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The uncomplemented subspace $K(E, F)$

by

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1. Introduction. For Banach spaces E and F, let $B(E, F)$ denote the space of bounded linear operators from E to F and $K(E, F)$ the subspace of compact operators. Give $B(E, F)$ and $K(E, F)$ the operator norm. In this paper we study the following question: When is $K(E, F)$ complemented in $B(E, F)$? That is, if $K(E, F)$ is a proper subspace of $B(E, F)$ we try to determine the existence of a bounded projection from $B(E, F)$ onto $K(E, F)$. This question has been studied by Thorp in [8] and Arterburn and Whitley in [1]. In all known cases the answer is either that $K(E, F) = B(E, F)$ or that there does not exist a bounded projection. However, there were simple cases for which the question was unanswered. For example Arterburn and Whitley [1] pointed out that in the case where E is the space of all bounded sequences $\boldsymbol{m}$ and F the space of convergent sequences c, the question remained open. In this paper we solve the problem for a large class of Banach spaces not only including the simple case mentioned above, but also for most examples, where E is $C(S)$, the space of continuous functions on a compact Hausdorff space S, and where F is an appropriate l_p space. The answer (see Corollary to Theorem 6 and Theorem 13) is given in the same form as all previously known cases, i.e., either $K(E, F) = B(E, F)$ or $K(E, F)$ is not complemented in $B(E, F)$, and there are still no known examples where this division of possibilities does not occur.

Using direct computations analogous to Pitt [3], we also show that if $E = C(S)$ (where S is compact and dispersed) and if $F = l_p$ $(1 \leqslant p < \infty)$, then every bounded operator from E into F is compact. See [2] for terminology and notation used below.

2. Stable Operators. Let F be a Banach space with Schauder basis $\{f_j, f'_j\}$. We denote by Π_n: $F \to F$ the operator defined by:

$$\Pi_n(y) = \sum_{j=1}^{n} (y, f'_j) f_j .$$

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To introduce the notion of stable operator we begin with a proposition which is well known and is the original motivation for defining the concept of stable operator.

PROPOSITION 1. *Suppose F has a Schauder basis $\{f_j, f'_j\}$. Let $T \in B(E, F)$. Then the following statements are equivalent*:

1. *T is a compact operator,*
2. $\lim_n \|T - \Pi_n T\| = 0$,
3. $\lim_{n,m} \|\Pi_n T - \Pi_m T\| = 0$.

The proof is an immediate consequence of Theorem 9.6, p. 115, of Schaefer [7].

DEFINITION. Let F be a Banach space with Schauder basis $\{f_j, f'_j\}$. A sequence of indices $1 = m(1) \leqslant n(1) < m(2) \leqslant n(2) < \ldots < m(k) \leqslant n(k) < \ldots$ is said to be a *stability sequence* for an operator $T \in B(E, F)$ if there exists an $\varepsilon > 0$ such that $\|(\Pi_{n(k)} - \Pi_{m(k)-1})T\| > \varepsilon$ for all k. Denote $\Pi_{n(k)} - \Pi_{m(k)-1}$ by $\Pi_{[k]}$ so that for each $y \in F$ we have $\Pi_{[k]}(y) = \sum_{j=m(k)}^{n(k)} (y, f'_j) f_j$. An operator T for which there exists a stability sequence is said to be a *stable operator*.

PROPOSITION 2. *Let F have a Schauder basis $\{f_j, f'_j\}$. Then $T \in B(E, F)$ is stable if and only if T is non-compact.*

Proof. This follows directly from the definition and part 3 of Proposition 1.

We need one more preliminary result about Schauder bases.

THEOREM 3. *If F has an unconditional basis $\{f_j, f'_j\}$, then F can be given an equivalent norm, $\|\cdot\|$, so that:*

if $\sum_{j=1}^{\infty} c_j f_j \in F$ *and if* $\sup\{|a_j|: j = 1, 2, \ldots\} < \infty$, *then* $\sum_{j=1}^{\infty} a_j c_j f_j \in F$

and $\|\sum_{j=1}^{\infty} a_j c_j f_j\| \leqslant \sup\{|a_j|\} \, \|\sum_{j=1}^{\infty} c_j f_j\|$.

Proof. The proof consists in a reformulation of Theorem 1, p. 73 together with Remark 4, p. 60, of Day [2].

Hereafter we assume the norm on a Banach space with unconditional basis satisfies the inequality of Theorem 3 above.

Let T be a stable operator in $B(E, F)$, where F has a Schauder basis. Let $\Pi_{[k]}$ be as described in the definition of a stable operator. For each $a = (a_1, a_2, \ldots, a_k, \ldots) \in \boldsymbol{m}$, let $T_a(x) = \sum_k a_k \Pi_{[k]} \circ T(x)$, for each $x \in E$.

PROPOSITION 4. *If F has an unconditional basis and $T \in B(E, F)$ is stable, then the mapping $j\colon \boldsymbol{m} \to B(E, F)$ defined by $j(a) = T_a$ defines a norm isomorphism from $\boldsymbol{m}$ into $B(E, F)$. Moreover, if $\varepsilon > 0$ is the number given by the definition of stable operator, then*: $\varepsilon \sup\{|a_k|: k = 1, 2, \ldots\} \leqslant \|j(a)\| \leqslant \|T\| \sup\{|a_k|: k = 1, 2, \ldots\}$.

Proof. By Theorem 3, $T_a(x) \in F$ for each $x \in F$. Applying the inequality of Theorem 3 we have:

$$\|T_a(x)\| = \Big\| \sum_k a_k \Pi_{[k]} \circ T(x) \Big\| = \Big\| \sum_k a_k \sum_{j=m(k)}^{n(k)} (T(x), f'_j) f_j \Big\|$$
$$\leqslant \sup\{|a_k|: k = 1, 2, \ldots\} \Big\| \sum_k \sum_{j=m(k)}^{n(k)} (T(x), f'_j) f_j \Big\|$$
$$\leqslant \sup\{|a_k|: k = 1, 2, \ldots\} \|T(x)\|.$$

(The last inequality follows from Theorem 3.)

Thus $\|T_a\| \leqslant \|T\| \sup\{|a_k|: k = 1, 2, \ldots\}$, and j is a bounded map from $\boldsymbol{m}$ into $B(E, F)$. To obtain the other inequality note that

$$\|T_a(x)\| \geqslant \|a_k \Pi_{[k]} \circ T(x)\| = |a_k| \, \|(\Pi_{n(k)} - \Pi_{m(k)-1}) T(x)\| \geqslant \varepsilon |a_k| \, \|x\|$$

for each $k = 1, 2, \ldots$; i.e.

$$\|T_a\| \geqslant \varepsilon \sup\{|a_k|: k = 1, 2, \ldots\}.$$

Since it is clear that j is a linear map, it follows from the above inequalities that j is a one-one norm isomorphism of $\boldsymbol{m}$ into $B(E, F)$.

To complete the description we need of the injection j, we must describe what the image of c_0 looks like in $B(E, F)$.

PROPOSITION 5. *Let F be as in Proposition 4. Then*

$$j(c_0) = j(\boldsymbol{m}) \cap K(E, F).$$

Proof. The proof consists of a direct verification using Theorem 3 and Propositions 1 and 4.

We are now in a position to establish the main result of this section.

THEOREM 6. *Let F be a Banach space with unconditional basis. If $B(E, F)$ contains a stable map T, then $K(E, F)$ is not complemented in $B(E, F)$.*

Proof. As will become clear shortly, all that remains is to construct a bounded projection $Q\colon K(E, F) \to j(c_0)$, where j is the injection described earlier. We do this by defining

$$Q(S) = j\big((\Pi_{[1]} \circ S(x_1), y'_1), (\Pi_{[2]} \circ S(x_2), y'_2), \ldots, (\Pi_{[k]} \circ S(x_k), y'_k), \ldots\big),$$

where $\Pi_{[k]}$ is the usual map associated with a stable operator T and x_k, y'_k are respective elements of E and F', chosen so that $\|x_k\| < \dfrac{1}{\varepsilon}$, $\|y'_k\| = 1$ and so that $(\Pi_{[k]} \circ T(x_k), y'_k) = 1$.

The choices of x_k and y'_k are possible since T is a stable map. Now for $S \in K(E, F)$, we have from Proposition 1 that $\|\Pi_{[k]} \circ S\| \to 0$, so that $Q(S) \in j(c_0)$. Thus Q is linear map from $K(E, F)$ into $j(c_0)$. It is a simple verification that $Q(j(a)) = j(a)$ for $a \in c_0$. Thus Q is a linear projection. Finally

$$\|(\Pi_{[k]} \circ S(x_k), y'_k)\| \leqslant \|\Pi_{[k]} \circ S\| \, \|x_k\| \, \|y'_k\| \leqslant \|S\| \frac{1}{\varepsilon},$$

$$\|Q(S)\| = \|j(\beta)\| \leqslant \|T\| \sup\{|b_k|: k = 1, 2, \ldots\} \leqslant \|T\| \, \|S\| \frac{1}{\varepsilon},$$

where $\beta = (b_1, b_2, \ldots, b_k, \ldots)$ and $b_k = (\Pi_{[k]} \circ S(x_k), y'_k)$.

Using the fixed stable operator T we thus can construct

1. an injection $j: \boldsymbol{m} \to B(E, F)$ satisfying $j(c_0) = j(\boldsymbol{m}) \cap K(E, F)$,
2. a bounded projection $Q: K(E, F) \to j(c_0)$.

Now suppose $K(E, F)$ is complemented in $B(E, F)$. Then we have the following diagram, where P denotes the hypothesized bounded projection

$$\begin{array}{ccc} B(E, F) & \xrightarrow{P} & K(E, F) \\ {\scriptstyle j}\uparrow & & \downarrow{\scriptstyle j^{-1} \circ Q} \\ \boldsymbol{m} & & c_0 \end{array}$$

If we consider the map $P_0 = j^{-1} \circ Q \circ P \circ j$, it is immediate from 1 and 2 that P_0 is a bounded projection from $\boldsymbol{m}$ onto c_0, which is impossible by Phillips' Theorem. Hence P cannot exist and the theorem follows.

Combining Proposition 2 and Theorem 6 we obtain the

COROLLARY. *Let F be a Banach space with unconditional basis. Then either $B(E, F) = K(E, F)$ or $K(E, F)$ is not complemented in $B(E, F)$.*

3. The case: S is non-dispersed. It is clear from the corollary above that its usefulness depends on being able to determine, for given Banach spaces E and F, when there exists a non-compact, or stable, operator in $B(E, F)$. If S is an infinite compact Hausdorff space and $E = C(S)$, the Banach space of continuous functions on S with the supreme norm, and if F is an appropriate l_p space, it is possible to produce such an operator. We split the spaces $C(S)$ into two types — the cases where S is non-dispersed and where S is dispersed. Recall that a compact Hausdorff space is dispersed if and only if it does not support a continuous regular Borel measure [3].

THEOREM 7. *Let S be an infinite non-dispersed compact Hausdorff space and let $E = C(S)$. If $F = c_0$ or $F = l_p$, $2 \leqslant p < \infty$, then:*

1. *$B(E, F)$ has a stable map;*
2. *$K(E, F)$ is not complemented in $B(E, F)$.*

Proof. That F has an unconditional basis is well known. If S is not dispersed, the existence of a stable map in $B(E, F)$ is known (e.g. see Pełczyński [5]). Part 2 then follows from the Corollary to Theorem 6.

Theorem 7 provides a negative answer to the simple case mentioned by Arterburn and Whitley in [1] and discussed in the introduction, as follows:

COROLLARY. *Let $E = \boldsymbol{m}$ and $F = c_0$ or l_p, $2 \leqslant p < \infty$. Then $K(E, F)$ is not complemented in $B(E, F)$.*

Proof. The Banach space of all bounded sequences can be identified with $C(\beta N)$, where βN denotes the Stone-Čech compactification of the integers N. It is well known (see [4]) that βN is not dispersed. Under this identification, the Corollary becomes a special case of the theorem.

In the case where E is as in Theorem 7 and $F = l_p$, $1 < p < 2$, the existence of a stable map in $B(E, F)$ is, as far as we know, unknown. In the case $F = l_1$, it is an easy consequence of another theorem due to Phillips that $B(E, F) = K(E, F)$, where $E = C(S)$ and S is *any* compact Hausdorff space.

4. The case: S is dispersed. In this section, we study the case where $E = C(S)$, S is compact Hausdorff and dispersed and $F = c_0, l_p$ $(1 \leqslant p < \infty)$.

For the case where S is dispersed and $F = c_0$ the following theorem obtains, the proof of which is due to A. Pełczyński and was communicated to us by him.

THEOREM 8. *If S is dispersed, then $C(S)$ contains a complemented copy of c_0. Thus setting $E = C(S)$ and $F = c_0$, $B(E, F)$ has a stable map and $K(E, F)$ is not complemented in $B(E, F)$.*

Proof. Let $S^{(1)}$ denote the set of non-isolated points of S. Since S is dispersed and of second category, $S^{(1)}$ is non-empty, compact and dispersed. Therefore we can choose a point $s_0 \in S^{(1)}$ which is isolated in $S^{(1)}$. Choose a closed neighborhood U of s_0 in S such that $U \cap S^{(1)} = \{s_0\}$. Since s_0 is not isolated in S there exists a sequence $\{s_n\}$ of distinct points in $U \setminus \{s_0\}$. Since U is closed and, except for s_0, consists of isolated points of S, the only possible cluster point of the sequence $\{s_n\}$ is s_0. Since S is compact it follows that $s_n \to s_0$.

If c is the subspace of $C(S)$ consisting of all functions f such that $f(S \setminus \{s_n\}) = 0$, then it is a simple verification that c is complemented in $C(S)$ and that c is isomorphic to c_0.

The case where S is dispersed and $F = l_p$, $1 \leqslant p < \infty$ will be approached differently. Whereas previously we proved that $K(E, F)$ was a proper subspace of $B(E, F)$ in order to apply the Corollary to Theorem 6, we now use an argument similar to the proof of the Littlewood-Pitt theorem [6] which asserts (as a special case) that every bounded operator from c_0 to l_p $(1 \leqslant p < \infty)$ is compact in order to prove that $K(E, F) = B(E, F)$ for the appropriate spaces E and F. Our result states that for the case of a compact dispersed space S, every bounded operator from $C(S)$ into l_p is compact. Throughout the remainder of this section we assume $1 \leqslant p < \infty$.

Let $T\colon C(S) \to l_p$ be a bounded operator. Then there are measures $\mu_1, \mu_2, \ldots, \mu_i, \ldots$ on S so that:

$$T(f) = \Big(\int f d\mu_1, \int f d\mu_2, \ldots, \int f d\mu_i, \ldots\Big) \in l_p$$

for every $f \in C(S)$. If S is dispersed, every measure on S is atomic. Thus, throughout this section, we will study operators $T\colon C(S) \to l_p$, where S is compact Hausdorff and where the measures $\mu_1, \mu_2, \ldots, \mu_i, \ldots$ are atomic. Let $s_1, s_2, \ldots, s_j, \ldots$ be the points in S which have non-zero mass for some μ_i and write $\mu_i = \sum_j c_{ij}\delta(s_j)$ for appropriate scalars c_{ij} ($\delta(s_j)$ denotes unit point mass at s_j. Then it follows that

$$\int f d\mu_i = \sum_j c_{ij} f(s_j) \quad \text{for each } i \text{ and for every } f \in C(S).$$

Set $\lambda_s = \{(a_1, a_2, \ldots, a_j, \ldots)\colon a_j = f(s_j) \text{ for all } j \text{ and for some } f \in C(S)\}$. Then, λ_s is a linear space of sequences on which we define the norm, $\|\cdot\|$, by setting $\|(a_1, a_2, \ldots, a_j, \ldots)\| = \sup\{|a_j|\colon j = 1, 2, \ldots\}$. It is easy to check that λ_s is a Banach space under this norm. Hereafter an element $(a_1, a_2, \ldots, a_j, \ldots)$ of λ_s is denoted by a.

LEMMA 9. *Let $T\colon C(S) \to l_p$ be a bounded operator as above. Then there exist bounded operators $P\colon C(S) \to \lambda_s$ and $M_s\colon \lambda_s \to l_p$ so that $\|T\| = \|M_s\|$ and so that the following diagram commutes:*

$$\begin{array}{ccc} C(S) & \xrightarrow{T} & l_p \\ & {\scriptstyle P}\searrow \quad \nearrow{\scriptstyle M_s} & \\ & \lambda_s & \end{array}$$

Proof. The mapping P is defined by $P(f) = \big(f(s_1), f(s_2), \ldots, f(s_j), \ldots\big)$ and M_s is defined as the map from λ_s into l_p satisfying

$$M_s(a) = \Big(\sum_j c_{1j}a_j, \sum_j c_{2j}a_j, \ldots, \sum_j c_{ij}a_j, \ldots\Big) \in l_p$$

for each $a \in \lambda_s$. It is easy to check that $T = M_s \circ P$, that $\|P\| = 1$ and that $\|T\| = \|M_s\|$.

We will show that M_s is necessarily a compact operator. But since the method of proof has other applications (see Theorem 12) than the special case in question, we formulate our approach abstractly in making the following definition.

DEFINITION. Let λ be a Banach space of sequences. We say λ has the *finite monotonicity property* (*f.m.p.*) if:

1. For each coordinate i there is an $a \in \lambda$, $a = (a_1, a_2, \ldots, a_n, \ldots)$ with $a_i \neq 0$.

2. Whenever $a \in \lambda$, $a = (a_1, a_2, \ldots, a_n, \ldots)$ and $\gamma = (c_1, \ldots, c_N, \ldots \ldots, c_j, \ldots)$ is a sequence for which $c_j = 0$ for $j > N$, then there is a sequence $\beta = (b_1, b_2, \ldots, b_n, \ldots) \in \lambda$ satisfying $b_j = a_j c_j$ for $j = 1, 2, \ldots, N$ and $\|\beta\| \leqslant \|a\| \sup\{|c_j|\colon j = 1, 2, \ldots\}$.

3. Whenever $a' = (a'_1, a'_2, \ldots, a'_j, \ldots)$ is a sequence which defines a continuous linear functional on λ by the equation $a'(a) = \sum_j a'_j a_j$, then $\lim_n \|a' - a'_n\| = 0$, where $a'_n = (a'_1, \ldots, a'_n, 0, \ldots, 0, \ldots)$.

Note that λ need not contain all finite sequences but rather, we require that λ has a "norm preserving extension" of any finite sequence.

LEMMA 10. *λ_s has the finite monotonicity property.*

Proof. 1 is obvious, 2 follows from the Tietze extension theorem, and 3 follows from the Riesz Representation Theorem.

Hereafter we only consider sequence spaces with f.m.p.

DEFINITION. Let λ be a Banach space of sequences. Let $M = (c_{ij})$ be a matrix. We say that M is a *matrix map* if the equation

$$M(a) = \Big(\sum_j c_{1j}a_j, \sum_j c_{2j}a_j, \ldots, \sum_j c_{ij}a_j, \ldots\Big)$$

defines a bounded linear operator from λ into l_p.

LEMMA 11. *Let $M = (c_{ij})\colon \lambda \to l_p$ be a matrix map. Then*

1. *$\gamma^{[i]} = (c_{i1}, c_{i2}, \ldots, c_{ij}, \ldots)$ defines, for each i, a continuous linear functional on λ, where $\gamma^{[i]}(a) = \sum_j c_{ij}a_j$.*

2. *$\sum_i |c_{ij}|^p < \infty$ for each j.*

3. *Let J be a finite set of indices. Let $M\Pi_J$ denote the matrix consisting of c_{ij} if $j \in J$ and of zeros if $j \notin J$. Let $M(1-\Pi_J)$ denote the matrix consisting of zeros if $j \in J$ and of c_{ij} if $j \notin J$. Then $M\Pi_J$ and $M(1-\Pi_J)$ are matrix maps satisfying*

$$\|M\Pi_J\| \leqslant \|M\|, \qquad \|M(1-\Pi_J)\| \leqslant \|M\|.$$

4. *If R_n is the matrix consisting of c_{ij} when $i, j \geqslant n$ and of zeros otherwise, then R_n is a matrix map from λ to λ_p satisfying $\|R_n\| \leqslant \|M\|$. (We call R_n the n-th remainder matrix of M.)*

5. *M is a compact matrix map if and only if $\lim_n \|R_n\| = 0$.*

Proof. 1. Clearly $\gamma^{[i]}$ is the linear functional given by $e_i'\circ M$, where $e_i'\colon l_p\to R$ is the continuous linear functional given by evaluation: $e_i'(x)=x_i$ for all $x=(x_1,x_2,\ldots,x_i,\ldots)$ in l_p. This proves 1.

2. The proof consists in a tedious but direct verification using part (2) in the definition of f.m.p.

3. For any $a=(a_1,a_2,\ldots a_j,\ldots)\in\lambda$ and any $\varepsilon>0$ we will show that there exists a $\beta\in\lambda$ with $\|\beta\|\leqslant\|a\|$ and satisfying $\|M(\beta)\|_p\geqslant\|M\Pi_J(a)\|_p-\varepsilon$. Indeed, β will be chosen so that $\|M(\beta)\|_p\geqslant\|\pi_N M\pi_J(a)\|_p-\varepsilon/2$, for all N; we deduce from this that $M\pi_J(a)\in l_p$. Let $j_0=\max\{j\colon j\in J\}$. For each N, let n be sufficiently large so that $n>j_0$ and $\|\gamma^{[i]}-\gamma_n^{[i]}\|<\varepsilon/4(N-1)\|a\|$, $i=1,2,\ldots,N-1$. Here, $\gamma_n^{[i]}$ denotes $(c_{i1},\ldots,c_{in},0,0,\ldots)$. Choose $\beta=(b_1,b_2,\ldots,b_j,\ldots)\in\lambda$ so that $\|\beta\|\leqslant\|a\|$, $b_j=a_j$ if $j\in J$ and $b_j=0$ if $j\notin J$ but $j\leqslant n$. Then, using an argument similar to that needed in the proof of 2 above,

$$\begin{aligned}\|M(\beta)\|_p&\geqslant\Big\|\big(\gamma_n^{[1]}(\beta),\ldots,\gamma_n^{[i]}(\beta),\ldots,\gamma_n^{[N-1]}(\beta),\gamma^{[N]}(\beta),\ldots\big)\Big\|_p-\varepsilon/2\\&\geqslant\|\gamma_n^{[1]}(\beta),\ldots,\gamma_n^{[N-1]}(\beta),0,0,\ldots)\|_p-\varepsilon/2\\&\geqslant\Big\|\Big(\sum_{j\in J}c_{1j}a_j,\ldots,\sum_{j\in J}c_{N-1j}a_j,0,0,\ldots\Big)\Big\|_p-\varepsilon/2\\&\geqslant\|M\Pi_J(a)\|_p-\varepsilon.\end{aligned}$$

Thus $\|M\|\geqslant\|M\Pi_J\|$.

Similarly, suppose $a\in\lambda$ is a unit vector so that $\|M(1-\Pi_J)(a)\|_p>\|M\|$, where the left-hand side is possibly infinite. Then there is a sufficiently large n so that

$$\Big\|\Big(\sum_{j\notin J}c_{1j}a_j,\ldots,\sum_{j\notin J}c_{nj}a_j,0,0,\ldots\Big)\Big\|_p>\|M\|.$$

We can choose a finite subset J_1 of indices disjoint from J so that

$$\Big\|\Big(\sum_{j\in J_1}c_{1j}a_j,\ldots,\sum_{j\in J_1}c_{nj}a_j,0,0,\ldots\Big)\Big\|_p>\|M\|.$$

Repeating the argument above we can find $\beta\in\lambda$, $\|\beta\|\leqslant\|a\|=1$ with

$$\|M(\beta)\|_p\geqslant\Big\|\Big(\sum_{j\in J_1}c_{1j}a_j,\ldots,\sum_{j\in J_1}c_{nj}a_j,0,0,\ldots\Big)\Big\|_p-\varepsilon>\|M\|$$

for sufficiently small ε. This, of course, is impossible.

4. Observe that $R_n=(1-\Pi_J)\big(M(1-\Pi_J)\big)$, where $J=\{1,2,\ldots,n\}$ and $(1-\Pi_J)$ on the left is the projection of l_p which maps e_i into 0 if $1\leqslant i\leqslant n$ and maps e_i to e_i if $i>n$. Thus

$$\|R_n\|\leqslant\|(1-\Pi_J)\|\;\|M(1-\Pi_J)\|\leqslant\|M\|.$$

5. If M is compact, then Proposition 1 proves: $\lim_n\|M-\Pi_nM\|=0$.

Since $R_n=(M-\Pi_{J(n)}M)(1-\Pi_{J(n)})$, where $J(n)=\{1,2,\ldots,n\}$, this together with part (3) above shows: $\lim_n\|R_n\|=0$. Conversely, assume $\varepsilon>0$ is given. Since $R_n=M-M\Pi_n-\Pi_nM(1-\Pi_n)$, by setting $F=\Pi_{N_0}M\Pi_n+\Pi_nM(1-\Pi_n)$ we can verify that $\|M-F\|\leqslant\|M\Pi_n-\Pi_{N_0}M\Pi_n\|+\|R_n\|<\varepsilon$, for a suitable choice of n and N_0.

For notational convenience in the subsequent proofs, we introduce the following terminology: Let $D=(d_{ij})$ and $M=(c_{ij})$ be matrices. We say that D is a *submatrix* of M if and only if $d_{ij}\neq0$ implies $d_{ij}=c_{ij}$. For a given sequence of pairs of integers $1\leqslant m(1)\leqslant n(1)<\ldots<m(k)\leqslant n(k)<\ldots$, by the $(m(k),n(k))$ *diagonal block*, denoted B_k, of M we mean the submatrix whose entries are c_{ij} if $m(k)\leqslant i,j\leqslant n(k)$ and are 0 otherwise. By the *k-th truncated diagonal block*, denoted D_k, of a matrix M we mean the submatrix of M consisting of $B_1,\ldots,B_k$ and of zeros elsewhere.

THEOREM 12. *Let $M=(c_{ij})\colon\lambda\to l_p$ be a non-compact matrix map. Then there exist indices $1=m(1)\leqslant n(1)<\ldots<m(k)\leqslant n(k)<\ldots$ so that for some $\varepsilon>0$, $\|B_k\|>\varepsilon$ and $\|D_k\|\leqslant2\|M\|$ for all $k=1,2,\ldots$*

Proof. By Part 5 of Lemma 11 there is an $\varepsilon>0$ so that $\|R_i\|>\varepsilon$ for all $i\in I$, where I is a subsequence of the positive integers. We may also assume $\|M\|>\varepsilon$. We choose the sequence of pairs of integers $m(k)$, $n(k)$ inductively as follows. Choose $m(1)=1$. It is not difficult to show, using Parts 2 and 3 of Lemma 11, that if $n(1)$ is sufficiently large, then $\varepsilon<\|\Pi_{n(1)}M\Pi_{n(1)}\|\leqslant\|M\|$. Set $B_1=D_1=\Pi_{n(1)}M\Pi_{n(1)}$. Choose $m(2)>n(1)$ so that

(a) $\|\Pi_{J(1)}M\Pi_{I(1)}\|<\frac12\|M\|$,

(b) $\|\Pi_{I(1)}M\Pi_{J(1)}\|<\frac12\|M\|$,

(c) $\|R_{m(2)}\|>\varepsilon$,

where $J(1)=\{i\colon m(1)\leqslant i<n(1)\}$ and $I(1)=\{i\colon i\geqslant m(2)\}$. This is possible by part 2 of Lemma 11. Assume now that for $l=1,2,\ldots,k-1$ we have chosen $m(1)\leqslant n(1)<\ldots<m(k-1)\leqslant n(k-1)<m(k)$ so that

(i) $\|\Pi_{J(l)}M\Pi_{I(l)}\|<\|M\|/2^{l+1}$,

(ii) $\|\Pi_{I(l)}M\Pi_{J(l)}\|=\|M\|/2^{l+1}$,

(iii) $\|R_{m(l+1)}\|>\varepsilon$,

where $J(l)=\{i\colon m(l)\leqslant i\leqslant n(l)\}$ and $I(l)=\{i\colon i\geqslant m(l+1)\}$. Since $\|R_{m(k)}\|>\varepsilon$ and since l_p has a Schauder basis, we can find a sufficiently large N so that if $J_N=\{1,2,\ldots,N\}$, then $\|\Pi_JR_{m(k)}\|>\varepsilon$ for any J with $J_N\subset J$. Repeating an argument similar to that used in the proof of Part 3 of Lemma 11 we can find a sufficiently large n so that if $J_n=\{1,2,\ldots,n\}$, then $\|\Pi_JR_{m(k)}\Pi_{J'}\|>\varepsilon$ for any J,J' with $J_N\subset J$, $J_n\subset J'$.

Choose $n(k)$ satisfying $m(k) \leqslant n(k)$ and so that $J_N \subset J_{n(k)}$, $J_n \subset J_{n(k)}$. Then if $B_k = \Pi_{J_{n(k)}} R_{m(k)} \Pi_{J_{n(k)}}$, we have $\|B_k\| > \varepsilon$. To see that $\|D_k\| \leqslant 2\|M\|$ apply (i) and (ii) of the induction hypothesis above.

THEOREM 13. *Let $M = (c_{ij})$: $\lambda_s \to l_p$ be a matrix map. Then M is compact.*

Proof. If M is not compact, then by Proposition 12 there is an $\varepsilon > 0$ and indices $1 = m(1) \leqslant n(1) < \ldots < m(k) \leqslant n(k) < \ldots$ so that $\|B_k\| > \varepsilon$ and $\|D_k\| \leqslant 2\|M\|$ for all k. Let $a^{[1]}, a^{[2]}, \ldots, a^{[k]}$ be unit vectors in λ satisfying $\|B_k(a^{[k]})\|_p = \|B_k\|$. This choice is possible because each B_k is a finite matrix. Let $\xi = (1, 1, \ldots, 1, \ldots)$. Clearly ξ is a unit vector of λ_s. By f.m.p. we may choose $\beta^{[t]} \epsilon \lambda$, where $\beta^{[t]} = (b_1^{[t]}, \ldots, b_j^{[t]}, \ldots)$ satisfies:

$$b_i^{[t]} = \begin{cases} a_i^{[k]} & \text{if } m(k) \leqslant i \leqslant n(k), \quad \text{where } k = 1, 2, \ldots, t, \\ 0 & \text{if } n(k) < i < m(k+1), \quad \text{where } k = 1, 2, \ldots, t-1, \end{cases}$$

and $\|\beta^{[t]}\| \leqslant \|\xi\| = 1$.

Thus, $\|D_t(\beta^{[t]})\|_p = \big(\sum_{k=1}^{t} \|B_k\|^p\big)^{1/p} = t^{1/p}\varepsilon$.

A contradiction now results because $2\|M\| \geqslant \|D_t\| \geqslant \|D_t(\beta^{[t]})\|_p = t^{1/p}\varepsilon$, where t is arbitrary.

Remark. It is not hard to check that the l_q spaces satisfy f.m.p. if $1 < q < \infty$ and that c_0 also satisfies f.m.p. Arguing as in Theorem 13 we may apply Theorem 12 to show that there are no non-compact operators T: $l_q \to l_p$ or T: $c_0 \to l_p$, if $1 \leqslant p < q < \infty$. This is the Littlewood-Pitt Theorem [6].

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Factorable and strictly singular operators, I

by

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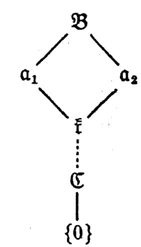
Let X and Y be Banach spaces; a linear bounded operator T: $X \to Y$ is *strictly singular* [5] if the subspaces $Z \subset X$ for which the restriction $T|_Z$: $Z \to T(Z)$ has a bounded inverse $T(Z) \to Z$ are necessarily finite dimensional. Assume X is the cartesian product $X = E_1 \times E_2$ of the Banach spaces E_1 and E_2. An operator T: $X \to X$ is *factorable* through E_i (for $i = 1, 2$) if it is of the form $T = SQ$ for suitable $Q: X \to E_i$ and S: $E_i \to X$. Denote by $\mathfrak{a}_i$ the (uniform) closure of the set of operators factorable through E_i. We prove below that under special conditions, an operator T is strictly singular if and only if $T \epsilon \mathfrak{a}_1 \cap \mathfrak{a}_2$. This is applied notably to the case $E_1 = \ell^p$ and $E_2 = \ell^q$ for $p \neq q$ (and more particularly, when $p > q = 2$) to obtain various relations between these types of operators. As a consequence, it can be seen ((3.1. e) below) that in this case, if T_1 and T_2 are strictly singular, then $T_1 T_2$ is compact.

Finally, the lattice of all closed two-sided ideals of the algebra of all bounded linear operators on $X = \ell^p \times \ell^2$, for $p > 2$, is studied in (3.4): it consists of six elements 0, $\mathfrak{C}$, $\mathfrak{k}$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{B}$ ordered as follows:



where solid lines mean that there are no ideals strictly between the ideals joined by them.