PROBLEM 1. Is the negation of the condition given by (4.18) also a necessary condition for z being an $\mathcal{LC}$-singular element?

We conjecture that the answer is in positive.

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Semi-continuous linear lattices

by

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In [1] an element x of a linear lattice L is called *normalable* if $L = \{x\}^{\perp\perp} \oplus \{x\}^{\perp}$. It can be shown (see [1], Theorem 6.14, p. 28) that if a linear lattice L is sequentially continuous, then every element of L is normalable. In this paper a linear lattice L is said to be *semi-continuous* if each element of L is normalable.

If every uniform Cauchy sequence in a linear lattice L is order convergent in L, then L is said to be *complete for uniform convergence*. A sequence $x_\nu \in L$ $(\nu = 1, 2, 3, \ldots)$ is called a *uniform Cauchy sequence* if there is $0 \leqslant k \in L$ such that, for each $\varepsilon > 0$, there exists $\nu(\varepsilon)$ for which $\mu, \nu \geqslant \nu(\varepsilon)$ implies $|x_\mu - x_\nu| \leqslant \varepsilon k$. According to Theorems 6.14, 3.3 of [1], every sequentially continuous linear lattice is semi-continuous and complete for uniform convergence. The converse of this statement is the main result (Theorem 4) of this paper. It is also shown that every Banach lattice is complete for uniform convergence. One can then apply these results to show that a Banach lattice is sequentially continuous iff it is semi-continuous.

Much of this paper makes use of spectral theory for linear lattices as developed in [1], §§ 4-12. Therefore we use the terminology and theorems of [1].

First we prove some theorems concerning projection operators on semi-continuous linear lattices. It is well known ([1], Theorem 5.28, p. 23) that if L is any linear lattice and P, P_λ $(\lambda \in \Lambda)$ are projection operators, then $Pz = \bigwedge_{\lambda\in\Lambda} P_\lambda z$ for all $z \geqslant 0$ implies $P = \bigwedge_{\lambda\in\Lambda} P_\lambda$; and $Pz = \bigvee_{\lambda\in\Lambda} P_\lambda z$ for all $z \geqslant 0$ implies $P = \bigvee_{\lambda\in\Lambda} P_\lambda$. By assuming semi-continuity of L we can also obtain the converse implications.

THEOREM 1. *If L is a semi-continuous linear lattice and P, P_λ $(\lambda \in \Lambda)$ are projection operators on L, then*

(i) $P = \bigwedge_{\lambda\in\Lambda} P_\lambda$ *implies* $Pz = \bigwedge_{\lambda\in\Lambda} P_\lambda z$ *for all* $z \geqslant 0$,

(ii) $P = \bigvee_{\lambda\in\Lambda} P_\lambda$ *implies* $Pz = \bigvee_{\lambda\in\Lambda} P_\lambda z$ *for all* $z \geqslant 0$.

PROOF. (i) It suffices to consider the case $P = 0$ since $P = \bigwedge_{\lambda\in\Lambda} P_\lambda$ iff $0 = \bigwedge_{\lambda\in\Lambda} (P_\lambda - P)$. Thus, assume $0 = \bigwedge_{\lambda\in\Lambda} P_\lambda$. For $z \geqslant 0$, we clearly have $P_\lambda z \geqslant 0$ for each $\lambda\in\Lambda$. If $y \leqslant P_\lambda z$ for all $\lambda\in\Lambda$, then $y^+ \leqslant P_\lambda z$ and $[y^+] \leqslant [P_\lambda z] = P_\lambda [z] \leqslant P_\lambda$ for all $\lambda\in\Lambda$. Thus, by assumption, we have $[y^+] = 0$ and consequently $y^+ = 0$. That is, $y = -(y^-) \leqslant 0$. Therefore $0 = \bigwedge_{\lambda\in\Lambda} P_\lambda z$.

(ii) If $P = \bigvee_{\lambda\in\Lambda} P_\lambda$, then $1-P = \bigwedge_{\lambda\in\Lambda} (1-P_\lambda)$, and by (i) we have $z - Pz = \bigwedge_{\lambda\in\Lambda} (z - P_\lambda z)$ for all $z \geqslant 0$; that is, $Pz = \bigvee_{\lambda\in\Lambda} P_\lambda z$ for all $z \geqslant 0$.

THEOREM 2. *If L is a semi-continuous linear lattice, then $x = \bigvee_{\lambda\in\Lambda} x_\lambda$ for $x_\lambda \geqslant 0$ $(\lambda\in\Lambda)$ implies $[x] = \bigvee_{\lambda\in\Lambda} [x_\lambda]$.*

Proof. Clearly $\{x\}^\perp = \{x_\lambda\colon \lambda\in\Lambda\}^\perp$ and thus $\{x\}^{\perp\perp} = \{x_\lambda\colon \lambda\in\Lambda\}^{\perp\perp}$. Consequently, for $z \geqslant 0$, we have $[x]z = [\{x_\lambda\colon \lambda\in\Lambda\}]z = \bigvee_{\lambda\in\Lambda} [x_\lambda]z$. The last equality holds by virtue of [1], Theorem 5.26. Now applying Theorem 5.28 of [1] we obtain $[x] = \bigvee_{\lambda\in\Lambda} [x_\lambda]$.

Recalling the definition of "completeness for uniform convergence" as given above, we may consider

THEOREM 3. *If every increasing uniform Cauchy sequence in a linear lattice L is order convergent, then L is Archimedean and complete for uniform convergence.*

Proof. L is Archimedean by the following argument. Let $y\in L^+$ and let ε be any positive real number. For any natural number $\nu(\varepsilon)$ such that $1/\nu(\varepsilon) \leqslant \varepsilon$ we have $\mu, \nu \geqslant \nu(\varepsilon)$ implies $|(1/\mu)y - (1/\nu)y| \leqslant \varepsilon y$. Thus $\{(1/\nu)y\}$ is a decreasing uniform Cauchy sequence and, by the hypothesis, there exists $z\in L$ such that $(1/\nu)y \downarrow_\nu z$. Consequently, $(2/\nu)y \downarrow_\nu 2z$. But $\bigwedge_\nu (1/\nu)y = \bigwedge_\nu (2/\nu)y$. Hence $z = 2z$. That is, $z = 0$.

Now we show that L is complete for uniform convergence. Let $\{a_\nu\}$ be any uniform Cauchy sequence in L. The sequence $\{\bigvee_{\nu=1}^{\eta} a_\nu\}_{\eta=1}^{\infty}$ obviously increases with η. It is also a uniform Cauchy sequence. For suppose ε is any positive number. Then, since $\{a_\nu\}$ is a uniform Cauchy sequence, there is $k\in L^+$ and $\nu(\varepsilon)$ such that $|a_\mu - a_\nu| \leqslant \varepsilon k$ for all $\mu, \nu \geqslant \nu(\varepsilon)$. Thus, for $\tau \geqslant \eta \geqslant \nu(\varepsilon)$, we have $0 \leqslant \bigvee_{\nu=1}^{\tau} a_\nu - \bigvee_{\nu=1}^{\eta} a_\nu \leqslant \bigvee_{\nu=\eta}^{\tau} a_\nu - a_\eta = \bigvee_{\nu=\eta}^{\tau} (a_\nu - a_\eta) \leqslant \varepsilon k$. It follows that there exists $b_1\in L$ such that $b_1 = \bigvee_{\nu\geqslant 1} a_\nu$. In general, by the same type of argument as just given, there exists $b_\varrho\in L$ such that $b_\varrho = \bigvee_{\nu\geqslant\varrho} a_\nu$ for $\varrho = 1, 2, 3, \ldots$ Clearly $\{b_\varrho\}_{\varrho=1}^{\infty}$ is a decreasing sequence.

It is also a uniform Cauchy sequence since $\varrho \geqslant \tau \geqslant \nu(\varepsilon)$ implies that, for any $\eta \geqslant \varrho$, we have $0 \leqslant \bigvee_{\nu=\tau}^{\eta} a_\nu - \bigvee_{\nu=\varrho}^{\eta} a_\nu \leqslant \bigvee_{\nu=\tau}^{\varrho} a_\nu - a_\varrho = \bigvee_{\nu=\tau}^{\varrho} (a_\nu - a_\varrho) \leqslant \varepsilon k$. Thus, by letting $\eta \to \infty$, we have $0 \leqslant b_\tau - b_\varrho \leqslant \varepsilon k$ for $\varrho \geqslant \tau \geqslant \nu(\varepsilon)$. It then follows from the hypothesis that there exists $b\in L$ such that $b = \bigwedge_{\varrho\geqslant 1} b_\varrho = \bigwedge_{\varrho\geqslant 1} \bigvee_{\nu\geqslant\varrho} a_\nu$.

By applying the above results to $\{-a_\nu\}$ we have that, for any $\varrho = 1, 2, 3, \ldots$, there exists $c_\varrho = \bigwedge_{\nu\geqslant\varrho} a_\nu$ and there exists $c = \bigvee_{\varrho\geqslant 1} c_\varrho$.

Finally, $b = c$, as shown by the following discussion. For $\eta \geqslant \varrho \geqslant \nu(\varepsilon)$ we have $0 \leqslant \bigvee_{\nu=\varrho}^{\eta} a_\nu - \bigwedge_{\mu=\varrho}^{\eta} a_\mu = \bigvee_{\nu=\varrho}^{\eta} (a_\nu - \bigwedge_{\mu=\varrho}^{\eta} a_\mu) = \bigvee_{\nu=\varrho}^{\eta} \big(\bigvee_{\mu=\varrho}^{\eta} (a_\nu - a_\mu)\big) \leqslant \varepsilon k$. Again letting $\eta \to \infty$ we obtain $0 \leqslant b_\varrho - c_\varrho \leqslant \varepsilon k$ for $\varrho \geqslant \nu(\varepsilon)$. Thus we have $0 \leqslant b - c \leqslant \varepsilon k$ by taking the limit as $\varrho \to \infty$. But ε is arbitrary and L is Archimedean. Hence $b = c$. This establishes the order convergence of $\{a_\nu\}$ and our proof is complete.

In order to obtain the main results of this paper we need a few lemmas.

LEMMA 1. *If a linear lattice L is semi-continuous and complete for uniform convergence, then $[y] \geqslant [x_\nu]\uparrow_\nu$ implies $[x_\nu]\uparrow_\nu [z]$ for some $z\in L$.*

Proof. Set $z_\mu = \sum_{\nu=1}^{\mu} (1/\nu)([x_\nu] - [x_{\nu-1}])|y|$ for $\mu = 1, 2, 3, \ldots$, where x_0 is taken to be 0. Clearly $0 \leqslant z_\mu\uparrow_\mu$. Let ε be any positive number and select a natural number τ such that $1/\tau \leqslant \varepsilon$. Then, for $\mu, \eta \geqslant \tau$, we have $|z_\mu - z_\eta| \leqslant \varepsilon |y|$. This is shown as follows. Suppose, without loss of generality, that $\mu > \eta \geqslant \tau$. Then $|z_\mu - z_\eta| = z_\mu - z_\eta = \sum_{\nu=\eta+1}^{\mu} (1/\nu)([x_\nu] - [x_{\nu-1}])|y| \leqslant (1/\tau) \sum_{\nu=\eta+1}^{\mu} ([x_\nu] - [x_{\nu-1}])|y| = (1/\tau)([x_\mu] - [x_\eta])|y| \leqslant (1/\tau)|y| \leqslant \varepsilon |y|$.

But L is complete for uniform convergence. Thus there exists $z\in L$ such that $z_\mu\uparrow_\mu z$. By applying Theorem 2 we see that $[z_\mu]\uparrow_\mu [z]$.

Let $t_\nu = (1/\nu)([x_\nu] - [x_{\nu-1}])|y|$ for $\nu = 1, 2, 3, \ldots$ Then $t_\nu \perp t_\mu$ for $\nu \neq \mu$ since $[t_\nu] = ([x_\nu] - [x_{\nu-1}])[y] = [x_\nu] - [x_{\nu-1}]$, and thus $\nu \neq \mu$ implies $[t_\nu][t_\mu] = 0$ by a routine computation. Hence, for each μ, $[z_\mu] = [\sum_{\nu=1}^{\mu} t_\nu] = \sum_{\nu=1}^{\mu} [t_\nu] = [x_\mu]$. Therefore $[x_\mu]\uparrow_\mu [z]$.

The following lemma involves the concept of a σ-universal space. A topological space is said to be *σ-universal* if the closure of every open F_σ set is an open set.

LEMMA 2. *If a linear lattice L is semi-continuous and complete for uniform convergence, then*

(i) *its proper space E_L is a σ-universal space,*

(ii) *$U_{[z]}$ is a σ-universal space for each $0 \neq z\in L$.*

Proof. (i) This result is proved in [2], Theorem 16.7, p. 82, for sequentially continuous linear lattices. The same proof is available to us since the only aspect of sequential continuity used there is the convergence property of projectors established in Lemma 1 above.

(ii) This result is immediate since $U_{[z]}$ is open and closed in E_L, and therefore a subset of $U_{[z]}$ is open or closed or F_σ with respect to the relative topology on $U_{[z]}$ if and only if it is respectively open or closed or F_σ in E_L.

LEMMA 3. *If a linear lattice L is semi-continuous and complete for uniform convergence, $z > 0$ and $L_z = \{x: |x| \leqslant az$ for some real number $a\}$, then L_z is isomorphic to the linear lattice C of all continuous real-valued functions on $U_{[z]}$.*

Proof. Let $\tau\colon L_z \to C$ by $\tau(x) = (x/z,\ \ )$, where $(x/z,\ \ )$ denotes that function on $U_{[z]}$ whose value at $P \in U_{[z]}$ is $(x/z, P)$. By using Theorems 10.5, 10.6 of [1] we have $\tau(ax+\beta y) = ((ax+\beta y)/z,\ \ ) = a(x/z,\ \ )+ +\beta(y/z,\ \ ) = a\tau(x)+\beta\tau(y)$ for x, y in L_z and real a, β.

Also, for $x, y \in L_z$ we have $x \leqslant y$ implies $\tau(x) \leqslant \tau(y)$ by [1], Theorem 10.8, and $\tau(x) \leqslant \tau(y)$ implies $[z]x \leqslant [z]y$ by [1], Theorem 11.3. But $[z]k = k$ for each $k \in L_z$, since $|k| \leqslant az$ implies $[k] \leqslant [az] = [z]$. Thus we have shown $x \leqslant y$ if and only if $\tau(x) \leqslant \tau(y)$.

Finally, if $\Phi \in C$, then $\Phi = \tau(x)$ for the element $x = \int_{[z]} \Phi(P)\,dPz$ in view of Theorem 12.10 of [1]. Hence τ is the required isomorphism.

We are now able to obtain our main result.

THEOREM 4. *A linear lattice L is sequentially continuous if and only if it is semi-continuous and complete for uniform convergence.*

Proof. If L is sequentially continuous, then L is semi-continuous by [1], Theorem 6.14, p. 28, and complete for uniform convergence by [1], Theorem 3.3, p. 13. Conversely, assume L is semi-continuous and complete for uniform convergence. Consider any sequence $\{x_\nu\}$ such that $0 \leqslant x_\nu \downarrow_\nu$. Letting $z = x_1$, it is clear that $\{x_\nu\}$ is contained in $L_z = \{y\colon |y| \leqslant az$ for some real number $a\}$. But L_z is isomorphic to the linear lattice C of all continuous real-valued functions on $U_{[z]}$ by Lemma 3. Also, C is a sequentially continuous linear lattice by [2], Theorem 41.1, p. 214, since $U_{[z]}$ is a σ-universal space by Lemma 2. Thus L_z is a sequentially continuous linear lattice. Consequently, there exists $x \in L_z$ such that $x \leqslant x_\nu$ for each ν, and $x \geqslant y$ for any $y \in L_z$ such that $y \leqslant x_\nu$ for each ν. Now suppose $k \in L$ and $k \leqslant x_\nu$ for each ν. Then $k^+ \leqslant x_\nu$ for each ν. But $k^+ \in L_z$ since $k^+ \leqslant x_1 = z$. Thus $x \geqslant k^+ \geqslant k$. This implies that $x = \bigwedge_\nu x_\nu$ in the linear lattice L.

THEOREM 5. *Every Banach lattice L is complete for uniform convergence.*

Proof. Let $\{x_\nu\}$ be any increasing uniform Cauchy sequence in L. Then there is $0 \leqslant k \in L$ such that, for every positive number ε, there exists $\nu(\varepsilon)$ for which $\mu, \nu \geqslant \nu(\varepsilon)$ implies $|x_\nu - x_\mu| \leqslant \varepsilon k$. Thus, $\|x_\nu - x_\mu\| \leqslant \varepsilon\|k\|$ for $\mu, \nu \geqslant \nu(\varepsilon)$. But L is complete as a metric space, and therefore there exists $x \in L$ such that $\lim_{\nu\to\infty} \|x_\nu - x\| = 0$. Consequently, $\{x_\nu\}$ order converges to x by [1], Theorem 30.1, p. 126. Now, by invoking Theorem 3 above, we see that L is complete for uniform convergence.

As an application of the main results, one may consider the following theorem which is seen to be an immediate consequence of Theorems 4 and 5.

THEOREM 6. *A Banach lattice is sequentially continuous if and only if it is semi-continuous.*

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