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On permanently singular elements in commutative m-convex locally convex algebras

by

W. ŻELAZKO (Warszawa)*

1. Introduction. All algebras in this paper are assumed to be commutative. By a *superalgebra* of a topological algebra A we mean any topological algebra having a subalgebra topologically isomorphic to A and having the same unit element provided A possessed one. If k is a class of topological algebras, then an element $x \in A \in k$ is said to be *permanently singular* in the class k (or shortly, k-singular) if for any superalgebra A_1 of A, belonging to the class k, x is singular in A_1. In this paper we deal with the classes of topological algebras, locally convex algebras and multiplicatively convex algebras denoted respectively by $\mathcal{T}$, $\mathcal{LC}$, $\mathcal{M}$.

In paper [1] Arens give a characterization of permanently singular elements in the class of Banach algebras. He proved that an element $x \in A \in \mathcal{B}$ ($\mathcal{B}$-class of Banach algebras) is $\mathcal{B}$-singular if and only if it is a topological divisor of zero (and, consequently, if and only if it is $\mathcal{T}$-singular). In this paper we study a concept of $\mathcal{M}$-singularity and we show that none of these statements is true for multiplicatively convex locally convex algebras (shortly m-convex algebras). In section 3, we give a characterization of $\mathcal{M}$-singularity and show by an example that there are $\mathcal{M}$-singular elements, which are even $\mathcal{T}$-singular but are not topological divisors of zero. In section 4 we give our main result stating that there are $\mathcal{M}$-singular elements which are non-$\mathcal{LC}$-singular. This example shows also that all non-zero elements of the radical of a certain m-convex algebra can be invertible in some its locally convex extension. Moreover, both algebras are B_0-algebras (= complete metric $\mathcal{LC}$-algebras).

We cannot give characterization of $\mathcal{LC}$-singularity but we pose a conjecture about it (cf. Problem 1).

2. Prerequisites. By a *topological algebra* we mean a topological linear space over complex or real scalars in which is defined a jointly

* This work was done during the author's stay in Aarhus.

continuous multiplication which is compatible with scalar multiplication. So if A is a topological algebra, then for every neighbourhood V of the origin there is a neighbourhood $U \ni 0$ such that

$$U^2 \subset V. \tag{2.1}$$

An element $x \in A$ is said to be a *topological divisor* of zero, if there is such a neighbourhood U of zero in A that

$$0 \in \overline{x(A \setminus U)}$$

(where $A \setminus U$ is the complement of U in A and bar denotes the closure in A).

A topological algebra A is called a *locally convex algebra* if it is locally convex topological linear space. Thus, by relation (2.1), for any continuous pseudonorm $|x|_\alpha$ in A, there exists a continuous pseudonorm $|x|_\beta$ such that

$$|xy|_\alpha \leqslant |x|_\beta |y|_\beta \tag{2.2}$$

for all $x, y \in A$. It can be assumed also that

$$|x|_\alpha \leqslant |x|_\beta$$

for each $x \in A$

A locally convex algebra is said to be *m-convex* if its topology can be given by means of a family of submultiplicative pseudonorms $\|x\|_\alpha$, i. e.

$$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha, \tag{2.3}$$

it can be assumed also

$$\|e\|_\alpha = 1 \tag{2.4}$$

in the case where the algebra in question possesses the unit e. The set of all continuous submultiplicative pseudonorms of an m-convex algebra A, which satisfy (2.4) if A possesses a unit e, will be denoted by $m(A)$. But we shall write rather $\|\cdot\|_\alpha$, $\alpha \in m(A)$, instead of $\|\cdot\| \in m(A)$ and we will treat $m(A)$ as set of indexes. For $\alpha, \beta \in m(A)$ we write $\alpha \geqslant \beta$ if

$$\|x\|_\beta \leqslant \|x\|_\alpha$$

for every $x \in A$. Thus $m(A)$ is a partially ordered set and for all $\alpha, \beta \in m(A)$, there is a $\gamma \in m(A)$ such that $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$. Any subset $m'(A) \subset m(A)$ will be called *fundamental* if it is cofinal with $m(A)$, i. e. for any $\alpha \in m(A)$ there is a $\beta \in m'(A)$ such that $\beta \geqslant \alpha$. If $m'(A)$ is a fundamental system, then the set $(\|x\|_\alpha)$, $\alpha \in m'(A)$, gives on A its topology.

For any $\alpha \in m(A)$ the set

$$N_\alpha = \{x \in A \colon \|x\|_\alpha = 0\}$$

is a closed ideal in A. We denote by A_α a Banach algebra, which is obtained by completion of A/N_α in the norm $\|\cdot\|_\alpha$. Any m-convex algebra is a subalgebra of the cartesian product $\prod_{\alpha \in m'(A)} A_\alpha$, where $m'(A)$ is a fundamental family for A. The embedding is given by $x \to (\pi_\alpha(x))_{\alpha \in m'(A)}$, where $\pi_\alpha(x)$ denotes the natural projection of A into A_α. It is also $\|x\|_\alpha = \|\pi_\alpha(x)\|_{(\alpha)}$, where $\|\cdot\|_{(\alpha)}$ is the Banach algebra norm in A_α.

Let A be a complete m-convex algebra.

An element $x \in A$ is invertible in A if and only if $\pi_\alpha(x)$ is invertible in A_α for every $\alpha \in m(A)$ (= if and only if $\pi_\alpha(x)$ is invertible in A_α for every $\alpha \in m'(A)$, where $m'(A)$ is a fundamental system for A).

An element $x \in A \in \mathscr{M}$ (A-complete) is in the radical rad A of A if and only if $\pi_\alpha(x)$ is in the radical rad A_α for every $\alpha \in m(A)$. Here, as before, we can take instead of $m(A)$ any fundamental system $m'(A)$.

For the details on m-convex and locally convex algebras the reader is referred to [3] and [5].

We assume in the sequel all algebras to be complete. We can do it without loss of generality since completion of a topological algebra is its superalgebra and a k-singular element in A is k-singular in any k-superalgebra of A (and we are considering classes of algebras which are closed under taking the completion).

3. A characterization of $\mathscr{M}$-singularity.

THEOREM 1. *Let A be a commutative m-convex algebra and let $x \in A$. Then x is $\mathscr{M}$-singular if and only if there exists an $\alpha_0 \in m(A)$ such that $\pi_\alpha(x)$ is a permanently singular element (= is topological divisor of zero) of A_α for each $\alpha \geqslant \alpha_0$, $\alpha \in m(A)$.*

Proof. Suppose that all $\pi_\alpha(x)$ are permanently singular for $\alpha \geqslant \alpha_0$, $\alpha, \alpha_0 \in m(A)$. If x is non-$\mathscr{M}$-singular then there exists an m-convex superalgebra $\tilde{A}$ of A in which x is an invertible element. Since pseudonorms in $m(\tilde{A})$ restricted to A form a fundamental system for A there is $\beta \in m(\tilde{A})$ such that $\beta | A \geqslant \alpha_0$. The element $\pi_\beta(x)$ is invertible in $\tilde{A}_\beta$, on the other hand $\pi_\beta(x) \in A_{\beta|A} \subset \tilde{A}_\beta$ and it is assumed to be permanently singular. The contradiction proves that x is an $\mathscr{M}$-singular element.

On the other hand, suppose that for every $\alpha \in m(A)$ there is a $\beta \geqslant \alpha$ such that $\pi_\beta(x)$ is not permanently singular. Thus the system

$$m'(A) = \{\alpha \in m(A) \colon \pi_\alpha(x) \text{ is non } \mathscr{B}\text{-singular in } A_\alpha\}$$

is a fundamental system for A. For every $a \in m'(A)$ there is a superalgebra $\tilde{A}_a$ of A_a in which $\pi_a(x)$ is an invertible element. We have now

$$A \subset \prod_{a \in m'(A)} A_a \subset \prod_{a \in m'(A)} \tilde{A}_a = \tilde{A}$$

and the natural imbeddings are homeomorphic isomorphisms. Moreover, x is an invertible element in $\tilde{A}$ and so it is not $\mathcal{M}$-singular.

Remark 1. An element $x \in A \in \mathcal{M}$ is $\mathcal{M}$-singular if and only if it is a topological divisor of zero in sense of Michael (cf. [3]).

The topological divisors of zero in sense of this paper are called in [3] strong topological divisors of zero.

In order to show that $\mathcal{M}$-singular elements need not be topological divisors of zero we recall an example given by Kuczma in [2]. Let $\mathcal{K}$ denote the linear space of all complex sequences $x = (x_i)_0^\infty$ provided with the pseudonorms

(3.1) $$\|x\|_k = |x_1| + \ldots + |x_k|, \quad k = 0, 1, \ldots$$

$\mathcal{K}$ is an m-convex algebra with convolution multiplication and with pseudonorms (3.1) which form a fundamental system in $m(\mathcal{K})$. An element $x \in \mathcal{K}$ can be regarded as formal power series $x = \sum_0^\infty x_i t^i$.

In [2] there is shown that in $\mathcal{K}$ there are no topological divisors of zero. On the other hand, it is an algebra with one maximal ideal

$$M = \{x \in \mathcal{K}:\ x_0 = 0\}$$

which is its radical (any element $x \in \mathcal{K}$ with $x_0 \neq 0$ is there invertible). So for every $x \in M$, $\pi_n(x)$ belongs to the radical of $\mathcal{K}_n$ and, consequently, is permanently singular. By Theorem 1 all elements in M are $\mathcal{M}$-singular. We have even a stronger result:

Proposition 1. *All elements in M are $\mathcal{T}$-singular.*

Proof. Put $t = (\delta_{i,1})_0^\infty$, where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise. We have $M = t\mathcal{K}$ and so it is sufficient to show that t is a $\mathcal{T}$-singular element in $\mathcal{K}$. If not, then there is a topological algebra $\tilde{A}$, which is a superalgebra for $\mathcal{K}$ in which t has the inverse t^{-1}. Since topology of $\tilde{A}$ restricted to $\mathcal{K}$ gives its topology, there is in $\tilde{A}$ a neighbourhood V of the origin such that

(3.2) $$V_{|\mathcal{K}} \subset \{x \in \mathcal{K}:\ \|x\|_0 < 1\}.$$

On the other hand, there is in $\tilde{A}$ a neighbourhood of zero U such that $U^2 \subset V$. We can find an $\varepsilon > 0$ and an integer m in such a way that

(3.3) $$\{x \in \mathcal{K}:\ \|x\|_m < \varepsilon\} \subset U.$$

Thus for any scalar λ we have $\lambda t^{m+1} \in U$, since $\|t^{m+1}\|_m = 0$. Since U is a neighbourhood of zero in $\tilde{A}$, there is a scalar $\mu > 0$ such that $\mu t^{-m-1} \in U$.

We have then

$$e = \mu t^{-m-1} \cdot \mu^{-1} t^{m+1} \in U^2 \subset V.$$

But it is in contradiction with (3.2), since $\|e\|_0 = 1$, so t is a $\mathcal{T}$-singular element in $\mathcal{K}$.

4. Another example. We shall construct now an m-convex algebra A_0 and an element $z \in A_0$ such that z is $\mathcal{M}$-singular but not $\mathcal{LC}$-singular. It would be algebra of sequences defined by means of certain matrices and we start with the construction of these matrices.

Lemma 1. *There exists a non-decreasing sequence $(\varphi_n)_0^\infty$ of non-negative integers, $\varphi_0 = 0$, such that*

(4.1) $$\varphi_p + \varphi_q \leqslant \varphi_{p+q} \quad \textit{for } p, q \geqslant 0;$$

(4.2) $$\lim_p \frac{\varphi_p}{p} = \infty;$$

(4.3) $$\sup_i (a\varphi_{k+i} - \varphi_i) = M_k^{(a)} < \infty$$

for $0 < a < 1$ *and* $k = 0, 1, \ldots$;

(4.4) $$M_k^{(a)} \geqslant M_k^{(\beta)} \quad \textit{for } a \geqslant \beta;$$

(4.5) $$M_p^{(a)} + M_q^{(a)} \geqslant M_{p+q}^{(a^2)}.$$

Proof. We shall find φ_n in the form $\varphi_n = n \cdot s_n$, where s_n is a non-decreasing sequence of non-negative integers, which tends to an infinity. We have then $\varphi_0 = 0$ and

$$\varphi_p + \varphi_q = ps_p + qs_q \leqslant ps_{p+q} + qs_{p+q} = \varphi_{p+q},$$

which proves (4.1). We have also (4.2) since $\lim s_n = \infty$.

We choose now s_n to be constant on "blocks" of increasing length and we put

$$s_k = n \quad \text{for } 2^n \leqslant k < 2^{n+1}$$

so that for $i \geqslant k$ it is

(4.6) $$s_{k+i} \leqslant s_i + 1.$$

Let $0 < a < 1$, and let k be a fixed integer. For sufficiently large i, so that $i > k$, and $\varepsilon i > k$, where $(1+\varepsilon)a < 1$ we have in view of (4.6)

(4.7) $$a\varphi_{k+i} - \varphi_i = a(k+i)s_{k+i} - is_i \leqslant a(k+i)(s_i+1) - is_i \leqslant ak + i[a + s_i(a(1+\varepsilon) - 1)]$$

and the right-hand side of (4.7) tends to $-\infty$ if $i \to \infty$ which proves (4.3).

Inequality (4.4) is an immediate consequence of formula (4.3) since φ_k is a non-decreasing sequence.

In order to have (4.5) we write for $0<\alpha<1$

$$M_q^{(\alpha)} > \alpha M_q^{(\alpha)} \geqslant \alpha^2\varphi_{p+q+i} - \alpha\varphi_{p+i},$$

$$M_p^{(\alpha)} \geqslant \alpha\varphi_{p+i} - \varphi_i$$

and so

$$M_q^{(\alpha)} + M_p^{(\alpha)} \geqslant \alpha^2\varphi_{p+q+i} - \varphi_i.$$

Taking the supremum with respect to i we obtain (4.5), q. e. d.

Let γ_n, $0<\gamma_n<\gamma_{n+1}<1$, be a sequence tending to 1. We define

$$a_k^{(n,\alpha)} = \begin{cases} \gamma_n^{\varphi_k} & \text{for } k \geqslant 0, \\ (1-\gamma_n)^{\frac{1}{\alpha}M_{-k}^{(\alpha)}} & \text{for } k \leqslant 0 \end{cases} \tag{4.8}$$

for $n=1,2,\ldots$, $0<\alpha<1$, and $k=0,\pm1,\pm2,\ldots$

LEMMA 2. *For any (m,α), there is (n,β), $n\geqslant m$, $\beta\geqslant\alpha$, such that*

$$a_{p+q}^{(m,\alpha)} \leqslant a_p^{(n,\beta)}\cdot a_q^{(n,\beta)} \tag{4.9}$$

for all $p,q=0,\pm1,\pm2,\ldots$

Moreover, for all $p,q\geqslant 0$ it is

$$a_{p+q}^{(m,\alpha)} \leqslant a_p^{(m,\alpha)}\cdot a_q^{(m,\alpha)}. \tag{4.10}$$

Proof. We shall consider four cases with respect to signs of p,q and $p+q$.

First we establish (4.10), which is a special case of (4.9) for $p,q\geqslant 0$. It is an immediate consequence of (4.1) since

$$a_{p+q}^{(m,\alpha)} = \gamma_m^{\varphi_{p+q}} \leqslant \gamma_m^{\varphi_p+\varphi_q} = a_p^{(m,\alpha)}a_q^{(m,\alpha)}.$$

Suppose now that $p,q\leqslant 0$ and fix (m,α).

We can rewrite formula (4.5) as

$$\frac{1}{\alpha}M_{-q}^{(\alpha^{1/2})} + \frac{1}{\alpha}M_{-q}^{(\alpha^{1/2})} \geqslant \frac{1}{\alpha}M_{-p-q}^{(\alpha)},$$

and so

$$(1-\gamma_m)^{-\frac{1}{\alpha}M_{-p}^{(\alpha^{1/2})}}(1-\gamma_m)^{-\frac{1}{\alpha}M_{-q}^{(\alpha^{1/2})}} \geqslant (1-\gamma_m)^{-\frac{1}{\alpha}M_{-p-q}^{(\alpha)}} = a_{p+q}^{(m,\alpha)}. \tag{4.11}$$

We can find now an $n'>m$ in such a way that

$$(1-\gamma_m)^{\alpha^{-1/2}} > (1-\gamma_{n'}). \tag{4.12}$$

Setting $\beta=\beta'=\alpha^{1/2}$ and $n=n'$, we obtain (4.9) from (4.11) and (4.12).

Suppose now that $p+q<0$ and $q<0$ while $p\geqslant 0$ and fix an (m,α).

First we prove

$$\frac{1}{\alpha}M_k^{(\alpha)} - \varphi_s \geqslant \frac{1}{\alpha}M_{k-s}^{(\alpha)} \tag{4.13}$$

for $k\geqslant s$ ($\geqslant 0$). By (4.1) we can write $\varphi_{k+i}-\varphi_s\geqslant\varphi_{k-s+i}$, and so

$$\varphi_{k+i} - \frac{1}{\alpha}\varphi_i - \varphi_s \geqslant \varphi_{k-s+i} - \frac{1}{\alpha}\varphi_i.$$

Taking supremum with respect to i first on the left-hand side and then on the right-hand side we obtain (4.13). Setting $s=p$, $k=-q$, we have $k-s=-p-q$ and so, by (4.13),

$$-\frac{1}{\alpha}M_{-q}^{(\alpha)} + \varphi_s \leqslant -\frac{1}{\alpha}M_{-p-q}^{(\alpha)}.$$

Thus

$$(1-\gamma_m)^{-\frac{1}{\alpha}M_{-q}^{(\alpha)}}(1-\gamma_m)^{\varphi_p} \geqslant (1-\gamma_m)^{-\frac{1}{\alpha}M_{-p-q}^{(\alpha)}} = a_{p+q}^{(m,\alpha)}. \tag{4.14}$$

We find now n'' in such a way that $\gamma_{n''}\geqslant\max[(1-\gamma_m),\gamma_m]$, so $\gamma_{n''}\geqslant(1-\gamma_m)$ and $(1-\gamma_{n''})\leqslant(1-\gamma_m)$ so that

$$(1-\gamma_{n''})^{-\frac{1}{\alpha}M_{-q}^{(\alpha)}}\gamma_{n''}^{\varphi_p} \geqslant (1-\gamma_m)^{-\frac{1}{\alpha}M_{-q}^{(\alpha)}}(1-\gamma_m)^{\varphi_p}$$

which together with (4.14) gives (4.9) with $\beta=\beta''=\alpha$ and $n=n''$.

We have to consider the remaining case when $p+q\geqslant 0$, $p\geqslant 0$, $q<0$. From (4.3) we have $\alpha\varphi_p - M_{-q}^{(\alpha)} \leqslant \varphi_{p+q}$, which implies

$$\gamma_m^{\alpha\varphi_p}\gamma_m^{-M_{-q}^{(\alpha)}} \geqslant \gamma_m^{\varphi_{p+q}} = a_{p+q}^{(\alpha,m)}. \tag{4.15}$$

We find now an integer $n'''\geqslant n$ in such a way that $\gamma_{n'''}\geqslant\gamma_m^{\alpha}$ and $1-\gamma_{n'''}\leqslant\gamma_m^{\alpha}$, which implies

$$\gamma_{n'''}^{\varphi_p} \geqslant \gamma_m^{\alpha\varphi_p} \quad\text{and}\quad (1-\gamma_{n'''})^{-\frac{1}{\alpha}M_{-q}^{(\alpha)}} \geqslant \gamma_m^{-M_{-q}^{(\alpha)}}$$

and together with (4.15) gives (4.9) with $\beta=\beta'''=\alpha$ and $n=n'''$.

Let us remark now that, by (4.3), we have

$$\frac{1}{\alpha}M_k^{(\alpha)} \geqslant \frac{1}{\beta}M_k^{(\beta)} \quad \text{for } \alpha>\beta$$

and so, since also γ_n is increasing with respect to n we see that

$$a_p^{(n,\beta)} \geqslant a_p^{(m,\alpha)}$$

for $n\geqslant m$ and $\beta\geqslant\alpha$. This follows that if we set $n=\max(m,n',n'',n''')$ and $\beta=\max(\alpha,\beta',\beta'',\beta''')$ we have obtained (4.9) in general, q. e. d.

We define now a locally convex algebra A. It consists of all complex sequences $x = (x_i)_{i=-\infty}^{\infty}$ such that

$$\|x\|_{(m,a)} = \sum_{p=-\infty}^{\infty} a_p^{(m,a)} |x_p| < \infty. \tag{4.16}$$

The multiplication in A is defined as convolution, or interpreting x as a formal Laurent series,

$$x = \sum_{i=-\infty}^{\infty} x_i t^i,$$

the product xy is the Cauchy product of two power series. It follows from (4.9) that the multiplication is well defined in A, and A is an $\mathscr{LC}$-algebra. We have namely

$$\|xy\|_{(m,a)} = \sum_i a_i^{(m,a)} \sum_{p+q=i} |x_p y_q| \leqslant \sum_i \sum_{p+q=i} a_p^{(n,\beta)} |x_p|\, a_q^{(n,\beta)} |y_q|$$
$$= \sum_p a_p^{(n,\beta)} |x_p| \sum_q a_q^{(n,\beta)} |y_q| = \|x\|_{(n,\beta)} \|y\|_{(n,\beta)}.$$

If $A_0 = \{x \in A\colon x_i = 0 \text{ for } i < 0\}$, then it is a subalgebra of A, which, by (4.10), is a multiplicatively convex algebra (pseudonorms $\|x\|_{(m,a)}$, are submultiplicative and do not depend upon a).

Lemma 3. *Let $z = (\delta_{1,n})_{n=-\infty}^{\infty}$ where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise. Then z is an $\mathscr{M}$-singular element of A_0.*

Proof. Put $\|x\|_n = \|x\|_{(n,a)}$ for $x \in A_0$. For a fixed n we have

$$\|z^k\|_n^{1/k} = (a_k^{(n,a)})^{1/k} = \gamma_n^{\varphi_k/k} \underset{k}{\to} 0$$

by formula (4.2). Thus if $|\cdot|_a$ is an arbitrary continuous submultiplicative pseudonorm on A_0, then for some n and $C > 0$ we have $|x|_a \leqslant C\|x\|_n$ for all $x \in A_0$. This follows

$$|x^k|_a^{1/k} \leqslant \sqrt[k]{C\|x^k\|_n} \underset{k}{\to} 0, \tag{4.17}$$

and $\pi_a(z)$ is in radical of A_a, so it is permanently singular in A_a. By Theorem 1 the element z is $\mathscr{M}$-singular.

Since z is clearly invertible in A we have our main result.

Theorem 2. *There exists an m-convex algebra A_0 and an element $z \in A_0$ which is $\mathscr{M}$-singular, but not $\mathscr{LC}$-singular.*

Remark 2. The algebra A of theorem 2 is a B_0-algebra, i. e. it is a complete metric $\mathscr{LC}$-algebra. It follows from the fact that its topology can be given by means of a denumerable sequence of pseodonorms $\|\cdot\|_{(n,1-1/n)}$, $n = 1, 2, \ldots$

By relation (4.17) the element z is in the radical rad A_0 of the algebra A_0. So if we put $M = zA_0$, we have $M = \text{rad } A_0$. On the other hand,

if $x \in M$, $x \neq 0$, then $x = z^k(\lambda e + zu)$, where $\lambda \neq 0$, $u \in A_0$, and k is a natural number. But z is invertible in A and $\lambda e + zu$ is invertible in A_0. Consequently, every non-zero element of M is invertible in A. So we have

Corollary 1. *There exists an m-convex B_0-algebra M which is a radical algebra and which possesses a B_0-extension A such that every non-zero element in M is invertible in A.*

From corollary 1 it follows immediately.

Corollary 2. *The algebra A contains the field of rational functions (which is the field of all rational expressions of the element z).*

Remark 3. Williamson constructed in [4] an example of B_0-algebra which contains the field of rational functions. Similarly as our algebra A, it consists of Laurent series with coefficients summable with weights defined by a matrix. In the Williamson's example the subalgebra consisting of series with vanishing non-positively indexed coefficients is also a radical algebra. It turns out, however, that this subalgebra is not an m-convex algebra. Also the subalgebra consisting of series with vanishing positively indexed coefficients is not an m-convex algebra. So this example cannot serve for our purposes.

We cannot give a characterization of $\mathscr{LC}$-singularity, we have only a sufficient condition which can be formulated as follows:

Proposition 2. *Let $z \in A \in \mathscr{LC}$. If z is not $\mathscr{LC}$-singular, then for every continuous pseudonorm $|\cdot|_a$ on A there is a continuous pseudonorm $|\cdot|_\beta$ such that*

$$\inf_{|x|_a \geqslant 1} |z^n x|_\beta > 0 \tag{4.18}$$

for $n = 1, 2, \ldots$

Proof. Let $\tilde{A}$ designate a locally convex extension of A, in which z is an invertible element, and denote its inverse by t. Let $|x|_a$ be a continuous pseudonorm on A and let $\|x\|_a$ denote a continuous pseudonorm in $\tilde{A}$ such that for every $x \in A$ it is $\|x\|_a \geqslant |x|_a$. Since $\tilde{A}$ is locally convex there is another continuous pseudonorm $\|x\|_\beta$ such that $\|uv\|_a \leqslant \|u\|_\beta \|v\|_\beta$. Setting $v = z^n x$, $u = t^n$ we have

$$\|x\|_a = \|uv\|_a \leqslant \|u\|_\beta \|v\|_\beta = \|u\|_\beta \|z^n x\|_\beta.$$

It follows that if $\|x\|_a \geqslant 1$, then $\|u\|_\beta > 0$ and $\|z^n x\|_\beta > 0$ and

$$\|z^n x\|_\beta \geqslant \|u\|_\beta^{-1} > 0.$$

But $|x|_a \geqslant 1$ implies $\|x\|_a \geqslant 1$ and so (4.18) holds if $|\cdot|_\beta$ denotes restriction of $\|\cdot\|_\beta$ to the algebra A.

PROBLEM 1. Is the negation of the condition given by (4.18) also a necessary condition for z being an $\mathscr{LC}$-singular element?

We conjecture that the answer is in positive.

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS OF THE UNIVERSITY OF AARHUS

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Semi-continuous linear lattices

by

BERNARD C. ANDERSON and HIDEGORO NAKANO (Wayne)

In [1] an element x of a linear lattice L is called *normalable* if $L = \{x\}^{\perp\perp} \oplus \{x\}^{\perp}$. It can be shown (see [1], Theorem 6.14, p. 28) that if a linear lattice L is sequentially continuous, then every element of L is normalable. In this paper a linear lattice L is said to be *semi-continuous* if each element of L is normalable.

If every uniform Cauchy sequence in a linear lattice L is order convergent in L, then L is said to be *complete for uniform convergence.* A sequence $x_\nu \in L (\nu = 1, 2, 3, \ldots)$ is called a *uniform Cauchy sequence* if there is $0 \leqslant k \in L$ such that, for each $\varepsilon > 0$, there exists $\nu(\varepsilon)$ for which $\mu, \nu \geqslant \nu(\varepsilon)$ implies $|x_\mu - x_\nu| \leqslant \varepsilon k$. According to Theorems 6.14, 3.3 of [1], every sequentially continuous linear lattice is semi-continuous and complete for uniform convergence. The converse of this statement is the main result (Theorem 4) of this paper. It is also shown that every Banach lattice is complete for uniform convergence. One can then apply these results to show that a Banach lattice is sequentially continuous iff it is semi-continuous.

Much of this paper makes use of spectral theory for linear lattices as developed in [1], §§ 4-12. Therefore we use the terminology and theorems of [1].

First we prove some theorems concerning projection operators on semi-continuous linear lattices. It is well known ([1], Theorem 5.28, p. 23) that if L is any linear lattice and $P, P_\lambda (\lambda \in \Lambda)$ are projection operators, then $Pz = \bigwedge_{\lambda\in\Lambda} P_\lambda z$ for all $z \geqslant 0$ implies $P = \bigwedge_{\lambda\in\Lambda} P_\lambda$; and $Pz = \bigvee_{\lambda\in\Lambda} P_\lambda z$ for all $z \geqslant 0$ implies $P = \bigvee_{\lambda\in\Lambda} P_\lambda$. By assuming semi-continuity of L we can also obtain the converse implications.

THEOREM 1. *If L is a semi-continuous linear lattice and P, P_λ $(\lambda \in \Lambda)$ are projection operators on L, then*

(i) $P = \bigwedge_{\lambda\in\Lambda} P_\lambda$ *implies* $Pz = \bigwedge_{\lambda\in\Lambda} P_\lambda z$ *for all* $z \geqslant 0$,

(ii) $P = \bigvee_{\lambda\in\Lambda} P_\lambda$ *implies* $Pz = \bigvee_{\lambda\in\Lambda} P_\lambda z$ *for all* $z \geqslant 0$.