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Existence and uniqueness of the solution to the critical problem in neutron transport theory

by

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I. Introduction. The critical problem in the neutron transport theory is usually formulated in the following way: How much should one increase (in the fictitious way) the number of secondary neutrons per fission to achieve the stationary neutron distribution in a given system? In other words, the fission neutron source is multiplied by $1/k$, where k is the so-called effective multiplication factor of the system and the positive value of k corresponding to the non-negative stationary solution is sought. The system is just critical if such a value of k is equal to unity.

The existence and uniqueness of the solution to the critical problem has been investigated for the transport equation by Shikhov and Shishkov [6]–[8] under a number of restrictive assumptions. In particular, they assumed that the scattering kernel is bounded for all finite velocities. At the same time, it is known that the diffusive model of scattering in liquids leads to a singular behavior of the scattering kernel.

In this paper it is proposed an approach to the critical problem different from that employed by Shikhov and Shishkov. It is based upon the theory of semigroups investigated in the neutron transport theory in connection with the time-dependent problems. In such a way the assumptions concerning the scattering kernel are less restrictive and, at the same time, the analysis becomes much simpler.

The neutron angular distribution is assumed to be square summable with respect to space and velocity variables. The reason for such assumption is that the analysis of the time dependent transport equation for square summable functions is almost complete so one can easily transfer the results to the stationary case.

For the sake of simplicity the homogeneous system is considered. The extension to heterogeneous systems is straightforward.

II. Formulation of the problem. The transport equation describing the stationary neutron distribution in a finite homogeneous convex body surrounded by vacuum has the following operational form:

$$T\Psi+K\Psi+\frac{1}{k}F\Psi=0. \tag{2.1}$$

The neutron distribution $\Psi(\vec{r}, \vec{v})$ depends on the position vector $\vec{r}$ and velocity vector $\vec{v}$. The operators in equation (2.1) are, successively, the streaming operator

$$(T\Psi)(\vec{r}, \vec{v}) = -\vec{v} \cdot \operatorname{grad}_{\vec{r}} \Psi(\vec{r}, \vec{v}) - v\sigma(v)\Psi(\vec{r}, \vec{v}), \tag{2.2}$$

where $v = |\vec{v}|$ and $\sigma(v)$ is total macroscopic cross section; the scattering operator

$$(K\Psi)(\vec{r}, \vec{v}) = \int_{\omega} d_3\vec{v}' K(\vec{v}, \vec{v}')\Psi(\vec{r}, \vec{v}'), \tag{2.3}$$

where

$$K(\vec{v}, \vec{v}') = \sqrt{\frac{m(v')}{m(v)}}\, v'\sigma_s(v' \to \vec{v}), \tag{2.4}$$

is the scattering kernel symmetrized by the Maxwelian $m(v) = e^{-v^2}$; and, finally, the fission operator including prompt and delayed neutrons

$$(F\Psi)(\vec{r}, \vec{v}) = \int_{\omega} d_3\vec{v}' F(v, v')\Psi(\vec{r}, \vec{v}'), \tag{2.5}$$

where the isotropic fission kernel is given by

$$F(v, v') = \frac{1}{4\pi}\beta(v)v'\sigma_f(v')\sqrt{\frac{m(v')}{m(v)}}. \tag{2.6}$$

Here $\beta(v)$ is the fission spectrum normalized to the total number of secondary neutrons per fission and $\sigma_f(v)$ is the fission cross-section.

The scattering kernel defined by equation (2.4) is symmetric only if the inelastic scattering on heavy atoms is absent in the system. It seems that the inclusion of such scattering would not lead to any serious difficulties. However, it would require the extension of the results obtained so far in the time-dependent transport theory and will not be considered in this paper.

The symmetrization of the scattering kernel $v'\sigma_s(\vec{v}' \to \vec{v})$ leads to the inconvenient behavior of the fission kernel for large v since $\beta(v)/\sqrt{m(v)}$ diverges at infinity. This is the reason that the upper bound of neutron velocities will be assumed so that the integration with respect to v is performed in equations (2.3) and (2.5) over the sphere ω with the finite radius, say, v_M.

The complex-valued function $\Psi(\vec{r}, \vec{v})$ will be defined and square summable over the domain $V \times \omega$, where V is the volume of the body.

Thus Ψ is assumed to belong to a complex Hilbert space H with the following scalar product and norm:

$$(f, g) = \int_V d_3\vec{r} \int_{\omega} d_3\vec{v} f(\vec{r}, \vec{v})\overline{g(\vec{r}, \vec{v})}; \qquad \|f\| = \sqrt{(f, f)}. \tag{2.7}$$

The operator K is bounded and defined for all $f \in H$ for all practical models of scattering in moderators. The proof is given by Borysiewicz and Mika [2]. The same will be obviously true for the operator F if the upper bound for neutron velocities is assumed since both $\beta(v)/\sqrt{m(v)}$ and $v\sigma_f(v)\sqrt{m(v)}$ are bounded for finite v.

The operator T is closed and densely defined. Its domain $D(T)$ consists of all functions $\Psi(\vec{r}, \vec{v})$ admitting the directional derivative along $\vec{v}$, satisfying the boundary condition that no neutrons are entering the system from the outside and such that $T\Psi = H$. The detailed analysis of the operator T is given by Vidav [9].

It is obvious that the solution to equation (2.1) will be sought among the functions $\Psi(\vec{r}, \vec{v})$ belonging to the domain of T since the remaining operators K and F do not imply any restrictions upon $\Psi(\vec{r}, \vec{v})$ and $D(A) = D(T)$.

III. Reduction of the critical problem to the eigenvalue problem. The important step in the analysis of equation (2.1) is to reduce it to the eigen, value equation, preferably for the compact operator. This task can be carried out with the help of the results obtained so far in the time-dependent neutron thermalization theory based upon the theory of semigroups in the Banach space.

The following theorem of Hille and Yosida [3] is applied: If the operator M is closed and densely defined in the Banach space E, its residual spectrum is empty and the inequality

$$\|(\lambda I - M)f\| \geqslant (\lambda - \gamma)\|f\| \tag{3.1}$$

holds for all $f \in D(M)$ and some real γ and every $\lambda > \gamma$, then M is an infinitesimal generator of a strongly continuous semigroup of operators $\{G_M(t)\}$ uniformly bounded on E. The semigroup $\{G_M(t)\}$ admits an estimate:

$$\|G_M(t)\| \leqslant e^{\gamma t}; \qquad t \geqslant 0. \tag{3.2}$$

The resolvent operator $R(\lambda; M) = (\lambda I - M)^{-1}$ exists and is bounded and defined everywhere for $\operatorname{Re}\lambda > \gamma$. If $\{G_M(t)\}$ is a semigroup of positive operators, then $R(\lambda, M)$ is also a positive operator for $\lambda > \gamma$. The operator in a Banach function space is positive if it leaves invariant the cone C of non-negative functions.

The resolvent operator and the semigroup are related to each other by the formula

$$R(\lambda, M)f = \int_0^\infty dt e^{-\lambda t} G_M(t)f; \quad f \in E; \ \mathrm{Re}\lambda > \gamma. \tag{3.3}$$

Consider first the operator T. Its properties depend on the behavior of the function $v\sigma(v)$ representing the total collision rate. It will be assumed that

$$\begin{aligned} &\text{(i)} && 0 < v\sigma(v) < \infty, \quad v \in [0, v_M]; \\ &\text{(ii)} && \inf_{v \in [0, v_M]} v\sigma(v) = \lim_{v \to 0} v\sigma(v) = \lambda^* > 0. \end{aligned} \tag{3.4}$$

Conditions (3.4) are satisfied in all physically important cases.

LEMMA 1. *If conditions* (3.4) *are satisfied, then the operator T defined by equation* (2.2) *in the Hilbert space H is an infinitesimal generator of a strongly continuous semigroup of positive operators $\{G_T(t)\}$. The semigroup is given by the formula:*

$$(G_T(t)f)(\vec{r}, \vec{v}) = \begin{cases} f(\vec{r} - \vec{v}t, \vec{v}) e^{-v\sigma(v)t}; & \vec{r} - \vec{v}t \in V; \\ & \qquad t \geqslant 0; \\ 0; & \vec{r} - \vec{v}t \notin V; \quad f \in H. \end{cases} \tag{3.5}$$

The semigroup $\{G_T(t)\}$ is uniformly bounded:

$$\|G_T(t)\| \leqslant e^{-\lambda^* t}; \quad t \geqslant 0. \tag{3.6}$$

The resolvent operator $R(\lambda, T) = (\lambda I - T)^{-1}$ exists for $\mathrm{Re}\lambda > -\lambda^*$ *and is given by*

$$\big((\lambda I - T)^{-1} f\big)(\vec{r}, \vec{v}) = \int_0^{s(\vec{r}, \vec{v})} ds' e^{-s'(\lambda + v\sigma(v))} f(\vec{r} - \vec{v}s', \vec{v}); \qquad f \in H; \ \mathrm{Re}\lambda > -\lambda^*; \tag{3.7}$$

where $s(\vec{r}, \vec{v})$ is the time required for a neutron to travel from the boundary to the point $\vec{r}$ in the direction of $\vec{v}$ with velocity v. In particular, $(-T)^{-1}$ exists as a bounded positive and defined everywhere operator on H.

Proof of Lemma 1 is easily obtained by the application of the Hille-Yosida theorem. The details can be found in the paper of Vidav [9].

LEMMA 2. *If the operator K is bounded, then the operator $A = T + K$ is an infinitesimal generator of a strongly continuous semigroup of operators $\{G_A(t)\}$ given for $t \geqslant 0$ and $f \in H$ by the perturbation series:*

$$G_A(t)f = \sum_{k=0}^{\infty} G_A^{(n)}(t)f; \quad G_A^{(0)}(t)f = G_T(t)f; \tag{3.8}$$

$$G_A^{(n)}(t)f = \int_0^t ds\, G_T(t-s) K G_A^{(n-1)}(s)f; \quad n \geqslant 1.$$

The series in equation (3.8) *converges absolutely and uniformly in t for each finite interval $[0, \tau]$, where $\tau > 0$. The semigroup $\{G_A(t)\}$ is uniformly bounded*

$$\|G_A(t)\| \leqslant e^{(-\lambda^* + \|K\|)t}; \quad t \geqslant 0. \tag{3.9}$$

Proof of Lemma 2 can be easily given by observing that the addition of the bounded operator to T does not change any of its properties required by the Hille-Yosida theorem. The series in equation (3.9) is obtained by the usual perturbation procedure [3]. The estimate (3.9) follows from (3.6) and (3.8).

It is seen from equation (3.3) together with (3.9) that the resolvent of A exists for $\mathrm{Re}\lambda > -\lambda^* + \|K\|$ and for $\lambda > -\lambda^* + \|K\|$ is a positive operator. However, this result is not very practical since if the norm of K is greater than λ^*, then the existence of $R(0, A) = (-A)^{-1}$ does not follow from the above analysis. Thus it is very important to find another estimate for the semigroup $\{G_A(t)\}$.

The total collision rate $v\sigma(v)$ consists of the absorption rate (including fission) and the scattering rate:

$$v\sigma(v) = v\sigma_a(v) + v\sigma_s(v). \tag{3.10}$$

It will be assumed that the minimum value of $v\sigma_a(v)$ is greater than zero:

$$\lambda_a^* = \inf_{v \in [0, v_M]} v\sigma_a(v) > 0. \tag{3.11}$$

Physically, it means that neutrons are absorbed in the system at all velocities.

The scattering rate $v\sigma_s(v)$ is connected with the scattering kernel $v'\sigma_s(\vec{v}' \to \vec{v})$ and with the symmetrized kernel $K(\vec{v}, \vec{v}')$ by the formula

$$v\sigma_s(v) = \int_\omega d_3\vec{v}'\, v\sigma_s(\vec{v} \to \vec{v}') = \int_\omega d_3\vec{v}'\, K(\vec{v}, \vec{v}') \sqrt{\frac{m(v')}{m(v)}}. \tag{3.12}$$

Lemma 3. *The semigroup* $\{G_A(t)\}$ *admits the estimate*

$$\|G_A(t)\| \leqslant e^{-\lambda_a^* t}; \qquad t \geqslant 0. \tag{3.13}$$

Proof of Lemma 3. Consider for real s and $f \in D(T)$ the expression:

$$\begin{aligned} &\|(\lambda I - T - K)f\| \cdot \|f\| \\ &\geqslant |((\lambda I - T - K)f, f)| \geqslant \operatorname{Re}((\lambda I - T - K)f, f) \\ &= \lambda\|f\|^2 + \int_V d_3\vec{r} \int_\omega d_3\vec{v}\ \vec{v} \cdot \operatorname{grad}_r |f(\vec{r}, \vec{v})|^2 + \int_V d_3\vec{r} \int_\omega d_3\vec{v}\, v\sigma_a(v)\, |f(\vec{r}, \vec{v})|^2 + \\ &\quad + \int_V d_3\vec{r} \int_\omega d_3\vec{v} \Big(v\sigma_s(v)\, |f(\vec{r}, \vec{v})|^2 - \int_\omega d_3\vec{v}\,' K(\vec{v}, \vec{v}\,') f(\vec{r}, \vec{v}\,') \overline{f(\vec{r}, \vec{v})} \Big). \end{aligned} \tag{3.14}$$

The second term in RHS is always non-negative which can be seen by the application of the Gauss theorem and the boundary condition satisfied by $f \in D(T)$. The same can be shown to be true for the last term in RHS after making use of equation (3.12) and the symmetry of $K(\vec{v}, \vec{v}\,')$. The details of the derivation are given by Kuščer and Vidav [5]. Finally, it follows from equation (3.14) that for real λ and $f \in D(T)$

$$\|(\lambda I - A)f\| \geqslant (\lambda + \lambda^*)\, \|f\|. \tag{3.15}$$

The comparison of equation (3.15) with (3.1) and (3.2) shows that the semigroup $\{G_A(t)\}$ admits an estimate given by equation (3.13).

Lemma 4. *The operator* $R(0, A) = (-A)^{-1}$ *exists as a bounded positive operator on* H.

Proof of Lemma 4 follows immediately from equations (3.3), (3.8), (3.13) and (3.11).

Lemma 5. *The operator* $(-T)^{-1} F$ *is compact.*

Proof of Lemma 5 can be obtained by the use of the theorem given by Borysiewicz and Mika [2] who have shown that if a certain operator, say M, is compact on a subspace S of H containing functions dependent only on the velocity $\vec{v}$ and $M(S) \subset S$, then the operator $(\lambda I - T)^{-M}$ is compact on H for $\lambda > -\lambda^*$. Now F is obviously compact on S since its kernel is bounded and the range of integration is finite. This completes the proof.

Theorem 1. *The critical problem given by equation* (2.1) *is equivalent to the eigenvalue problem*

$$K\Psi = B\Psi \tag{3.16}$$

for the compact positive operator B *given by the formulas*:

$$\begin{aligned} Bf &= (-T-K)^{-1}Ff = (I - (-T)^{-1}K)^{-1}(-T)^{-1}Ff \\ &= (-T)^{-1}(I - K(-T)^{-1})^{-1}Ff; \qquad f \in H. \end{aligned} \tag{3.17}$$

The eigenfunctions of B *necessarily belong to the domain of* T.

Proof of Theorem 1. The existence of the operators $(-T)^{-1}$ and $(I - (-T)^{-1}K)^{-1}$ follows from Lemmas 1 and 4. The compactness of B follows from Lemma 5. The third form of B given by equation (3.17) shows that the range of B consists of the functions belonging to the domain of T. The operator B is positive since F is positive and $(-A)^{-1} = (-T-K)^{-1}$ is positive.

IV. Existence of the non-negative solution to the critical problem. The existence of a non-negative eigenfunction of the operator B corresponding to a positive eigenvalue k being, at the same time, a solution to the critical problem given by equation (2.1), will be proved by making use of the Krein-Rutman theorem [4]:

Let G be a positive compact operator on a Banach function space E with a cone C of non-negative functions. Let for some $g \in C$ the function $Gg^p - ag \in C$ for some $a > 0$ and natural p. Then there exists $g_0 \in C$ such that $\lambda_0 g_0 = Gg_0$ and $\lambda_0 \geqslant a^{1/p}$ is no smaller than the modulus of any other eigenvalue. Similarly, there exists g_0^* such that $\lambda_0 g_0^* = G^* g_0^*$, where G^* is an operator adjoint to G.

Lemma 6. *The operator* B *can be written in the form*

$$Bf = (-T)^{-1}Ff + B_1 f; \qquad f \in H; \tag{4.1}$$

where B_1 *is a positive compact operator.*

Proof of Lemma 6. From equation (3.3) written for A and $\lambda \doteq 0$ it follows that for $f \in H$ and $\tau > 0$

$$(-A)^{-1}f = \int_0^\infty dt G_A(t) f = \int_0^\tau dt G_A(t) f + \int_\tau^\infty dt G_A(t) f. \tag{4.2}$$

The perturbation series representing $G_A(t)$ in equation (3.8) is absolutely and uniformly convergent in the interval $[0, \tau]$. Thus it may be integrated term by term in this interval to give for $f \in H$

$$(-A)^{-1}f = \lim_{\tau\to\infty} \Big\{ \int_0^\tau dt G_T(t) f + \sum_{n=1}^\infty \int_0^\tau dt G^{(n)}(t) f + \int_\tau^\infty dt G_A(t) f \Big\}. \tag{4.3}$$

Since

$$\lim_{\tau\to\infty} \int_0^\tau dt G_T(t) f = (-T)^{-1} f; \qquad f \in H;$$

equation (4.3) gives

$$(-A)^{-1}f = (-T)^{-1}f + R_1 f; \qquad f \in H; \tag{4.4}$$

where the operator

$$R_1 f = \lim_{\tau\to\infty}\Big\{\sum_{n=1}^{\infty}\int_0^{\tau} dt G^{(n)}(t)f + \int_{\tau}^{\infty} dt G_A(t)f\Big\} \tag{4.5}$$

is positive and bounded and defined for all $f \in H$. Thus the lemma is seen to be true with $B_1 = R_1 F$.

LEMMA 7. *The operator $(-T)^{-1}F$ on H satisfies the requirements of the Krein-Rutman theorem with $p = 1$.*

Proof of Lemma 7. As the function g it will be taken $\chi_\eta \in H$, the characteristic function of the set $\eta \subset V\times\omega$, such that $0 < v_0 \leqslant v \leqslant \leqslant v_1 \leqslant v_M$ and $\inf_{\vec{r}\in\eta,\, \vec{r}_0\in\Gamma} |\vec{r}-\vec{r}_0| \geqslant \delta > 0$ if Γ is the boundary of V. The velocities v_1 and v_1 are chosen so that $\sigma_f(v) > 0$ and $\beta(v) > 0$ for $v_0 \leqslant \leqslant v \leqslant v_1$. Using equation (4.1) and the explicit expressions for $(-T)^{-}$ and F given in equations (3.7) and (2.2) one obtains the inequality

$$(-T)^{-1}F\chi_\eta \tag{4.6}$$

$$= \int_0^{s(\vec{r},\vec{v})} ds' e^{-s'v\sigma(v)} \frac{\beta(v)}{\sqrt{m(v)}} \int_0^{v_M} v'^3 dv' \sigma_f(v')\sqrt{m(v')}\, \chi_\eta(\vec{r}-\vec{v}s', v')$$

$$= \frac{\beta(v)}{v\sqrt{m(v)}} \int_0^{vs(\vec{r},\vec{v})} ds'' e^{-s''\sigma(v)} \int_0^{v_M} v'^3 dv' \sigma_f(v')\sqrt{m(v')}\, \chi_\eta\Big(\vec{r}-\frac{\vec{v}}{v}s'', v'\Big)$$

$$\geqslant \frac{\delta\beta(v)}{v\sqrt{m(v)}} e^{-d\sigma(v)} \int_{v_0}^{v_1} v'^3 dv' \sigma_f(v')\sqrt{m(v')}$$

in which d is the maximum chord drawn inside the body. Write

$$a_1 = \inf_{v_0\leqslant v\leqslant v_1} \beta(v)/v\sqrt{m(v)} \quad \text{and} \quad a_2 = \inf_{v_0\leqslant v\leqslant v_1} \sqrt{m(v)}\,\sigma_f(v).$$

The choice of v_0 and v_1 implies that both parameters a_1 and a_2 are positive. Since $v\sigma(v)$ is bounded in ω, one has $a_3 = \sup_{v_0\leqslant v\leqslant v_1} \sigma(v) < \infty$. Now equation (4.6) yields

$$(-T)^{-1}F\chi_\eta \geqslant \tfrac{1}{4}\,\delta a_1 e^{-da_3}(v_1^4 - v_0^4)\, a_2 \chi_\eta = \mu\chi_\eta, \tag{4.7}$$

where $\mu > 0$. Thus it is seen that the operator $(-T)^{-1}F$ satisfies the requirements of Theorem 2 with $a = \mu$, $g = \chi_\eta$, and $p = 1$.

LEMMA 8. *The operator B has a non-negative eigenfunction in H, say Ψ_0, for the positive eigenvalue k_0 which is not smaller than the modulus of any other eigenvalue. Similarly, the adjoint operator B^* has for k_0 non-negative eigenfunction $\Psi_0{}^*$.*

Proof of Lemma 8 follows immediately from Lemmas 6 and 7 and the Krein-Rutman theorem.

THEOREM 2. *The critical problem (2.1) has a non-negative solution Ψ_0 and the eigenvalue k_0 represents the effective neutron multiplication factor. The value of k_0 is no smaller than the modulus of any other value of k for which equation (2.1) has a solution.*

Proof of Theorem 3 follows from Lemma 8 and Theorem 1.

V. Uniqueness of the non-negative solution to the critical problem. The uniqueness of the non-negative solution to the critical problem will be proved under some more restricted physical conditions.

It will be assumed that for $k > 0$ and almost all $\vec{v}, \vec{v}' \in \omega$

$$K(\vec{v}, \vec{v}') + \frac{1}{k}F(\vec{v}, \vec{v}') > 0. \tag{5.1}$$

Condition (5.1) is satisfied in all practical reactor systems.

Introduce the operators

$$Q(k)f = \Big(T + K + \frac{1}{k}F\Big)f; \quad f \in \Delta(T) \tag{5.2}$$

and

$$S(k)f = (-T)^{-1}\Big(K + \frac{1}{k}F\Big)f; \quad f \in H. \tag{5.3}$$

LEMMA 9. *If the operator B has an eigenvalue k, then $Q(k)$ has an eigenvalue zero and $S(k)$ an eigenvalue unity. The converse is also true. The corresponding eigenfunctions coincide.*

Proof of Lemma 9 follows directly from the definition of the operators $Q(k)$ and $S(k)$.

The operator $Q^*(k)$ adjoint to $Q(k)$ is defined as

$$Q^*(k)f = \Big(T^* + K + \frac{1}{k}F^*\Big)f; \quad f \in D(T^*), \tag{5.4}$$

where

$$(T^*f)(\vec{r}, \vec{v}) = \vec{v}\cdot\operatorname{grad}_{\vec{r}} f(\vec{r}, \vec{v}) - v\sigma(v)f(\vec{r}, \vec{v}); \quad f \in D(T^*), \tag{5.5}$$

and

$$(F^*f)(\vec{r}, \vec{v}) = \int_\omega d_3\vec{v}' F(v', v) f(\vec{r}, \vec{v}'); \quad f \in H. \tag{5.6}$$

The domain of $Q^*(k)$ is identical with $D(T^*)$ and consists of all functions $f \in H$, admitting the directional derivative along $\vec{v}$, satisfying the boundary condition that no neutrons are leaving the system and such that $T^* f \in H$.

Similarly, one can define an operator

$$S^*(k) f = \left(K + \frac{1}{k} F^*\right)(-T^*)^{-1} f, \qquad f \in H, \tag{5.7}$$

adjoint to $S(k)$.

LEMMA 10. *If B^* has an eigenvalue k with an eigenfunction Ψ^*, then $Q^*(k)$ has an eigenvalue zero with the same eigenfunction and $S^*(k)$ has an eigenvalue unity with an eigenfunction*

$$\tilde{\psi}^* = \left(K + \frac{1}{k} F^*\right) \Psi^*.$$

The converse is also true.

Proof of Lemma 10 follows directly from the definitions of the operators $Q^*(k)$ and $S^*(k)$.

LEMMA 11. *If condition* (5.1) *is satisfied, then any non-negative eigenfunction of B or B^* is positive almost everywhere.*

Proof of Lemma 11 follows from condition (5.1) and the properties of the operator $(-T)^{-1}$ given explicity by equation (3.7). It is seen that $(-T)^{-1} f > 0$ for almost all $\vec{r} \in V$ whenever $f > 0$ on a set of a positive measure in V.

LEMMA 12. *There are no non-negative eigenfunctions of B for $k \neq k_0$ if condition* (5.1) *is satisfied.*

Proof of Lemma 12. Assume that, besides k_0, there is for B another eigenvalue k_1 for which there exists a non-negative eigenfunction. Two following equations are satisfied:

$$\left(T^* + K + \frac{1}{k_0} F^*\right) \Psi_0^* = 0; \qquad \left(T + K + \frac{1}{k_1} F\right) \Psi_1 = 0. \tag{5.8}$$

By taking the scalar products and subtracting the obtained relations from each other, one gets

$$(k_0 - k_1)(\Psi_0^*, F\Psi_1) = 0. \tag{5.9}$$

It follows from Lemma 11 that Ψ_0^* is positive almost everywhere. Then, from equation (5.9), it follows that $F\Psi_1 = 0$ almost everywhere which, in turn, shows by the second of equations (5.8) that $(T+K)\Psi_1 = 0$ almost everywhere and Ψ_1 is an eigenfunction of the operator A corresponding to the eigenvalue zero. This is, however, in a contradiction with Lemma 4. Thus it is seen that the assumption that B has a non-negative eigenfunction for an eigenvalue k_1 different from k_0 leads to a contradiction.

LEMMA 13. *The eigenvalue k_0 is simple and the eigenfunction Ψ_0 is defined uniquely up to a multiplication constant if condition* (5.1) *is satisfied.*

Proof of Lemma 12 follows closely the argument used by Vidav [9] in a slightly different physical situation. Suppose, contrary to Lemma 13, that for k_0 there exists another eigenfunction, say φ_0. One can always take a linear combination $\tilde{\varphi}_0 = \varphi_0 + \beta\Psi_0$ such that $(\tilde{\Psi}_0^*, \tilde{\varphi}_0) = 0$, where $\tilde{\Psi}_0^*$ is the eigenfuction of $S^*(k)$. This is always possible since $(\tilde{\Psi}_0^*, \Psi_0) > 0$. The latter follows from the fact that both $\tilde{\Psi}_0^*$ and Ψ_0 are positive almost everywhere by Lemma 11. It is seen now that $\tilde{\varphi}_0$ is neither positive nor negative and $|\tilde{\varphi}_0(\vec{r}, \vec{v})| > \varphi_0(\vec{r}, \vec{v})$ on a set of positive measure. Since $S(k_0)$ is an integral operator with a positive kernel and $\tilde{\varphi}_0 = S(k_0)\tilde{\varphi}_0$, one gets

$$|\tilde{\varphi}_0(\vec{r}, \vec{v})| < \big(S(k_0)|\tilde{\varphi}_0|\big)(\vec{r}, \vec{v})$$

on a set of positive measure.

From this it is seen that

$$0 < (\tilde{\Psi}_0^*, |\tilde{\varphi}_0|) < \big(\tilde{\Psi}_0^*, S(k_0)|\tilde{\varphi}_0|\big) = \big(S^*(k_0)\tilde{\Psi}_0\, |\tilde{\varphi}_0|\big) = (\tilde{\Psi}_0^*, |\tilde{\varphi}_0|),$$

which is a contradiction. This shows that $\tilde{\varphi}_0$ has to be identically equal to zero and φ_0 proportional to Ψ_0.

It has been shown in Lemma 8 that the eigenvalue k_0 is not smaller than the modulus of any other eigenvalue. This result may be strengthened by modifying the approach applied by Borysiewicz [1] in a differenq physical situation.

LEMMA 14. *There are no eigenvalues of B on the circle $|k| = k_0$ if condition* (5.1) *is satisfied.*

Proof of Lemma 14. Suppose, contrary to the above conjecture, that there exists an eigenvalue k_1 of B corresponding to an eigenfunction Ψ_1 and such that $|k_1| = k_0$. From the equation $k_1\Psi_1 = B\Psi_1$ it follows that

$$|k_1|\,|\Psi_1(\vec{r}, \vec{v})| = |(B\Psi_1)(\vec{r}, \vec{v})| \leqslant (B|\Psi_1|)(\vec{r}, \vec{v}). \tag{5.10}$$

From equation (5.8) one gets

$$\begin{aligned} k_0(\Psi_0^*, |\Psi_1|) &= |k_1| \cdot (\Psi_0^*, |\Psi_1|) \leqslant (\Psi_0^*, B|\Psi_1|) \\ &= (B^*\Psi_0^*, |\Psi_1|) = k_0(\Psi_0^*, |\Psi_1|). \end{aligned} \tag{5.11}$$

Since the function Ψ_0^* is positive almost everywhere, it follows from equation (5.11) that

$$k_0|\Psi_1| = B|\Psi_1| \tag{5.12}$$

almost everywhere. Thus $|\Psi_1|$ is an eigenfunction of B corresponding to k_0. Hence it is necessarily proportional to Ψ_0 and Ψ_1 can be written after omitting the arbitrary constant, in the form

$$\Psi_1(\vec{r}, \vec{v}) = \Psi_0(\vec{r}, \vec{v})\, e^{i\theta(\vec{r}, \vec{v})}. \tag{5.13}$$

Let $k_1 = k_0 e^{i\gamma}$; then the eigenvalue equations $k_1 \Psi_1 = B\Psi_1$ and $k_0\Psi_0 = B\Psi_0$ give

$$k_0 e^{i(\gamma+\theta)} \Psi_0 = B\Psi_1 = e^{i(\gamma+\theta)} B\Psi_0. \tag{5.14}$$

The last equation can be also written in the form

$$B\{\Psi_0(\vec{r}', \vec{v}')\,(1-e^{i(\gamma+\theta(\vec{r},\vec{v})-\theta(\vec{r}',\vec{v}'))})\} = 0. \tag{5.15}$$

Consider the real part of LHS of equation (5.15):

$$B\{\Psi_0(\vec{r}', \vec{v}')\,\big(1-\cos(\gamma+\theta(\vec{r},\vec{v})-\theta(\vec{r}',\vec{v}'))\big)\} = 0. \tag{5.16}$$

Since the kernel of B and Ψ_0 is positive almost everywhere, it follows from equation (5.16) that the non-negative function

$$1-\cos\big(\gamma+\theta(\vec{r},\vec{v})-\theta(\vec{r}',\vec{v}')\big) = 0 \tag{5.17}$$

almost everywhere. This is possible only if $\theta(\vec{r},\vec{v}) = 0$ almost everywhere and $\gamma = 0$. This implies, however, that $k_1 = k_0$ and k_0 is the only eigenvalue on the circle $|k| = k_0$ with the eigenfunction Ψ_0.

The results of this chapter, together with those obtained in the previous one, may be summarized in the following.

MAIN THEOREM. *If the reactor system satisfies condition* (5.1), *then the critical problem has always a positive solution unique in a cone of non-negative functions in the Hilbert space H for a positive value of the effective neutron multiplication factor k_0. The solution for k_0 is unique up to a multiplication constant. The modulus of any other value of the parameter k for which the critical problem has a solution is less than k_0.*

The results obtained in the paper not only prove that the positive solution to the critical problem exists, but also show that is has properties which allow to perform successfully the numerical computations. These properties are: the uniqueness of the solution and the fact that k_0 is the only eigenvalue on the circle $|k| = k_0$.

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