The uncomplemented subspace $K(E, F)$

by

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1. Introduction. For Banach spaces $E$ and $F$, let $B(E, F)$ denote the space of bounded linear operators from $E$ to $F$ and $K(E, F)$ the subspace of compact operators. Give $B(E, F)$ and $K(E, F)$ the operator norm. In this paper we study the following question: When is $K(E, F)$ complemented in $B(E, F)$? That is, if $K(E, F)$ is a proper subspace of $B(E, F)$ we try to determine the existence of a bounded projection from $B(E, F)$ onto $K(E, F)$. This question has been studied by Thorpe in [8] and Arterburn and Whiteley in [1]. In all known cases the answer is either that $K(E, F) = B(E, F)$ or that there does not exist a bounded projection. However, there were simple cases for which the question was unanswered. For example Arterburn and Whiteley [1] pointed out that in the case where $E$ is the space of all bounded sequences $m$ and $F$ the space of convergent sequences $w$, the question remained open. In this paper we solve the problem for a large class of Banach spaces not only including the simple case mentioned above, but also for most examples, where $E$ is $C(S)$, the space of continuous functions on a compact Hausdorff space $S$, and where $F$ is an appropriate $l_p$ space. The answer (see Corollary to Theorem 6 and Theorem 13) is given in the same form as all previously known cases, i.e., either $K(E, F) = B(E, F)$ or $K(E, F)$ is not complemented in $B(E, F)$, and there are still no known examples where this division of possibilities does not occur.

Using direct computations analogous to Pitt [3], we also show that if $E = C(S)$ (where $S$ is compact and dispersed) and if $F = l_p(1 < p < \infty)$, then every bounded operator from $E$ into $F$ is compact. See [2] for terminology and notation used below.

2. Stable Operators. Let $F$ be a Banach space with Schauder basis \{$f_i$\}. We denote by $\Pi_n: F \to F$ the operator defined by:

$$\Pi_n(y) = \sum_{i=1}^{n} (y, f_i)f_i.$$
To introduce the notion of stable operator we begin with a proposition which is well known and is the original motivation for defining the concept of stable operator.

**Proposition 1.** Suppose $F$ has a Schauder basis $\{f_j, f_j'\}$. Let $T \in B(E, F)$. Then the following statements are equivalent:
1. $T$ is a compact operator,
2. $\lim_{n \to \infty} \|T - \Pi_n T\| = 0$,
3. $\lim_{n \to \infty} \|T - \Pi_n T\| = 0$.

The proof is an immediate consequence of Theorem 9.6, p. 110, of Schaefer [7].

**Definition.** Let $F$ be a Banach space with Schauder basis $\{f_j, f_j'\}$. A sequence of indices $1 = m(1) < m(2) < m(3) < \ldots < m(k) < \ldots$ is said to be a stability sequence for an operator $T \in B(E, F)$ if there exists an $\varepsilon > 0$ such that $\|T - \Pi_{m(k)} T\| > \varepsilon$ for all $k$. Denote $\Pi_{m(k)}^s - \Pi_{m(k-1)}^s$ by $\Pi_{m(k)}^s$ so that for each $\varepsilon \in F$ we have $\Pi_{m(k)}^s(y) = \sum_{j=m(k-1)+1}^{m(k)} \langle y, f_j \rangle f_j$. An operator $T$ for which there exists a stability sequence is said to be a stable operator.

**Proposition 2.** Let $F$ have a Schauder basis $\{f_j, f_j'\}$. Then $T \in B(E, F)$ is stable if and only if $T$ is non-compact.

The proof follows directly from the definition and part 3 of Proposition 1.

We need one more preliminary result about Schauder bases.

**Theorem 3.** If $F$ has an unconditional basis $\{f_j, f_j'\}$, then $F$ can be given an equivalent norm, $\| \cdot \|_1$, so that:
1. $\sum_{j=1}^{m} a_j f_j \in F$ and $\sup_{j \geq 1} |a_j| < \infty$, then $\sum_{j=1}^{m} a_j f_j \in F$ and $\| \sum_{j=1}^{m} a_j f_j \|_1 \leq \sup_{j \geq 1} |a_j| \sum_{j=1}^{m} \| f_j \|_1$.

The proof consists of a reformulation of Theorem 1, p. 73 together with Remark 4, p. 60, of Day [2].

Hereafter we assume the norm on a Banach space with unconditional basis satisfies the inequality of Theorem 3 above.

Let $T$ be a stable operator in $B(E, F)$, where $F$ has a Schauder basis. Let $\Pi_{m(k)}^s$ be as described in the definition of a stable operator. For each $a = (a_1, a_2, \ldots, a_k) \in m$, let $T_a = m(a) \Pi_{m(k)}^s T(x_a)$, for each $a \in A$.

**Proposition 4.** If $F$ has an unconditional basis and $T \in B(E, F)$ is stable, then the mapping $\varphi : m \to B(E, F)$ defined by $\varphi(a) = T_a$ defines a norm isomorphism from $m$ into $B(E, F)$. Moreover, if $\varepsilon > 0$ is the number given by the definition of stable operator, then $\varepsilon \sup_{k \geq 1} |a_k| < \frac{1}{\varepsilon}$.

**Proof.** By Theorem 3, $T_a \in B(E, F)$ for each $a \in A$. Applying the inequality of Theorem 3 we have:

$$
\|T_a(x)\| = \| \sum_{j=m(k-1)+1}^{m(k)} \langle x, f_j \rangle f_j \| = \| \sum_{j=m(k-1)+1}^{m(k)} \langle x, f_j \rangle f_j \| \\
\leq \sup_{k \geq 1} |a_k| \| \sum_{j=m(k-1)+1}^{m(k)} \langle x, f_j \rangle f_j \| \\
\leq \sup_{k \geq 1} |a_k| \| \sum_{j=m(k-1)+1}^{m(k)} \langle x, f_j \rangle f_j \| < \varepsilon \sup_{k \geq 1} |a_k|.
$$

(The last inequality follows from Theorem 3.)

Thus $\|T_a(x)\| < \varepsilon \sup_{k \geq 1} |a_k| = \varepsilon \| x \|_1$, and $T_a$ is a bounded map from $m$ into $B(E, F)$. To obtain the other inequality note that

$$
\|T_a(x)\| > |a_k| \| \sum_{j=k(1)+1}^{k(2)} \langle x, f_j \rangle f_j \| > \varepsilon |a_k| \| x \|_1
$$

for each $k = 1, 2, \ldots$, i.e.,

$$
\|T_a\| > \varepsilon \sup_{k \geq 1} |a_k| = \varepsilon \| x \|_1.
$$

Since it is clear that $j$ is a linear map, it follows from the above inequalities that $j$ is a one-one norm isomorphism of $m$ into $B(E, F)$.

To complete the description we need of the injection $j$, we must describe what the image of $c_0$ looks like in $B(E, F)$.

**Proposition 5.** Let $F$ be as in Proposition 4. Then

$$
\varphi(c_0) = \varphi(m) \cap K(E, F).
$$

The proof consists of a direct verification using Theorem 3 and Propositions 1 and 4.

We are now in a position to establish the main result of this section.

**Theorem 6.** Let $F$ be a Banach space with unconditional basis. If $B(E, F)$ contains a stable map $T$, then $K(E, F)$ is not complemented in $B(E, F)$.

The proof as will become clear shortly, all that remains is to construct a bounded projection $Q : K(E, F) \to j(c_0)$, where $j$ is the injection described earlier. We do this by defining

$$
Q(x) = f_1 \sum_{i=1}^{\infty} f_i (x_i, y_i), f_2 \sum_{i=1}^{\infty} f_i (x_i, y_i), \ldots, f_k \sum_{i=1}^{\infty} f_i (x_i, y_i), \ldots,
$$

where $f_k$ is the usual map associated with a stable operator $T$ and $x_i, y_i$ are respectively elements of $E$ and $F'$, chosen so that $|x_i| < \frac{1}{k}$, $|y_i| = 1$ and so that $\langle T(x), y_i \rangle = 1$. 

Uncomplemented subspace $K(E, F)$
The choices of $a_0$ and $y_0'$ are possible since $T$ is a stable map. Now for $S \in K(E, F)$, we have from Proposition 1 that $||\Pi_{\overline{S}} \circ S|| \to 0$, so that $Q(S) = j_*(\alpha)$. Thus $Q$ is a linear map from $K(E, F)$ into $j(Q)$. It is a simple verification that $Q(j_*(\alpha)) = j(\alpha)$ for $\alpha \in c_0$. Thus $Q$ is a linear projection.

Finally

$$||\Pi_{\overline{S}} \circ S(a_0), y_0'|| \leq ||\Pi_{\overline{S}} \circ S|| \cdot ||a_0|| \cdot ||y_0'|| \leq ||S|| \cdot \frac{1}{\varepsilon},$$

$$Q(S) = \frac{\|\cdot\|}{\|\cdot\|} \sup \{||\beta|| : \beta = (b_1, b_2, \ldots, b_k) \leq ||\beta|| \leq ||T|| \leq \frac{1}{\varepsilon},$$

where $\beta = (b_1, b_2, \ldots, b_k)$ and $b_k = ||\Pi_{\overline{S}} \circ S(a_0), y_0'||$.

Using the fixed stable operator $T$ we thus can construct

1. an injection $j : m \to B(E, F)$ satisfying $j(\alpha) = j(m) \cap K(E, F)$;
2. a bounded projection $Q : K(E, F) \to j(\alpha)$.

Now suppose $K(E, F)$ is complemented in $B(E, F)$. Then we have the following diagram, where $P$ denotes the hypothesized bounded projection

$$B(E, F) \overset{P}{\longrightarrow} K(E, F) \overset{j}{\longrightarrow} \overset{\alpha}{m} \overset{i}{\longrightarrow}$$

If we consider the map $P_j = j^{-1}Q \circ P \circ j$, it is immediate from 1 and 2 that $P_j$ is a bounded projection from $m$ onto $c_0$, which is impossible by Phillips' Theorem. Hence $P$ cannot exist and the theorem follows.

Combining Proposition 2 and Theorem 6 we obtain the

**COROLLARY.** Let $E$ be a Banach space with unconditional basis. Then either $B(E, F) = K(E, F)$ or $K(E, F)$ is not complemented in $B(E, F)$.

3. **The case: $S$ is non-dispersed.** It is clear from the corollary above that its usefulness depends on being able to determine, for given Banach spaces $E$ and $F$, when there exists a non-compact, or stable, operator in $B(E, F)$, and $S$ is a non-dispersed Banach space of continuous functions on $S$ with the compact norm, and if $F$ is a non-compact Hausdorff space, it is possible to produce such an operator. We split the spaces $C(S)$ into two types — the cases where $S$ is non-dispersed and where $S$ is dispersed. Recall that a compact Hausdorff space is dispersed if and only if it does not support a compact regular Borel measure [3].

**THEOREM 7.** Let $S$ be an infinite non-dispersed compact Hausdorff space and let $E = C(S)$. If $F = c_0$ or $F = l_p$, $2 \leq p < \infty$, then:

1. $B(E, F)$ has a stable map;
2. $K(E, F)$ is not complemented in $B(E, F)$.

**Proof.** That $F$ has an unconditional basis is well known. If $S$ is not dispersed, the existence of a stable map in $B(E, F)$ is known (see Pełczyński [5]). Part 2 then follows from the Corollary to Theorem 6.

Theorem 7 provides a negative answer to the simple case mentioned by Arterburn and Whitley in [1] and discussed in the introduction, as follows:

**COROLLARY.** Let $E = m$ and $F = c_0$ or $l_p$, $2 \leq p < \infty$. Then $K(E, F)$ is not complemented in $B(E, F)$.

**Proof.** The Banach space of all bounded sequences can be identified with $C(\beta N)$, where $\beta N$ denotes the Stone-Čech compactification of the integers $\mathbb{N}$. It is well known (see [4]) that $\beta N$ is not dispersed. Under this identification, the Corollary becomes a special case of the theorem.

In the case where $E$ is as in Theorem 7 and $F = l_p$, $1 < p < 2$, the existence of a stable map in $B(E, F)$ is, as far as we know, unknown. In the case $F = l_1$, it is an easy consequence of another theorem due to Phillips that $B(E, F) = K(E, F)$, where $E = C(S)$ and $S$ is any compact Hausdorff space.

4. **The case: $S$ is dispersed.** In this section, we study the case where $E = C(S)$, $S$ is compact Hausdorff and dispersed and $F = c_0$, $l_p$ ($1 \leq p < \infty$).

For the case where $S$ is dispersed and $F = c_0$, the following theorem obtains, the proof of which is due to A. Pełczyński and was communicated to us by him.

**THEOREM 8.** If $S$ is dispersed, then $C(S)$ contains a complemented copy of $c_0$. Thus setting $E = C(S)$ and $F = c_0$, $B(E, F)$ has a stable map and $K(E, F)$ is not complemented in $B(E, F)$.

**Proof.** Let $S^{(1)}$ denote the set of non-isolated points of $S$. Since $S$ is dispersed and of second category, $S^{(1)}$ is non-empty, compact and dispersed. Therefore we can choose a point $a \in S^{(1)}$ which is isolated in $S^{(1)}$.

Choose a closed neighborhood $U$ of $a$ in $S$ such that $U \cap S^{(1)} = \{a\}$. Since $a_0$ is not isolated in $S$ there exists a sequence $(s_n)$ of distinct points in $U$. Since $U$ is closed and, except for $s_n$, consists of isolated points of $S$, the only possible cluster point of the sequence $(s_n)$ is $s_0$. Since $S$ is compact it follows that $s_n \to s_0$.

If $c$ is the subspace of $C(S)$ consisting of all functions $f$ such that $f(S \setminus \{a_0\}) = 0$, then it is a simple verification that $c$ is complemented in $C(S)$ and that $c$ is isomorphic to $c_0$.

The case where $S$ is dispersed and $F = l_p$, $1 < p < \infty$ will be approached differently. Whereas previously we proved that $K(E, F)$ was not isomorphic to $C(E, F)$ in order to apply the Corollary to Theorem 6, we now use an argument similar to the proof of the Littlewood-Pitt
theorem [6] which asserts (as a special case) that every bounded operator from $c_0$ to $l_p$ ($1 \leq p < \infty$) is compact in order to prove that $\mathcal{K}(E, F) = B(E, F)$ for the appropriate spaces $E$ and $F$. Our result states that for the case of a compact dispersed space $E$, every bounded operator from $C(S)$ into $l_p$ is compact. Throughout the remainder of this section we assume $1 \leq p < \infty$.

Let $T: C(S) \rightarrow l_p$ be a bounded operator. Then there are measures $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ on $S$ so that:

$$T(f) = (\int f d\mu_1, \int f d\mu_2, \ldots, \int f d\mu_n, \ldots)$$

for every $f \in C(S)$. If $S$ is dispersed, every measure on $S$ is atomic. Thus, throughout this section, we will study operators $T: C(S) \rightarrow l_p$, where $S$ is compact Hausdorff and where the measures $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ are atomic. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be the points in $S$ which have non-zero mass for some $\mu_i$ and write $\mu_i = \sum c_i \delta(\xi_i)$ for appropriate scalars $c_i$ (where $\delta(\xi_i)$ denotes unit point mass at $\xi_i$). Then it follows that

$$\int f d\mu_i = \sum c_i f(\xi_i) \quad \text{for each } i \text{ and for every } f \in C(S).$$

Set $\lambda_a = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$: $\lambda_a = f(\xi_i)$ for all $i$ and for some $f \in C(S)$.

Then, $\lambda_a$ is a linear space of sequences on which we define the norm, $||\cdot|| = \sup \{|a_j|: j = 1, 2, \ldots\}$. It is easy to check that $\lambda_a$ is a Banach space under this norm. Hereafter an element $(a_1, a_2, a_3, \ldots)$ of $\lambda_a$ is denoted by $a$.

**Lemma 9.** Let $T: C(S) \rightarrow l_p$ be a bounded operator as above. Then there exist bounded operators $P: C(S) \rightarrow \lambda_a$ and $M_a: \lambda_a \rightarrow l_p$ so that $||T|| = ||M_a||$ and so that the following diagram commutes:

$$\xymatrix{ C(S) \ar[r]^T & l_p \ar@/_0.5pc/[l]_P \ar@/^0.5pc/[l]^{M_a} }$$

Proof. The mapping $P$ is defined by $P(f) = (f(\xi_1), f(\xi_2), \ldots, f(\xi_n), \ldots)$ and $M_a$ is defined as the map from $\lambda_a$ into $l_p$ satisfying

$$M_a(a) = \left( \sum c_1 a_1, \sum c_2 a_1, \ldots, \sum c_n a_1, \ldots \right) \in l_p$$

for each $a \in \lambda_a$. It is easy to check that $T = M_a \circ P$, that $||T|| = 1$ and that $||P|| = ||M_a||$.

We will show that $M_a$ is necessarily a compact operator. But since the method of proof has other applications (see Theorem 12) the special case in question, we formulate our approach abstractly in making the following definition.

**Definition.** Let $\lambda$ be a Banach space of sequences. We say $\lambda$ has the finite monotonity property (f.m.p.) if:

1. For each coordinate $i$ there is an $a \in \lambda$, $a = (a_1, a_2, \ldots, a_n, \ldots)$ with $a_i \neq 0$.

2. Whenever $a \in \lambda$, $a = (a_1, a_2, \ldots, a_n, \ldots)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots)$ is a sequence for which $\gamma_j = 0$ for $j > N$, then there is a sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_n, \ldots)$ satisfying $\beta_j = a_j \gamma_j$ for $j = 1, 2, \ldots, N$ and $||\beta|| \leq ||a|| \sup \{|\gamma_j|: j = 1, 2, \ldots\}$.

3. Whenever $\alpha = (a_1, a_2, a_3, \ldots)$ is a sequence which defines a continuous linear functional on $\lambda$ by the equation $\sigma'(\alpha) = \sum a_i \theta_i$, then $\lim \|\alpha' - a_i\| = 0$, where $\alpha' = (a_1, \ldots, a_i, 0, \ldots, 0, \ldots)$.

Note that $\lambda$ need not contain all finite sequences but rather, we require that $\lambda$ has a “norm preserving extension” of any finite sequence.

**Lemma 10.** $\lambda_a$ has the finite monotonity property.

Proof. 1 is obvious, 2 follows from the Tietze extension theorem, and 3 follows from the Riesz Representation Theorem.

Hereafter we only consider sequence spaces with f.m.p.

**Definition.** Let $\lambda$ be a Banach space of sequences. Let $M = (m_{ij})$ be a matrix. We say that $M$ is a matrix map if the equation

$$M(a) = \left( \sum_{j} c_j a_1, \sum_{j} c_j a_2, \ldots, \sum_{j} c_j a_n, \ldots \right)$$

defines a bounded linear operator from $\lambda$ into $l_p$.

**Lemma 11.** Let $M = (m_{ij}): \lambda \rightarrow l_p$ be a matrix map. Then

1. $\lambda^{(0)} = (c_1, c_2, \ldots, c_n, \ldots)$ defines, for each $i$, a continuous linear functional on $\lambda$, where $\lambda^{(i)}(a) = \sum c_j a_j$.

2. $\sum |c_j| < \infty$ for each $j$.

3. Let $I$ be a finite set of indices. Let $M_I$ denote the matrix consisting of $c_i$ if $j \in I$ and of zeros if $j \notin I$. Let $M(1-I)$ denote the matrix consisting of zeros if $j \notin I$ and of $c_i$ if $j \in I$. Then $M_I$ and $M(1-I)$ are matrix maps satisfying $||M_I|| \leq ||M||$, $||M(1-I)\| \leq ||M||$.

4. If $R_n$ is the matrix consisting of $c_i$ when $i, j \geq n$ and of zeros otherwise, then $R_n$ is a matrix map from $\lambda$ to $l_p$ satisfying $||R_n|| \leq ||M||$. (We call $R_n$ the $n$-th remainder matrix of $M$.)

5. $M$ is a compact matrix map if and only if $\lim_{n \to \infty} ||R_n|| = 0.$
Proof. 1. Clearly $\psi^0$ is the linear functional given by $\phi' \circ M$, where $\phi': L_{p} \to B$ is the continuous linear functional given by evaluation: $\phi'(a) = a$ for all $z = (a_1, a_2, \ldots, a_n, \ldots)$ in $L_p$. This proves 1.

2. The proof consists in a tedious but direct verification using part (2) in the definition of f.m.p.

3. For any $a = (a_1, a_2, \ldots, a_n, \ldots)$ and any $\alpha > 0$ we will show that there exists a $\beta \in \mathbb{C}$ of $|\beta| \leq |\alpha|$ and satisfying $\|M(\beta)\|_p = \|M\|_p \beta$. Indeed, $\beta$ will be chosen so that $\|M(\beta)\|_p \geq \|M\|_p |\beta| - \alpha/2$, for all $\alpha$. We deduce this from the fact that $M(\beta) = c_1$. Let $\beta = \max(|\beta|, \epsilon J)$. For each $N$, let $a$ be sufficiently large so that $n > \beta$ and $|\|M(\beta)\|_p - \epsilon/2| < \epsilon(1/2)$. Choose $\beta = (b, b, \ldots, b, \ldots) \in \mathbb{C}$ so that $\|\beta\| \leq |\alpha|$, $b = 0$ if $j \in J$ and $b = 0$ if $j \notin J$. Then, using an argument similar to that needed in the proof of 2 above,

$$\|M(\beta)\|_p \geq \|M(\beta_1) \cdots \beta_n\|_p \geq \|M\|_p \|\beta\|_p - \epsilon/2 \geq \|M\|_p \epsilon/2 - \epsilon/2$$

Thus $\|M\| \geq M\|_p$.

Similarly, suppose $\alpha \in \mathbb{C}$ is a unit vector so that $\|M(1 - H_j)(a)\|_p \geq \|M\|_p \delta$ where the left-hand side is possibly infinite. Then there is a sufficiently large $n$ so that

$$\\left\| \sum_{i} a_i \delta_i \right\|_p \geq \|M\|_p \|\alpha\|_p$$

We can choose a finite subset $J_\alpha$ of indices disjoint from $J$ so that

$$\\left\| \sum_{i} a_i \delta_i \right\|_p \geq \|M\|_p$$

Repeating the argument above we can find $\beta \in \mathbb{C}$ with $\|\beta\| \leq |\alpha| = 1$ with freedom small $\epsilon$. This of course is possible.

4. Observe that $R_\alpha = (1-H_j) M(1-H_j)$, where $J = \{1, 2, \ldots, n\}$ and $1-H_j$ on the left is the projection of $1$ which maps $e_i$ into $0$ if $1 \leq i \leq n$ and maps $e_i$ to $e_i$ if $i > n$. Thus

$$\|R_\alpha\| \leq \|1-H_j\|_p \leq \|M\|_p$$

5. If $M$ is compact, then Proposition 1 proves $\lim \|M - H_j M(1-H_j)\| = 0$.

Since $R_\alpha = (M - H_j M(1-H_j))$, where $\alpha = (1, 2, \ldots, n)$, this together with part (3) above shows: $\lim \|R_\alpha\| = 0$. Conversely, assume $\epsilon > 0$ is given. Since $R_\alpha = M - M(1-H_j)M(1-H_j)$, by setting $R = R_\alpha M(1-H_j)$ we can verify that $\|M - R\| \leq \|M\|_p \alpha$, which is eventually closed in $\mathbb{C}$ for each $n$. For notational convenience in the subsequent proofs, we introduce the following terminology: Let $D = (d_{ij})$ and $M = (m_{ij})$. We say that $D$ is a submatrix of $M$ if and only if $d_{ij} = 0$ implies $m_{ij} = 0$. For a given sequence of pairs of integers $1 \leq m(1) < n(1) < \ldots < m(k) < n(k) < \ldots$ by the $(m, n)$ diagonal block, denoted $B_{mn}$, of $M$ we mean the submatrix whose entries are $c_{ij}$ if $m(k) < i, j < n(k)$ and are 0 otherwise. By the $k$-th truncated diagonal block, denoted $B_{mn}$, of $M$ we mean the submatrix of $M$ consisting of $B_{mn}$, $B_{m(n+1)}$, and of zeros elsewhere.

Theorem 12. Let $\alpha = (d_{ij}: i \to j)$ be a non-compact matrix map. Then there exist indices $1 \leq m(1) < n(1) < \ldots < m(k) < n(k) < \ldots$ so that for some $\lambda > 0$, $|B_{mn}| > \lambda$ and $|D_{mn}| \geq \lambda |B_{mn}|$ for all $k = 1, 2, \ldots$.

Proof. By Part 2 of Lemma 11 there is an $\epsilon > 0$ so that $\|R_\alpha\| > \epsilon$ for all $\alpha \in \mathbb{C}$, where $R_\alpha$ is a submatrix of the positive integers. We may also assume $\|M\| > \epsilon$. We choose the sequence of pairs of integers $m(k), n(k)$ inductively as follows. Choose $m(1) = 1$. It is not difficult to show, using parts 2 and 3 of Lemma 11, that if $m(k)$ is sufficiently large, then $\epsilon > \|R_{m(k), n(k+1)}\| > \epsilon$. Let $J = 1 - H_{m(k)}$. Choose $m(k+1) > n(k)$ so that

(a) $\|R_{m(k), n(k+1)}\| > \frac{\lambda}{2} |M||$,

(b) $\|R_{m(k+1), n(k+1)}\| > \frac{\lambda}{2} |M||$,

(c) $\|R_{m(k), n(k)}\| > \epsilon$, where $J = (i: m(1) < i < n(1))$ and $J(1) = (i: i > m(2))$. This is possible by part 2 of Lemma 11. Assume now that for $1 \leq i < k - 1$ we have chosen $m(1) < n(1) < \ldots < m(k-1) < n(k-1) < m(k)$ so that

(i) $\|R_{m(k-1), n(k)}\| > \frac{\lambda}{2} |M|^{k+1}$,

(ii) $\|R_{m(k+1), n(k)}\| > \frac{\lambda}{2} |M|$,

(iii) $\|R_{m(k), n(k)}\| > \epsilon$, where $J = (i: m(1) < i < n(1))$ and $J(1) = (i: i > m(1+1))$. Since $\|R_{m(k), n(k)}\| > \epsilon$ and since $J$ has a Schauder basis, we can find a sufficiently large $i$ so that if $m(k+1) = (1, 2, \ldots, n(k))$, then $\|R_{m(k+1), n(k+1)}\| > \epsilon$ for any $J$ with $J < J$. Repeating an argument similar to that used in the proof of Part 3 of Lemma 11, we can find a sufficiently large $n(k)$ so that if $n(\alpha) = (1, 2, \ldots, n(k))$, then $\|R_{m(\alpha), n(\alpha)}\| > \epsilon$ for any $\alpha$ with $\alpha < J$. Thus $J_n = J < J_n = J'$.
Choose \( n(k) \) satisfying \( m(k) \leq n(k) \) and so that \( J_N = J_{m(k)} \), \( J_m = J_{n(k)} \). Then if \( B_k = \text{I}_{m(k)}R_k\text{I}_{n(k)} \), we have \( \|B_k\| > \varepsilon \). To see that \( \|B_k\| \leq 2\|M\| \) apply (i) and (ii) of the induction hypothesis above.

**Theorem 13.** Let \( M = (a_{ij}) \); \( \lambda_i \to l_p \) be a matrix map. Then \( M \) is compact.

**Proof.** If \( M \) is not compact, then by Proposition 12 there is an \( \varepsilon > 0 \) and indices \( 1 = m(1) < m(2) < \cdots < m(k) \leq n(k) < \cdots \) so that \( \|B_k\| > \varepsilon \) and \( \|D_k\| \leq 2\|M\| \) for all \( k \). Let \( d^{(1)}, d^{(2)}, \ldots, d^{(k)} \) be unit vectors in \( \lambda_i \) satisfying \( \|B_k(d^{(i)})\| = \|B_k\| \). This choice is possible because each \( B_k \) is a finite matrix. Let \( \xi = (\eta_1, \eta_2, \ldots, \eta_l) \). Clearly \( \xi \) is a unit vector of \( \lambda_i \). By f.m.p. we may choose \( \beta^{(k)} \xi \), where \( \beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_l^{(k)}) \) satisfies:

\[
\beta_1^{(k)} = \begin{cases} d^{(i)} & \text{if } m(k) \leq i \leq n(k), \\ 0 & \text{if } n(k) < i < m(k+1), \end{cases}
\]

where \( k = 1, 2, \ldots, t \), and \( \|\beta^{(k)}\| \leq \|\xi\| = 1 \).

Thus, \( \|D_k(\beta^{(k)})\| = \|B_k(\beta^{(k)})\| = \|B_k\| \).

A contradiction now results because \( 2\|M\| \geq \|\xi\| \geq \|D_k(\beta^{(i)})\| = \|B_k\| = \varepsilon \), where \( t \) is arbitrary.

Remark. It is not hard to check that the \( l_p \) spaces satisfy f.m.p. if \( 1 < p < \infty \) and that \( \xi_0 \) also satisfies f.m.p. Arguing as in Theorem 13 we may apply Theorem 12 to show that there are no non-compact operators \( T : l_p \to l_p \) or \( T : \xi_0 \to \xi_0 \), if \( 1 < p < q < \infty \). This is the Littlewood-Pitt Theorem [9].

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**Factorable and strictly singular operators, I**

by

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Let \( X \) and \( Y \) be Banach spaces; a linear bounded operator \( T : X \to Y \) is strictly singular [5] if the subspaces \( Z \subseteq X \) for which the restriction \( T|_Z : Z \to T(Z) \) has a bounded inverse \( T(Z) \to Z \) are necessarily finite dimensional. Assume \( X \) is the cartesian product \( X = E_1 \times F_2 \) of the Banach spaces \( E_1 \) and \( E_2 \). An operator \( T : X \to X \) is factorable through \( E_1 \) (for \( i = 1, 2 \)) if it is of the form \( T = SQ \) for suitable \( Q : X \to E_i \) and \( S : E_i \to X \). Denote by \( a \) the (uniform) closure of the set of operators factorable through \( E_1 \). We prove below that under special conditions, an operator \( T \) is strictly singular if and only if \( T \cap a \). This is applied notably to the case \( E_1 = l_p \) and \( E_2 = l_q \) for \( p \neq q \) (and more particularly, if \( p > q = 2 \)) to obtain various relations between these types of operators. As a consequence, it can be seen ([(3.1. e) below]) that in this case, if \( T_1 \) and \( T_2 \) are strictly singular, then \( T_1 \cap T_2 \) is compact.

Finally, the lattice of all closed two-sided ideals of the algebra of all bounded linear operators on \( X = l^p \times l^q \), for \( p > 2 \), is studied in (3.4): it consists of six elements \( 0, \mathfrak{F}, \ell, \mathfrak{I}, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B} \) ordered as follows: