u$. Recall that $m(f_u, y) = \chi_{(0,u)}(y)\,m(f, y)$, $m(f^u, y) = m(f, y+u)$. Here u is to be regarded as a monotone function of z, to be chosen.

To treat I, we have, since T is of type $(1, \infty)$,

$$\operatorname{ess\,sup}|Tf^u| \leqslant \|f^u\|_1,$$

so that $m(Tf^u, y) = 0$ if $y \geqslant z$ and

$$\|f^{u(z)}\|_1 = \int_{u(z)}^\infty m(f, t)\,dt \leqslant z.$$

Since $\int A(f^*) = \int_0^\infty a(t)\,m(f, t)\,dt \leqslant 1$, $\|f^u\|_1 \leqslant 1/a(u)$. Thus if we define $u(z)$ by

$$1/a\big(u(z)\big) = z \tag{3.5}$$

it follows that $I = 0$.

Since

$$\int_z^\infty ym(Tf_u, y)\,dy \leqslant \int_0^\infty ym(f_u, y)\,dy = \int_0^u ym(f, y)\,dy,$$

$$J \leqslant \int_0^\infty\int_0^{u(z)} ym(f, y)\,dy\,da(z) = \int_0^\infty y\int_0^{u^{-1}(y)} da(z)\,m(f, y)\,dy.$$

In case $\alpha = 0$, there is no term K, and

$$y\int_0^{u^{-1}(y)} da(z) = ya\big(u^{-1}(y)\big) \leqslant a(y)$$

if $a(y) \leqslant 1/y\bar{a}(1/y)$, that is, if

$$b(y) \leqslant 1/\bar{a}(1/y). \tag{3.6}$$

Hence in this case (3.4) holds (since

$$\int A^*(h) = \int_0^\infty m(h, y)\frac{dy}{\bar{a}(1/y)}).$$

Now suppose we define $b(y)$ by (3.6), with equality in place of inequality. Then

$$\alpha = \lim_{y\to 0} b(y)/y = \lim_{u\to\infty}\frac{a(u)}{u} = \inf_u\frac{a(u)}{u}.$$

In case $\alpha > 0$, we then have $A(x) \geqslant \frac{1}{2}\alpha x^2$, so that $f \in L^2$ and so since T is type $(2, 2)$ norm $\leqslant 1$,

$$K \leqslant \tfrac{1}{4}\alpha\int_0^\infty ym(f, y)\,dy < \infty.$$

Return to the estimate for J. When $\alpha > 0$, the quantity $y\int_0^{u^{-1}(y)} da(z) = y\big(a(u^{-1}(y)) - \alpha\big)$; the negative term and the finiteness of K allow some cancellation. That is

$$J+K \leqslant J+4K \leqslant \int_0^\infty a(y)\,m(f, y)\,dy,$$

so (3.4) holds. This completes the proof of the theorem.

Remarks. (i) In case $\bar{a}(1/t) = 1/\bar{a}(t)$, e.g. if A is a power, then $A^* = \bar{A}$.

(ii) Similar inequalities hold when we have only $|T(f+g)(x)| \leqslant K(|Tf(x)|+|Tg(x)|)$. The constant 1/2 is replaced then by $1/2K$ in the conclusion.

(iii) Let us write $v = \int_0^\infty A(f^*(x))\,dx$, and define $u(z)$ not by (3.5) but by

$$v/a(u(z)) = z.$$

Since $A(x)/x^2$ decreases, v is finite for $f \in L_A$. Then we continue as before to obtain

$$(3.7) \qquad v\int_0^\infty A^*\Big(\frac{1}{2v}(Tf)^*(x)\Big)\,dx \leqslant 1 \qquad \Big(v = \int_0^\infty A(f^*(x))\,dx\Big).$$

Corollary. *If in addition to the hypotheses of Theorem* (3.2) *we have* $|T(cf)| = |c|\,|Tf|$ *for each constant* c, *then*

$$(3.8) \qquad \int_0^\infty A^*(k(Tf)^*(x))\,dx < \infty \qquad \textit{for each } k > 0.$$

Proof. We have, since $a(0) = 0$, that $\lim_{x\to 0} A(x)/x = 0$. Thus in (3.7) we may replace f by rf, $r > 0$ and solve for r in $r/2v = k$ (v now depends on r). In case A^* is submultiplicative, or supermultiplicative, we modify (3.7) to obtain

(3.9) Corollary. *With v as in (3.7), if*

$$A^*(xy) \leqslant CA^*(x)A^*(y) \qquad \textit{for } x, y > 0,$$

then

$$\int_0^\infty A^*((Tf)^*(x))\,dx \leqslant C^2A^*(2)\frac{A^*(v)}{v} = \leqslant C^2A^*(2)/\bar{a}(1/v).$$

If $A^*(xy) \geqslant cA^*(x)A^*(y)$, $x, y > 0$, *then*

$$\int_0^\infty A^*((Tf)^*(x))\,dx \leqslant 1/cvA^*(1/2v) \leqslant 4\bar{a}(4v)/c.$$

In the next theorem $R\bar{A}R(x) = 1/\bar{A}(1/x)$, where $\bar{A}$ is the Young's complement of A (see (2.8)).

(3.10) Theorem. *Let $A(x)/x$ be non-decreasing and $A(x)/x^2$ non-increasing. If T is a sublinear operation of types $(1, \infty)$ and $(2, 2)$ there exists a constant $k > 0$ depending on A, T only such that*

$$\int_0^\infty R\bar{A}R(k(Tf)^*(x))\,dx \leqslant 1 \quad \textit{if} \quad \int_0^\infty A(f^*(x))\,dx \leqslant 1.$$

Proof. Let $C_0(x) = \int_0^x (A(t)/t)\,dt$, $C_1(x) = \int_0^x (C_0(t)/t)\,dt$. Then $C_0(x)/x^2$ is non-increasing and

$$(3.11) \qquad C_1(x) \leqslant A(x) \leqslant C_1(4x).$$

Case 1. $\lim_{x\to 0} A(x)/x = 0$, $\lim_{x\to\infty} A(x)/x = \infty$. Then C_1 satisfies the hypotheses of (3.2), as does kT for some constant $k > 0$. Thus by (3.2) and (3.11),

$$\int_0^\infty C_1^*(\tfrac{1}{2}k(Tf)^*(x))\,dx \leqslant \int_0^\infty A(f^*(x))\,dx$$

if the right-hand side is $\leqslant 1$. The proof for this case is now an application of the following lemma, which will be used again later (see (4.3)).

(3.12) Lemma. *If $0 \leqslant a(t)$ is non-decreasing, continuous from the left, and $\bar{a}(y) = \sup\{x: a(x) \leqslant y\}$ ($\sup\emptyset = 0$) is its left-continuous "inverse", then with $A(x) = \int_0^x a(t)\,dt$,*

$$1/\bar{A}(4/x) \leqslant \int_0^x \frac{dt}{\bar{a}(1/t)} \leqslant 1/\bar{A}(1/x),$$

where by convention $1/0 = +\infty$, $1/+\infty = 0$.

Proof. Since $\bar{A}(x) = \int_0^x \bar{a}(t)\,dt$,

$$\bar{A}(1/t) \leqslant \frac{1}{t}\bar{a}\Big(\frac{1}{t}\Big) \leqslant \bar{A}\Big(\frac{2}{t}\Big).$$

The first of these inequalities gives the second of the lemma since $1/t\bar{A}(1/t)$ is non-decreasing. The second, and integration from $x/2$ to x gives the first inequality of the lemma.

Return to the proof of (3.10).

Case 2. $A(x)/x$ is bounded away from 0 or away from ∞.

(a) If $A(x)/x \geqslant a > 0$, then $A(x)/x \to \infty$ as $x \to \infty$ unless $A(x)$ is equivalent to x, in which case there is nothing to prove. Define a new function A_0 as follows:

$$A_0(x) = \begin{cases} A(x)/A(1), & x \geqslant 1, \\ A(1)/A(1/x), & x < 1. \end{cases}$$

Note that A_0 satisfies the conditions of case 1. Now $A(x) \sim \max\big(A_0(x), x\big)$, so $R\bar{A}R(x) \sim \max(R\bar{A}_0R, x^\infty)$. Hence T maps $L_A = L_{A_0} \cap L^1$ into $L_{R\bar{A}_0R} \cap L^\infty = L_{R\bar{A}R}$, and we obtain the conclusion of (3.10) for this case.

(b) $A(x)/x \leqslant M < \infty$. We define $A_0(x)$ now in terms of $A(x)$ for $0 < x < 1$, and get that A_0 satisfies the conditions of case 1 (unless $A(x) \sim x$).

Now $A(x) \sim \min\big(A_0(x), x\big)$, so $R\bar{A}R(x) \sim \min\big(R\bar{A}_0R(x), x^\infty\big)$, and thus T maps $L_A = L_{A_0} + L^1$ into $L_{R\bar{A}_0R} + L^\infty = L_{R\bar{A}R}$, and (3.10) follows.

Remark. In (3.10) it suffices to have $A(x)/x$ non-decreasing and A equivalent to a function A_0 such that $A_0(x)/x^2$ is non-increasing. This is the case if and only if for some $K > 0$

$$A(x) \leqslant Kx^2 \inf_{y \leqslant x} A(y)/y^2.$$

(3.13) THEOREM. *Let T be sublinear, of weak type (2,2) and type $(1, \infty)$, with norms $\leqslant 1$. Let $B(x) = \int_0^x b(t)\,dt$, $0 \leqslant b$ non-decreasing. Suppose $A(x) = \int_0^x a(t)\,dt$, where $a(x)$ is continuous and strictly increasing from 0 to ∞ with x. Then if*

$$x \int_0^x \big(b(t)/t^2\big)\,dt \leqslant 1/2\bar{a}(1/x), \qquad x > 0,$$

$$\int_0^\infty B\big(\tfrac{1}{2}(Tf)^*(x)\big)\,dx \leqslant 1 \quad \textit{when} \quad \int_0^\infty A\big(f^*(x)\big)\,dx \leqslant 1.$$

Proof. As in Theorem (3.2) choose $u = u(y)$ so that $m(Tf^u, y) = 0$. Then

$$I \equiv \int_0^\infty m(Tf, 2y)\,b(y)\,dy \leqslant \int_0^\infty m(Tf_u, y)\,b(y)\,dy$$
$$\leqslant 2\int_0^\infty \int_0^{u(y)} tm(f, t)\,dt\,\frac{b(y)}{y}\,dy,$$

by weak type (2, 2) norm $\leqslant 1$.

This gives

$$I \leqslant 2\int_0^\infty t \int_0^{u^{-1}(t)} \frac{b(y)}{y^2}\,dy\,m(f, t)\,dt.$$

Now $u^{-1}(t) = 1/a(t)$ so

$$ta(t)\,\frac{1}{\bar{a}\big(1/u^{-1}(t)\big)} = a(t),$$

and thus by the hypothesis on $b(t)$, the result follows.

EXAMPLES. (i) Let $A(x) = x\log(x+1)$ (so that $L_A(0, 2\pi) = L\log^+ L$). Then

$$\bar{A}(x) \sim x(e^x - 1), \qquad R\bar{A}R(x) \sim \frac{x}{e^{1/x}-1} \sim \begin{cases} e^{-1/x} & \text{for small values,} \\ x^2 & \text{for large values.} \end{cases}$$

Since $A(x)/x^2$ decreases and $A(x)/x \to 0$ with x, we get for $f(x) \sim \sum_n c_n e^{inx}$, that for each $\varepsilon > 0$

$$\sum_n e^{-\varepsilon/|c_n|} < \infty \quad \text{whenever} \quad \int_0^{2\pi} |f(x)|\log^+|f(x)|\,dx < \infty$$

(Hardy and Littlewood [10]).

(ii) Let $A(x) = x/\log(1+1/x)$. Since RAR is A in the preceding example,

$$R\bar{A}R(x) \sim x(e^x - 1) \sim \begin{cases} x^2 & \text{for small values,} \\ e^x & \text{for large values.} \end{cases}$$

Again $A(x)/x^2$ decreases and $A(x)/x \to 0$ as $x \to 0$. Thus if

$$\sum_n |c_n|/\log(1+1/|c_n|) < \infty,$$

there exists $f \in L^1(0, 2\pi)$ such that $f(x) \sim \sum_n c_n e^{inx}$, and

$$\int_0^{2\pi} e^{k|f(x)|}\,dx < \infty$$

for each $k > 0$ (Hardy and Littlewood [9]).

(iii) The Fourier transformation is of types

$$\Big(x^2/\log(1+x),\ x^2/\log^+\frac{1}{x}\Big), \qquad \Big(x^2\log\Big(1+\frac{1}{x}\Big),\ x^2\log^+ x\Big).$$

More generally, the Fourier transformation is of type $(A, R\bar{A}R)$ whenever $A(x) = x^p\varphi(x)$, where $1 < p < 2$ and $\varphi(x)$ is "slowly varying" (see [35]), or when $p = 2$ and φ decreases.

4. Remarks on operations of types $(1, \infty)$ **and** $(2, 2)$. The sufficient condition in Theorem (3.1), that if $A(x) \geqslant mx^2$ near 0 a sublinear operation of types $(1, \infty)$ and $(2, 2)$ is of type $(A, R\bar{A}_1R)$ is, as we shall see (section 5) necessary for the boundedness of the Fourier transform. But it is not necessary for all operations of types $(1, \infty)$ and $(2, 2)$, as we shall show in this section by means of an example which has several uses, namely the operation which assigns to $f \in (L^1+L^2)(X, \mu)$ the function

$$(4.1)\qquad Uf(x) = \int_0^{1/x} f^*(t)\,dt = \frac{1}{x} f^{**}\left(\frac{1}{x}\right),$$

where $f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\,dt$. We recall (see e.g. [22]) that $(f+g)^{**} \leqslant f^{**}+ +g^{**}$, $(cf)^{**} = |c|f^{**}$.

The main theorem of this section is (4.6) which gives a characterization of operations of types $(1, \infty)$ and $(2, 2)$ in terms of U. It is immediate that U is of type $(1, \infty)$ norm 1. An application of Hardy's inequality (see [12] and [35], Ch. I) shows that U is of type $(2, 2)$ norm 2.

Example. Let $A(x) = \max(x^2, x^\infty)$. Then $A_1(x) = x^2$, so $R\bar{A}_1R(x) \sim x^2$. The following proposition shows that $UL_A \subset L^p, 1 < p < 2$; but $x^p \not\leqslant x^2$ near 0.

(4.2) Proposition. *If $f \in L^2 \cap L^\infty$, then $Uf \in L^p, 1 < p \leqslant 2$.*

Proof.

$$Uf(x) = \frac{1}{x} f^{**}\left(\frac{1}{x}\right) \leqslant \begin{cases} \frac{1}{x}\|f\|_\infty, \\ \frac{1}{x^{1/2}}\|f\|_2. \end{cases}$$

If U is of type (A, B), U is of type $(\bar{B}, \bar{A})$. The proof is based on the inequality

$$\left|\int_0^\infty f(y)g(y)\,dy\right| \leqslant \int_0^\infty f^*(x)g^*(x)\,dx \qquad (\text{see } [12]).$$

U is of type (A, B) if for some $\varepsilon > 0$

$$\int_0^\infty B\big(\varepsilon A^{-1}(x)/x\big)\,dx < \infty.$$

This follows from the inequality $f^{**}(x) \leqslant \|f\|_A A^{-1}(1/x)$.

It is convenient to express $R\bar{A}R$ in another form for what follows.

(4.3) Lemma. *The functions A^*, $R\bar{A}R$ and C, defined by* (3.3), (2.14), *and by*

$$C^{-1}(x) = xA^{-1}(1/x) \qquad \textit{respectively},$$

are equivalent to each other.

Note. $C(x)$ will have this meaning in the remainder of this paper.

Proof. That $A^* \sim R\bar{A}R$ is Lemma (3.12). To show $R\bar{A}R \sim C$ we use inequality (2.12):

$$x \leqslant A^{-1}(x)\,\bar{A}^{-1}(x) \leqslant 2x.$$

It follows that

$$\frac{1}{A^{-1}(1/x)} \leqslant \bar{C}^{-1}(x) \leqslant \frac{2}{A^{-1}(1/x)},$$

and hence that

$$\frac{x}{2} \leqslant \frac{1}{\bar{A}^{-1}(1/x)}\,\frac{1}{A^{-1}(1/x)} \leqslant (R\bar{A}R)^{-1}(x)\bar{C}^{-1}(x) \leqslant \frac{1}{\bar{A}^{-1}(1/x)}\,\frac{1}{A^{-1}(1/x)} \leqslant 2x.$$

From this the lemma follows.

(4.4) Theorem. *The operation U is of weak type (A, B) on $(0, \infty)$ with norm $\leqslant 1$ if and only if $B(u) \leqslant C(u)$ for all $u \geqslant 0$.*

Proof. Since $Uf(x) = \frac{1}{x}f^{**}(1/x)$ is a decreasing function of x,

$$m(Uf, y) = \sup\{x\colon Uf(x) > y\}.$$

Since

$$f^{**}(x) \leqslant \|f\|_A A^{-1}(1/x),$$

$Uf(x) > y$ implies $\|f\|_A C^{-1}(1/x) > y$. Thus

$$\begin{aligned} m(Uf, y) &\leqslant \sup\{x\colon \|f\|_A C^{-1}(1/x) > y\} \\ &= (\inf\{t\colon \|f\|_A C^{-1}(t) > y\})^{-1} = 1/C(y/\|f\|_A). \end{aligned}$$

Sufficiency follows since $1/C(u) \leqslant 1/B(u)$ for all u. Now set $f(x) = u\chi_{(0,u)}(x)$. Then $\int_{-\infty}^{\infty} A\big(f(x)\big)\,dx = uA(v) \leqslant 1$ if $v = A^{-1}(1/u)$.

Thus

$$\frac{1}{x}f^{**}\left(\frac{1}{x}\right) = Uf(x) = \begin{cases} v/x, & x > 1/u, \\ vu, & x \leqslant 1/u, \end{cases}$$

and hence

$$m(Uf, y) = \begin{cases} v/y, & y < uv, \\ 0, & \text{otherwise}. \end{cases}$$

Set now $y=\theta uv$, $0<\theta<1$; from $m(Uf,y)\leqslant 1/B(y)$ it then follows that $B(y)\leqslant y/v$ or $B(\theta uv)\leqslant \theta u$, then $\theta uv\leqslant B^{-1}(\theta u)$, or $\theta C^{-1}(u)\leqslant B^{-1}(\theta u)$ and since $1/\theta B^{-1}(\theta u)\to B^{-1}(u)$ as $\theta\uparrow 1$,

$$B(u)\leqslant C(u)\quad \text{for all } u\geqslant 0.$$

(4.5) THEOREM. *If $f\to Uf$ is of type (A,B) on $(0,\infty)$, norm $1/\varepsilon$, then*

$$B(x)+x\int_0^x \frac{B(t)}{t^2}\,dt\leqslant C\left(\frac{x}{\varepsilon}\right).$$

Proof. As above, set $f(x)=v\chi_{(0,u)}(x)$. Then $\|f\|_A\leqslant 1$ and

$$f^{**}(x)=\begin{cases} v, & 0<x<u,\\ \dfrac{1}{x}uv, & x\geqslant u.\end{cases}$$

Hence

$$1\geqslant\int_0^\infty B\big(\varepsilon xf^{**}(x)\big)\frac{dx}{x^2}=\int_0^u B(\varepsilon vx)\frac{dx}{x^2}+\int_u^\infty B(\varepsilon uv)\frac{dx}{x^2}$$

$$=\varepsilon v\int_0^{\varepsilon uv} B(t)\frac{dt}{t^2}+\frac{B(\varepsilon uv)}{u}.$$

After multiplying through by u and replacing uv by $C^{-1}(u)$, we replace u by $C(x)$ and use the inequality (see (2.11)) $C^{-1}\big(C(x)\big)\geqslant x$.

(4.6) THEOREM. *Let T be a sublinear operation. T is of type $(1,\infty)$ and $(2,2)$ if and only if for some constant K and each $f\in L^1+L^2$*

$$\int_0^x (Tf)^*(t)^2\,dt\leqslant K\int_0^x\left(\frac{1}{t}f^{**}\left(\frac{1}{t}\right)\right)^2dt,\qquad x>0,$$

where $f^{**}(x)=\dfrac{1}{x}\int_0^x f^*(t)\,dt.$

Proof. Let the condition hold. Then for $f\in L^1$,

$$x(Tf)^*(x)^2\leqslant K\int_0^x\left(\frac{1}{t}f^{**}\left(\frac{1}{t}\right)\right)^2dt$$

$$=K\int_0^x\left(\int_0^{1/t}f^*(s)\,ds\right)^2dt\leqslant Kx\|f\|_1^2.$$

For $f\in L^2$,

$$\int_0^\infty (Tf)^*(x)^2\,dx\leqslant K\int_0^\infty\left(\frac{1}{t}f^{**}\left(\frac{1}{t}\right)\right)^2dt=K\int_0^\infty f^{**}(x)^2\,dx,$$

which by Hardy's inequality is dominated by $4K\|f\|_2^2$. Thus T is of types $(1,\infty)$ and $(2,2)$. Assume now that T is of types $(1,\infty)$, $(2,2)$, norms $\leqslant 1$. Then with $f=f^u+f_u$, $f_u=\operatorname{sgn} f\min(u,|f|)$,

$$m(Tf,y)\leqslant m(Tf_u,y-\|f^u\|_1)$$

if $y>\|f^u\|_1$. It follows that for each u

$$(Tf)^*(x)\leqslant (Tf_u)^*(x)+\|f^u\|_1,\qquad x>0.$$

Thus

$$I\equiv\left(\int_0^x (Tf)^*(t)^2\,dt\right)^{1/2}\leqslant\left(\int_0^\infty f_u^*(t)^2\,d\right)^{1/2}+x^{1/2}\|f^u\|_1.$$

Set $u=f^*\left(\dfrac{1}{x}\right)$. Then by computations, using the relation $xf^{**}(x)=xf^*(x)+\int_{f^*(x)}^\infty m(f,y)\,dy$ (see [21]),

$$I\leqslant\left(\int_0^x\left(\frac{1}{t}f^*\left(\frac{1}{t}\right)\right)^2dt\right)^{1/2}+x^{-1/2}f^{**}\left(\frac{1}{x}\right).$$

Since $\dfrac{1}{x}f^{**}\left(\dfrac{1}{x}\right)^2\leqslant\int_0^x\left(\dfrac{1}{t}f^{**}\left(\dfrac{1}{t}\right)\right)^2dt$, we get

$$\int_0^x (Tf)^*(t)^2\,dt\leqslant 4\int_0^x\left(\frac{1}{t}f^{**}\left(\frac{1}{t}\right)\right)^2dt.$$

(4.7) THEOREM. *Let $B(x)/x^i$ be non-negative and non-decreasing, $i=1,2$. Let T be of types $(1,\infty)$ and $(2,2)$. There exists a constant k depending on T such that for $f\in L^1+L^2$,*

$$\int_0^x B\big((Tf)^*(t)\big)\,dt\leqslant\int_0^x B\left(k\frac{1}{t}f^{**}\left(\frac{1}{t}\right)\right)dt,\qquad x>0.$$

Proof. This is a consequence of (4.6) and Theorem 249 in Hardy-Littlewood-Polya [12] (see also [11]).

In order that

$$\int_a^b\varphi\big(g(x)\big)\,dx\leqslant\int_a^b\varphi\big(f(x)\big)\,dx$$

should be true for every convex and continuous φ it is necessary and sufficient that

$$\int_a^b (g(x)-y)^+ dx \leqslant \int_a^b (f(x)-y)^+ dx$$

for all y (here a^+ means $\max(a, 0)$).

As an application of (4.7) we take $B(x) = x^q$, $q \geqslant 2$, and apply Hardy's inequality, to deduce the following theorem of Hardy-Littlewood-Paley type:

$$\Big(\int_0^\infty (Tf)^*(x)^q dx\Big)^{1/q} \leqslant C_q \Big(\int_0^\infty f^*(x)^q x^{q-2} dx\Big)^{1/q},$$

where $C_q \leqslant kq$. In the terminology of Lorentz spaces $L_{p,q}$ (see [22]), T carries $L_{q',q}$ into $L_{q,q}$.

To complete the pair, we note that

$$f \to x(Tf)^{**}(x),$$

with measure dx/x^2 is sublinear of weak type $(1, 1)$ and type $(2, 2)$, whence by the Marcinkiewicz interpolation theorem, for $1 < p \leqslant 2$,

$$\Big(\int_0^\infty (Tf)^*(x)^p x^{p-2} dx\Big)^{1/p} \leqslant C_p \Big(\int_0^\infty f^*(x)^p dx\Big)^{1/p}.$$

That is, T carries $L_{p,p}$ into $L_{p',p}$.

Results of this nature might also be obtained by direct interpolation methods, as it was first observed in [34]. The following theorem can be shown by means of the general idea given in (3.2) and [21].

(4.8) THEOREM. *Let T be a sublinear operation of weak type* $(1, 1)$ *and of type* (s, s) $1 < s < \infty$. *If A is any Young's function such that*

(i) $\displaystyle\int_0^x \frac{A(t)}{t^2} dt = O\Big(\frac{A(x)}{x}\Big)$,

(ii) $\dfrac{A(x)}{x^s}$ *is a non-increasing function of* x,

T is of type (A, A).

For $s = 2$ this result is readily seen to generalize the ones stated in [34], in particular we note that Theorem (4.8) applies to $A(x) = x^2\varphi(x)$, $\varphi(x)$ slowly varying and non-increasing. Further applications are obtained by similar arguments to the ones given in [35], Examples XII 1, 2, 3, for general unbounded orthonormal systems.

5. Necessary conditions. Let G be a locally compact abelian group with dual Γ. In this section we show that if for some $\varepsilon > 0$, $f \in L^1(G) + L^2(G)$ and $\int_G A(|f(x)|) dx \leqslant 1$ together imply $\int_\Gamma B(\varepsilon|\hat f(x)|) dx \leqslant 1$, then $L_B(\Gamma) \supseteq L_{\hat A}(\Gamma)$, where $\hat A$ is the largest function D such that $D \leqslant R\bar A R$ in the appropriate sense and $D(x)/x^2$ increases (see (2.15)). We first prove this assertion for the special cases of the real line and the integers by means of lacunarity arguments; a duality argument gives the result for $G = T$ and for any compact group G once the result is known for its discrete dual Γ. The structure theorem leads to a reduction to the case of discrete groups G. There again examples based on the lacunarity argument applied to Z are used.

(a) The case $G = R$. The inequality $\int_\Gamma B(\varepsilon|\hat f(x)|) dx \leqslant 1$ is readily seen to imply

$$m(\hat f, y) \leqslant 1/B(\varepsilon y), \tag{5.1}$$

for $f = v\chi_E$, where E is a measurable set of finite measure u and $1 \geqslant \int_{-\infty}^{\infty} A(|f(x)|) dx = uA(v)$. This will hold if $v = A^{-1}(1/u)$ and in what follows v is always so chosen. We now apply the following lemma, to be proved shortly.

(5.2) LEMMA. *There exist constants $K > 0$, $\delta > 0$ such that if $u > 0$ and $t \geqslant 1$ are given, a set $E = E_{u,t}$ exists such that E has measure u and*

$$m\Big(\hat\chi_E, \frac{u}{Kt}\Big) \geqslant \delta t^2/u. \tag{5.3}$$

Let $f = v\chi_E$, combine (5.1) and (5.3) to obtain:

$$\frac{1}{B(\varepsilon uv/Kt)} \geqslant m\Big(v\hat\chi_E, \frac{uv}{Kt}\Big) = m\Big(\hat\chi_E, \frac{u}{Kt}\Big) \geqslant \frac{\delta t^2}{u}.$$

In particular, $B(\varepsilon uv/Kt)$ is finite (though this may not be true for all values of the argument), and so

$$B\Big(\frac{\varepsilon uv}{Kt}\Big) \leqslant \frac{u}{\delta t^2}$$

from which we deduce (by (2.11))

$$\frac{\varepsilon uv}{Kt} \leqslant B^{-1}\Big(\frac{u}{\delta t^2}\Big).$$

Recall that $uv = uA^{-1}(1/u) \equiv C^{-1}(u)$ (see (4.3)). The last inequality then assumes the form

$$\frac{\varepsilon}{Kt} C^{-1}(u) \leqslant B^{-1}(u/\delta t^2), \quad u > 0, t \geqslant 1. \tag{5.4}$$

(Note. If we let $u = t^{1/2}$, for $u \geqslant 1$, we get $A^{-1}(1/u) \leqslant \text{const}\cdot(1/u)^{1/2}$ so that $A(x) \geqslant mx^2$, for x near 0.) The taking of inverses, leads to

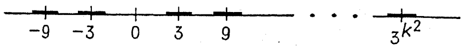
$$C\left(\frac{Ktx}{\varepsilon}\right) \geqslant \delta t^2 B(x), \qquad x > 0,\ t \geqslant 1,$$

thus

$$B(x) \leqslant \frac{1}{\delta}(Kx)^2 \inf_{y \geqslant Kx} \frac{C(y)}{y^2} \leqslant \hat{A}(kx)$$

for some $k > 0$, which implies $L_B(R) \supseteq L_{\hat{A}}(R)$ as desired ($\hat{A}$ is defined by (2.15)).

We turn now to the proof of the Lemma (5.2). By a change of variables one finds that it suffices to treat the case $u = 1$. Now for a particular value of t, let $k = [t]$. Let E_k be the set symmetric about the origin consisting of $2k^2$ intervals of length $1/2k^2$, centered at the points $\pm 3, \pm 9, \ldots, \pm 3^{k^2}$.



Then E_k has measure 1 and

$$|\hat{\chi}_{E_k}(x)| = \left|\frac{\sin x/4k^2}{x/4k^2}\right| \left|\frac{1}{2k^2}\sum_{n=1}^{k^2}\cos 3^n x\right|.$$

The first factor is greater than 1/2 in $(-k^2\pi, k^2\pi)$. The second factor is a lacunary trigonometric polynomial which therefore exceeds a fixed positive multiple of the l^2-norm of its coefficient sequence in a set of fixed positive measure in $(-\pi, \pi)$. (This is Theorem (8.25) in Chapter V of [35]. If the intervals of E_k were centered at $\pm 3^{n_0+1}, \ldots, \pm 3^{n_0+k^2}$ this would also follow from our extension — (Lemma (5.12) to discrete groups of Example 27 in Chapter V of [35]). Therefore, $|\hat{\chi}_{E_n}(x)| > \text{const}\cdot k^{-1}$ in a set of measure at least a constant times k^2 (by periodicity). Note that the constants do not depend on E_u. Then since $t/2 \leqslant k \leqslant t$ (5.3) follows (with $u = 1$).

(b) The case $G = Z$. As before we have (5.1) for $f = v\chi_E$, where χ_E is the characteristic function of a finite set (with u elements) and $v = A^{-1}(1/u)$. Since the dual group ($\Gamma = T$) is compact, we only need to show (5.4) for $u \geqslant 1, t \geqslant 1$. The lemma corresponding to (5.2) is

(5.5) Lemma. *Given $n > 0$, $k > 0$ there is a set E with $k^2 n$ elements such that the trigonometric polynomial $\hat{\chi}_E(x) = \sum_{n \in E} e^{inx}$ satisfies*

$$m\left(\hat{\chi}_E, \frac{kn}{K}\right) \geqslant \frac{\delta}{n}, \tag{5.6}$$

where K, $\delta > 0$ are independent of k and n.

Proof. Let $E_n = 0, \ldots, n-1$ and choose n_0 so large that two distinct translates $E_n + 3^{n_0+m}$, $m = 1, 2, \ldots, k^2$, of E are disjoint (later on n_0 will be chosen perhaps larger) with $E = \bigcup_{m=1}^{k^2}(E_n + 3^{n_0+m})$ we set

$$\hat{\chi}_E(x) = \sum_{\nu \in E} e^{i\nu x} = \sum_{\nu=0}^{n-1} e^{i\nu x}\cdot \sum_{m=1}^{k^2} e^{(3^{n_0}+m)ix},$$

so

$$|\hat{\chi}_E(x)| \geqslant \frac{|\sin\frac{1}{2}nx|}{|\sin\frac{1}{2}x|}\left|\sum_{m=1}^{k^2}\cos(3^{n_0+m})x\right|.$$

The first factor exceeds $\left|\frac{2n}{\pi 2}x\Big/\frac{x}{2}\right| = \frac{2n}{\pi}$ if $\left|\frac{n}{2}x\right| \leqslant \pi/2$, that is, in a set of measure π/n. The second factor is a lacunary polynomial. Therefore by Example 27, Chapter V of [35] (or by (5.12)), it exceeds a certain fixed multiple of k (which is the square root of the sum of the squares of its coefficients) in a certain fixed proportion of any fixed set, provided n_0 is large enough. That is, (5.6) holds for some such set E, and the lemma is proved.

To complete the proof of necesity in this case, we combine (5.6) and (5.1) with $u = k^2 n$ and $v = A^{-1}(1/u) = A^{-1}(1/k^2 n)$, to get

$$B\left(\frac{\varepsilon kn}{K}A^{-1}(1/k^2 n)\right) \leqslant \frac{n}{\delta}$$

which leads to

$$\frac{\varepsilon}{Kk}C^{-1}(k^2 n) \leqslant B^{-1}(n/\delta),$$

and then, with $k \leqslant t \leqslant 2k, n \leqslant x < 2n$ determined by given $x \geqslant 1$, $t \geqslant 1$, we get, since $C^{-1}(x)/x$ decreases,

$$\frac{\varepsilon}{2Kt}C^{-1}\left(\frac{t^2 x}{2}\right) \leqslant B^{-1}\left(\frac{x}{\delta}\right), \qquad t \geqslant 1, x \geqslant 1 \tag{5.7}$$

(compare with (5.4)).

Note. We need only consider a sequence

$$\{n_j\} \text{ of values of } n \text{ such that } n_j < n_{j+1} \leqslant q n_j,$$

where $q > 1$. The sequence $\{p^j\}$ is used in connection with (5.19). Thus

$$\delta B(x) \leqslant 2C\left(\frac{2Ktx}{\varepsilon}\right)/t^2, \qquad t \geqslant 1, x \geqslant B^{-1}(1/\delta);$$

it follows that $L_B(T) \supseteq L_{\hat{A}}(T) \equiv L_{\hat{A}}(0, 2\pi)$.

(5.8) Remark. In (5.7) take $x = 1$ to get

$$A^{-1}\left(\frac{2}{t^2}\right) \leqslant \text{const}\,\frac{1}{t}, \quad t \geqslant 1$$

which again gives $A(x) \geqslant mx^2$ near 0. This condition, which guarantees that $\hat{A}$ be non-trivial, means that $L_A \subseteq L_1 + L_2$ since $f = f^u + f_u$, where $f_u = (\operatorname{sgn} f)\min(u, |f|)$ and thus for some small $u > 0$

$$A(|f|) \geqslant A(|f_u|) + A(|f^u|) \geqslant m|f_u|^2 + |f^u|\frac{A(u)}{u}.$$

In particular the Fourier transform of each $f \in L_A$ is well-defined in terms of the Fourier transformation in L^1 and L^2.

c) The case $G = T$; a duality argument for compact groups. Throughout this section (g, γ) denotes the value of the character $\gamma \in \Gamma$ at $g \in G$ (Γ is the dual group $\hat{G}$ of G). The following lemma will be used later also.

(5.9) Lemma. *Let G be a compact abelian group, with (discrete) dual Γ. If for some $\varepsilon > 0$*

$$\int_G A(|f(x)|)\,dx \leqslant 1 \text{ implies } \sum_{\gamma \in \Gamma} B(\varepsilon |c_\gamma|) \leqslant 1,$$

where $c_\gamma = \int_G f(x)\overline{(x, \gamma)}\,dx = \hat{f}(\gamma)$, there exists $\delta > 0$ such that for every sequence $c = \{c_\gamma\}_{\gamma \in \Gamma}$ with only a finite number of non-zero terms

$$\sum_{\gamma \in \Gamma} \bar{B}(|c_\gamma|) \leqslant 1 \quad \text{implies} \quad \int_G \bar{A}\left(\delta \left|\sum_\gamma c_\gamma \overline{(x, \gamma)}\right|\right) dx \leqslant 1.$$

Hence (with $\hat{c}(x) \equiv \sum_\gamma c_\gamma \overline{(x, \gamma)}$) for $y > 0$

$$m(\hat{c}, y) \leqslant \frac{1}{\bar{A}(\delta y)} \quad \text{if} \quad \Sigma \bar{B}(|c_\gamma|) \leqslant 1. \tag{5.10}$$

Proof. Using (2.10), with $\|\cdot\|_A$ denoting the norm in $L_A(G)$ and $\|\cdot\|_{l_A}$ that in $L_A(\Gamma) \equiv l_A$, we have

$$\|\hat{c}\|_{\bar{A}} \leqslant \sup_{\|g\|_A \leqslant 1} \left| \int_G \hat{c}(x) g(x)\,dx \right| = \sup_{\|g\|_A \leqslant 1} \left| \sum_\gamma c_\gamma \hat{g}(\gamma) \right|$$
$$\leqslant \sup_{\|g\|_A \leqslant 1} 2\|c\|_{l_{\bar{B}}} \|\hat{g}\|_{l_B} \leqslant 2K\|c\|_{l_{\bar{B}}},$$

where K denotes the norm of the Fourier transformation. This proves the lemma.

Now when $G = T$, $\Gamma = Z$ we apply (5.5) and the derivation of (5.7) to obtain

$$\varepsilon t x \bar{B}^{-1}\left(\frac{1}{t^2 x}\right) = \frac{\varepsilon}{t} t^2 x \bar{B}^{-1}\left(\frac{1}{t^2 x}\right) \leqslant \bar{A}^{-1}\left(\frac{x}{\delta}\right), \quad t \geqslant 1, x \geqslant 1.$$

We let $t = 1/s$, $x = 1/y$, multiply both sides of this inequality by $B^{-1}(s^2 y) C^{-1}(y)$ and use (2.12), (4.3) to get for some constant $K > 0$

$$C^{-1}(y) \leqslant \frac{K}{s} B^{-1}(s^2 y), \quad s \leqslant 1, y \leqslant 1,$$

so that

$$\frac{1}{s^2} B\left(\frac{sy}{K}\right) \leqslant C(y), \quad y \leqslant C^{-1}(1), s \leqslant 1. \tag{5.11}$$

It follows from the definition of $\hat{A}$ (see (2.15b)) and its maximality with respect to the properties $D \leqslant C$, $D(x)/x^2$ increasing, that $B \leqslant \hat{A}$. Hence Theorem (2.16) holds for $G = T$.

(d) A property of certain trigonometric polynomials. In this subsection we show that for certain sets S, called "*dispersed*" sets (Definition (5.13)), we have the following lemma, based on a result of Zygmund (Example 27 in Chapter V of [35]) concerning lacunary series.

(5.12) Lemma. *Let S be a "dispersed" set in the discrete group G, and let $E \subseteq \Gamma$ (the dual group of G) have positive measure. Then there exists a finite set $F \subseteq S$ such that*

$$m(f\chi_E, \lambda C) \geqslant \mu m(E),$$

whenever $f(x) = \sum c_s \overline{(s, x)}$ is a trigonometric polynomial with coefficients $c_s = 0$ unless $s \in S \sim F$; $C = (\sum |c_s|^2)^{1/2}$; and λ, μ are positive constants depending on S but not on E or f.

Proof. By Lemmas (5.14) and (5.16), proved later, there exists a finite set $F \subseteq S$ such that for f as in the statement of the lemma,

$$\tfrac{1}{2} C^2 m(E) \leqslant \int_E |f(x)|^2\,dx \leqslant 2C^2 m(E),$$

$$\int_E |f(x)|^4\,dx \leqslant KC^4 m(E),$$

where K depends on S. By Hölder's inequality and the second of the preceding inequalities,

$$\int_E |f(x)|^2\,dx \leqslant \left(\int_E |f(x)|\,dx\right)^{2/3} \left(KC^4 m(E)\right)^{1/3}.$$

It follows that for a certain $\lambda > 0$

$$\int_E |f(x)|\,dx \geqslant 2\lambda C m(E).$$

The conclusion now follows with the aid of an inequality of Paley and Zygmund [35], Chapter V, (8,26):

If $g(x) \geqslant 0$ is measurable on E,

$$\int_E g(x)\,dx \geqslant Am(E) > 0, \quad \text{and} \quad \int_E g^2(x)\,dx \leqslant B^2 m(E),$$

then

$$m(g, \tfrac{1}{2}A) \geqslant \tfrac{1}{4}(A/B)^2 m(E).$$

We turn now to the definition of "dispersed" sets and the two lemmas alluded to before.

(5.13) Definition. An infinite set $S \subseteq G$ is "*dispersed*" if

(i) the number of solutions of

$$0 \neq g = s_1 - s_2, \quad s_1, s_2 \in S,$$

is bounded independent of $g \neq 0$;

(ii) The number of solutions of

$$0 \neq g = s - t + u - v$$

is bounded independent of $g \neq 0$, when s, t, u, v satisfy:

(1) $s-t \neq 0$, (2) $s+u \neq 0$, (3) $s-v \neq 0$, (4) $-t+u \neq 0$, (5) $-t-v \neq 0$, (6) $u-v \neq 0$;

(iii) $S = -S$ or $S \cap (-S) = \emptyset$.

We let Q denote the set of quadruples $(s, t, u, v) \in S^4$ which satisfy $0 \neq s-t+u-v$ and (1)-(6). If $F \subseteq S$ is a finite set, we let Q_F denote the set defined as before with $S \sim F$ in place of S (see (5.17) for examples).

In the following lemma only condition (i) of (5.13) is used; the argument is due to Zygmund (see [35], V. 6.5, and the remarks on Chapter V).

(5.14) Lemma. *Let S, E be as in* (5.12). *Let $v > 1$. There exists a finite set $F \subseteq S$ such that*

(5.15) $$v^{-1} C^2 m(E) \leqslant \int_E |f(x)|^2\,dx \leqslant v C^2 m(E),$$

when f and C are as in (5.12).

Proof. Expand $|f(x)|^2 = f\bar{f}$. Then

$$\int_E |f(x)|^2\,dx = \sum |c_s|^2 m(E) + \sum_{s\neq t} c_s \bar{c}_t \int_E \overline{(s-t, x)}\,dx = C^2 m(E) + R.$$

Denote the integrals in R by e_{s-t} $(= \hat{\chi}_E(s-t))$. By the Schwarz inequality

$$|R| \leqslant \Big(\sum_{s\neq t} |c_s|^2 |c_t|^2\Big)^{1/2} \Big(\sum_{s\neq t} |e_{s-t}|^2\Big)^{1/2} \leqslant C^2 \Big(\sum N |e_u|^2\Big)^{1/2},$$

where N is a bound on the number of ways $u \neq 0$ can be expressed as a difference of elements of S. The sum is taken over the non-zero elements of $T-T$, where $T = S \sim F$, with F a finite set to be chosen as follows.

Arrange in decreasing numerical order the numbers $|e_u|^2$. This induces an ordering (not unique) on the elements of $u \in G$ with $|e_u| \neq 0$. Let K_n be the set consisting of the first n such u. For a first r,

$$\Big(N \sum_{g \notin K_r} |e_g|^2\Big)^{1/2} \leqslant \Big(1 - \frac{1}{v}\Big) m(E).$$

Let K denote the first K_n with $n \geqslant r$ such that $|e_g| < |e_h|$ if $g \notin K_n$, $h \in K_n$ (that is, we place in K the elements u of K_r and also those h such that $|e_h|$ is equal to the least value $|e_u|$, $u \in K_r$). Now for each $0 \neq g$ in K there are only a finite number of elements of the "dispersed" set S which can be used to express g as $s_1 - s_2$ with s_1, s_2 in S. Let F denote the set of all $s \in S$ which enter into some representation of some $0 \neq g \in K$. Then

$$\Big(\sum_{0\neq u \in T-T} N |e_u|^2\Big)^{1/2} \leqslant (1 - 1/v) m(E),$$

and this gives the left-hand inequality (5.15). The other follows, using that $(1-1/v) < v-1$.

(5.16) Lemma. *Let S, E be as in* (5.12). *There exists a constant K depending only on S, and a finite set $F \subseteq S$ such that*

$$\int_E |f(x)|^4\,dx \leqslant K C^4 m(E)$$

when f and C are as in (5.12).

Proof. Notation is as in (5.14). Now $|f(x)|^2 = C^2 + \sum_{u\neq 0} d_u (u, -x)$, where $d_u = \sum_t c_{t+u} \bar{c}_t$. Then by the Parseval formula

$$\int_\Gamma |f(x)|^4\,dx = C^4 + \sum_{u\neq 0} |d_u|^2.$$

But

$$\sum_{u\neq 0} |d_u|^2 = \sum_{u\neq 0} \Big|\sum_t c_{u+t}\bar{c}_t\Big|^2 \leqslant \sum_{u\neq 0} N \sum_t |c_{u+t}|^2 |c_t|^2 \leqslant N C^4,$$

where N is a bound suitable for (5.13) (i). Thus with

$$|f(x)|^4 = \sum_{s,t,u,v} c_s \bar{c}_t c_u \bar{c}_v (s-t+u-v, -x),$$

we get

$$(N+1)C^4 \geqslant \int_{\Gamma} |f(x)|^4 dx = \sum_{s-t+u-v=0} c_s \bar{c}_t c_u \bar{c}_v.$$

Hence

$$\int_E |f(x)|^4 dx \leqslant (N+1) C^4 m(E) + \sum_{s-t+u-v\neq 0} c_s \bar{c}_t c_u \bar{c}_v (e_{s-t+u-v}).$$

Now by the Schwarz inequality

$$\Big|\sum_{Q} c_s \bar{c}_t c_u \bar{c}_v e_{s-t+u-v}\Big| \leqslant \Big(\sum_{Q_F} |c_s \bar{c}_t c_u \bar{c}_v|^2\Big)^{1/2} \Big(\sum_{Q_F} |e_{s-t+u-v}|^2\Big)^{1/2}$$
$$\leqslant \Big(\sum_{Q_F} |c_s \bar{c}_t c_u \bar{c}_v|^2\Big)^{1/2} \Big(M \sum |e_w|^2\Big)^{1/2}$$
$$\leqslant C^4 \Big(M \sum_{P_F} |e_w|^2\Big)^{1/2},$$

where Q_F is defined just after (5.13) (iii);

$$P_F = \{w \neq 0\colon\ w = s-t+u-v,\ (s, t, u, v) \in Q_F\} \cup$$
$$\cup \{w \neq 0\colon\ w = s-t,\ s, t \in S \sim F\};$$

M is a bound suitable for (5.13) (ii); and F is a finite set, to be chosen. The first requirement on F is that $\big(M \sum_{P_F} |e_w|^2\big)^{1/2} \leqslant m(E)$. A method of choosing such F is given in the proof (5.14).

It remains to consider the case: $s-t+u-v \neq 0$ and $(s, t, u, v) \notin Q$. Let R denote the set of such quadruples (here s, t, u, v each belong to $S \sim F$). Let R_k denote the set of (s, t, u, v) in R such that condition (k) of (5.13) (ii) fails, $k = 1, \ldots, 6$.

Then $R = R_1 \cup \ldots \cup R_6$ and

$$\Big|\sum_{R} c_s \bar{c}_t c_u \bar{c}_v e_{s-t+u-v}\Big| \leqslant \sum_{i=1}^{6} \sum_{R_i} |c_s \bar{c}_t c_u \bar{c}_v|\, |e_{s-t+u-v}|.$$

In R_1 (and similarly in R_3, R_4, R_6) $s = t$ so condition (6) of (5.13) (ii) must hold; otherwise $s-t+u-v = 0$. Thus

$$\sum_{R_1} \leqslant \Big(\sum_{s} |c_s|^2\Big) \Big(\sum_{u-v\neq 0} |c_u|\, |c_v|\, |e_{u-v}|\Big) \leqslant C^4 \Big(N \sum_{P_F} |e_w|^2\Big)^{1/2} \leqslant C^4 m(E)$$

for F (possibly) further restricted. The same argument gives estimates for R_3, R_4, R_6.

Now if $S \cap (-S) = \emptyset$, $R_2 = R_5 = \emptyset$, and the lemma holds in this case. Otherwise, by (5.13) (iii), $S = -S$, and

$$\sum_{R_2} = \sum_{s,t,v} |c_s|\, |c_{-s}|\, |c_t|\, |c_v|\, |e_{-t-v}|$$
$$\leqslant \Big(\sum_{s} |c_s|\, |c_{-s}|\Big) \Big(\sum_{t,v} |c_t|^2 |c_v|^2\Big)^{1/2} \Big(N \sum_{P_F} |e_w|^2\Big)^{1/2},$$

if F is enlarged so that $F = -F$. Thus $\sum_{R_2} \leqslant C^4 m(E)$. Lastly, $\sum_{R_5}$ is handled in a similar way. Hence

$$\int_E |f(x)|^4 dx \leqslant (N+8) C^4 m(E).$$

(5.17) EXAMPLES. $\{\pm 3^k\colon k \geqslant 0\}$. In $\sum_{n=1}^{\infty} Z_p$, p a prime, or in $\sum_{n=1}^{\infty} Z_{p_n}$, $2 < p_1 < p_2 < \ldots$ a strictly increasing sequence of primes, let S denote the set of basis elements $\delta_k = (\delta_{k1}, \ldots, \delta_{kn}, \ldots)$, $k = 1, 2, \ldots$

(e) The two remaining special cases. In this subsection we construct examples corresponding to (5.5) for discrete groups $\sum_{n=1}^{\infty} Z_p$, p a fixed prime, and $\sum_{m=1}^{\infty} Z_{p_m}$, where $p_1 < p_2 < \ldots$ is a strictly increasing sequence of primes. In subsection (f) the general case will be reduced to these and the preceding examples. In each case we let S be the collection $\{s_1, s_2, \ldots\}$ of sequences with a "1" in the n-th position and zeroes elsewhere. It is clear that S is a "dispersed" set.

In the case of a fixed prime consider the set F_n of members of G with support in $1 \leqslant m \leqslant n$ and the basis elements $s_{n_0+1}, \ldots, s_{n_0+k^2}$, where $n_0 > n$. Let

$$F = \{s + s_{n_0+j}\colon s \in F_n,\ 1 \leqslant j \leqslant k^2\}.$$

F has $k^2 p^n$ elements, and

$$\hat{\chi}_F(x) = \Big(\sum_{F_n} \overline{(s, x)}\Big) \Big(\sum_{j=1}^{k^2} \overline{(s_{n_0+j}, x)}\Big).$$

The first factor equals p^n in a set E of measure p^{-n}. The second factor, for n_0 sufficiently large, is by Lemma (5.12) greater than λC $\big(C = (\sum_{j=1}^{k^2} 1^2)^{1/2} = k\big)$ in a set of measure at least $\mu m(E) = \mu p^{-n}$. This gives

(5.18) LEMMA. *Let $G = \sum_{m=1}^{\infty} Z_p$, p a prime. There exist positive numbers λ, μ such that for any $n > 0$, $k > 0$ there is a set F with $k^2 p^n$ elements such that*

$$m(\hat{\chi}_F, \lambda k p^n) \geqslant \mu p^{-n}.$$

(5.19) Remark. The argument following (5.5) now applies, so Theorem (2.16) holds in this case.

(5.20) Lemma. *If* $G = \sum_{m=1}^{\infty} Z_{p_m}$, $p_1 < p_2 < \ldots$ *a strictly increasing sequence of primes, then given* $k > 0$, $n > 0$ *there exists a set* F *with* $k^2 n$ *elements such that*

$$m(\hat{\chi}_F, \lambda kn) \geqslant \mu/n,$$

where $\lambda > 0$, $\mu > 0$ *are independent of* k, n.

Proof. Let p be the first $p_m > 2n$, let Λ denote the annihilator of Z_p in Γ (the set of $x \in \Gamma$ such that $(g, x) = 1$ for all $g \in Z_p$). With g denoting a generator of Z_p consider

$$P_n(x) = \sum_{m=1}^{n} \overline{(mg, x)} = \sum_{m=1}^{n} \overline{(g, x)^m} = \begin{cases} \overline{(g, x)}\, \dfrac{\overline{(g, x)^n} - 1}{\overline{(g, x)} - 1}, & x \notin \Lambda, \\ n, & x \in \Lambda. \end{cases}$$

Number the cosets of Λ in Γ as $\Lambda_1, \Lambda_2, \ldots, \Lambda_p$ so that $\overline{(g, x)} = e^{2\pi i r/p}$ in Λ_r. Then

$$|P_n(x)| = \left| \frac{\sin \pi nr/p}{\sin \pi r/p} \right| \geqslant \frac{2}{\pi} n$$

if $|\pi nr/p| \leqslant \pi/2$. We can put $p - r$ in place of r so

$$|P_n(x)| \geqslant \frac{2n}{\pi} \quad \text{in } 1 + 2\left[\frac{p}{2n}\right] \text{ of the } \Lambda_r,$$

that is, in a set of measure $\geqslant 1/2n$ since each Λ_r has measure $1/p$.

Finally we complete the construction as in Lemma (5.18) using base elements $s_{n_0+1}, \ldots, s_{n_0+k^2}$ and Lemma (5.12).

(f) Reduction to previously considered cases. Let G be a locally compact abelian group. We begin with a lemma. For background see the first two chapters of Rudin's book [27].

(5.21) Lemma. *Let* H *be an open subgroup of* G. *If* $L_A(G)\hat{} \subseteq L_B(\hat{G})$, *then* $L_A(H)\hat{} \subseteq L_B(\hat{H})$.

Proof. Since H is open $L_A(H) \subseteq L_A(G)$ in a natural way. If f in $L_A(G)$ belongs to $L_A(H)$ (i.e., is $\equiv 0$ off H), then $\hat{f}$ is constant on the cosets of Λ, the annihilator of H (Λ is compact since $\Lambda = (G/H)\hat{}$ and G/H is discrete). Thus $\hat{f}$ determines a function h on $\Gamma/\Lambda \cong \hat{H}$. We assert that h is a multiple of the Fourier transform of f as an element of $L_A(\hat{H})$. This follows since each character on H can be extended to a character on G, and any two such extensions lie in the same coset of Λ.

Finally, we have, for some $\varepsilon > 0$

$$1 \geqslant \int_{\Gamma} B(\varepsilon|\hat{f}(x)|)\,dx = \int_{\Gamma/\Lambda} \Big(\int_{\Lambda} B(\varepsilon|\hat{f}(\xi + y)|)\,dy \Big)\,d\xi$$
$$= \text{const.} \int_{\hat{H}} B(\varepsilon'|\hat{f}(x)|)\,dx$$

(cf. Rudin [27] Chapter 2, Section 7). This proves the lemma.

We consider two cases. First, suppose G is not discrete. By the structure theorem ([27], Ch. 2, Section 4), G has an open subgroup H of the form $R^n \times C$, where $n \geqslant 0$ and C is compact. If $n > 0$ consider functions of the form

$$F(x, c) = f(x_1)\varphi(x_2, \ldots, x_n)\chi_C(c)$$

with for example φ the square of the Fourier transform of a characteristic function, in order to reduce to the case of the real line (5a). If $n = 0$, C is an infinite compact group, and we apply the duality argument (Lemma (5.9) and after) to reduce to the case of discrete groups.

Then consider an infinite discrete group G.

(i) If G contains an element g of infinite order we apply Lemma (5.12) to the (open) subgroup generated by g to reduce to the case $G = Z$ (5b).

(ii) If G contains no element of infinite order, but contains an infinite subgroup of bounded order, then G also contains $\sum_{n=1}^{\infty} Z_p$ for some prime p. We apply (5.21), (5.18), etc.

(iii) If G contains no element of infinite order, and no infinite subgroup of bounded order then G contains $\sum_{m=1}^{\infty} Z_{p_m}$, $p_1 < p_2 < \ldots$ for some strictly increasing sequence of primes. Apply (5.21) and (5.20).

This completes the proof of Theorem (2.16).

6. Notes and Remarks. In the issue of Comptes Rendus dated 1 July 1912 [31] there appeared a paper by W. H. Young, extending the Parseval theorem to the relation

$$\Big(\sum_n |c_n|^q \Big)^{1/q} \leqslant \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p\,dx \Big)^{1/p}$$

between a function f and its sequence $\{c_n\}$ of Fourier coefficients. The exponents p and q had to satisfy the relation $1/p + 1/q = 1$, with q a positive even integer. In the following year [32] Young published the dual result, in which the sum and integral switch sides in the inequality. Later, F. Hausdorff [13] showed that the only requirement on q was that q be at least 2. F. Riesz [24] showed that the trigonometric system could be replaced by any uniformly bounded orthonormal system. Titchmarsh [29] extended the inequalities to Fourier integrals. The interpolation theorem of M. Riesz [25] had all these results as a first application. For notes see the books of Zygmund [35], Edwards [7], and the Collected Papers of Hardy, vol III [6].

In 1918 Carleman [3] gave a continuous function with Fourier coefficients c_n such that

$$\sum |c_n|^{2-\delta} = +\infty \quad \text{for } \delta > 0.$$

Thus the condition that q exceed 2 was shown to be essential. Special examples ([35], Ch. XII, Sec. 2) show that $1/p+1/q = 1$ is the best possible relation for p, q. The case of Fourier integrals is covered in Chapter XVI, section 3 of Zygmund's book.

Mulholland [20] considered the generalization of the Hausdorff-Young inequalities which is obtained by replacing the powers p and q by an appropriately related pair of positive increasing functions. In this context the M. Riesz interpolation theorem did not apply. Also, Hardy and Littlewood had already obtained the inequalities for some important special cases [9], [10]. In contrast to Cooper's generalization of Hölder's inequality, Mulholland gave the relation

$$(*) \qquad \Phi^{-1}(x) = x\Psi^{-1}(1/x)$$

to interchange the behavior of Φ at zero with that of Ψ at infinity. He essentially proved the necessary condition of Theorem (4.5).

The Marcinkiewicz interpolation theorem, with powers replaced by convex functions, was applied to Fourier coefficients by Zygmund [34] and W. Riordan [26]. The methods used by Hardy and Littlewood [10], and by Zygmund [34] to derive the result of Example (i), section 3 actually give the stronger result that

$$\sum_1^\infty c_n^*/n$$

converges, where $\{c_n^*\}$ is the non-increasing rearrangement of the sequence of Fourier coefficients. The interpolation condition of Riordan's thesis [26] yields relation (*).

Integrability theorems for trigonometric transforms (see the book by Boas [1] with this title) of functions restricted by conditions of monotonicity, positivity, etc. have been discovered in great numbers. For some of these in the context of Orlicz spaces see Chen [4].

Integral transforms with the form

$$h(x) = \int_0^\infty F(xy)f(y)\,dy$$

were considered by O'Neil ([22], Theorem 10.11), with integrability conditions on $F(x)$ not satisfied by $\exp(ix)$. However, better results are then possible. For example, in the case of the Laplace transform, the exponent 2 in (4.6) may be replaced by 1.

Theorem (3.2) leaves the following question: is the factor 2 multiplying the norm necessary, or is it merely a consequence of the method of proof? The question of best possible constants for the L^p Hausdorff-Young inequalities on compact groups was treated by Hirschman [16], as was previously done for other contexts ([2] and [8]).

The operation U of section 4 arises from the estimate

$$\frac{1}{x}f\left(\frac{\pi}{x}\right) \leqslant \int_0^\infty \sin(xt)f(t)\,dt, \qquad x > 0,$$

when f is integrable near zero, convex, and decreases to zero at infinity.

The concept of "dispersed set" (section 5) was introduced in order to obtain inequalities such as in (5.16) including the factor $m(E)$ (cf. [15], and [28]).

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Components and open mapping theorems

by

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Introduction. In a recent paper of De Wilde [8], strictly netted locally convex spaces are defined and some closed graph theorems are proved. It seems, as it is not the strict net itself in such a space which plays the essential role, but that it is only a tool for constructing a general structure on the space, a structure, which is independent of the actual construction and from which the closed graph theorems are a consequence. This is the background for this paper.

After the preliminaries are given in section 0 and 1, section 2 brings a general open mapping theorem for closed mappings from a pre-(F)-sequence into a not necessarily metrizable topological group, a generalization of Theorem 2 in [6], and it is from this that all other open mapping — and closed graph theorems presented here will be derived.

In section 3 come the main notions, those of a component and an overwhelming set of components for a locally convex space. The existence of such sets is responsible for the validity of the open mapping and closed graph theorems proved here. The notions are slightly different from those introduced by Słowikowski [7]; this enables us to include new cases, for example, it turns out (section 4) that the structure of strictly netted spaces and of Souslin spaces gives the possibility of constructing sets of components, which overwhelme.

In section 4 also examples from [7] are taken, and it is indicated that the class $\mathscr{D}_0$ considered by Raikov [3] is contained in the class of spaces having an overwhelming set of components.

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0. Notation and Terminology. All vector spaces in consideration are supposed to be over the complex numbers, and all locally convex spaces are supposed to be Hausdorff, unless something else is stated.

As it is customary, we shall write "the topological space X" instead of "the topological space (X, τ)", where no confusion about the topology τ will arise.