*is a weak* exposed point of* $U_{E'}$, *then* $x'\circ\Phi_s$ *is a weak* exposed point of* $U_{X'}$.

Proof. Since $x'$ is a weak* exposed point of $U_{E'}$, we can find $x\in E$, $\|x\|=1$, such that $\{y\colon \|y'\|=y'(x)=1\}=\{x'\}$. Let $f\in X$ satisfy the hypotheses above. Define $H=\{\varphi\colon \varphi(f)=\|\varphi\|=1\}$. If $y'\circ\Phi_t\in H$, then $\|y'\|\leqslant 1$ and $y'(f(t))=1$, so $\|f(t)\|=1$. Hence, $\Phi_t=\pm\Phi_s$ and $f(t)=\pm f(s)=\pm x$. Thus, $y'(x)=\pm 1$, so $y'=\pm x'$. Hence, the only extreme point of $H$ is $x'\circ\Phi_s$.

Again, since $H$ is the weak* closed convex hull of $\mathrm{ext}(H)$, $H=\{x'\circ\Phi_s\}$.

COROLLARY 2.3. *Let* $A$ *be a subspace of* $C(S)$ *and* $X=A\otimes_\lambda E$ *(or* $A\tilde{\otimes}_\lambda E$*). If there is a function in* $A$ *that peaks at* $s$ *relative to* $A$ *and if* $x'$ *is a weak* exposed point of* $U_{E'}$, *then* $x'\circ\Phi_s$ *is a weak* exposed point of* $U_{X'}$.

Proof. Let $x\in E$, $\|x\|=1$. Let $g\in A$ peaks at $s$ relative to $A$. Then $f=g\cdot x$ satisfies the hypotheses of theorem 2.2, q. e. d.

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# On positive functionals on a group algebra multiplicative on a subalgebra

by

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This paper was motivated by two independent facts. One, observed by Thoma [13], was that if $G$ is a discrete group in which every element has finitely many conjugates and $\mathfrak{z}$ is the center of the $l_1$-group algebra of $G$, then a class-function which defines a multiplicative functional on $\mathfrak{z}$ is positive-definite because $\mathfrak{z}$ is symmetric. The other fact, observed by M. Moskovitz (oral communication), was that if $G$ is a locally compact group, $K$ a compact subgroup of $G$, then any bounded $K$-spherical function on $G$ is positive-definite, if $L_1(G)$ is symmetric.

Of course, if $L_1(G)$ is symmetric, then any Banach *-subalgebra of it is symmetric. Thus if one knows that $L_1(G)$ is symmetric, one can establish the positive-definiteness of certain functions by means of the facts revealed above. However, to decide whether $L_1(G)$ is symmetric may be difficult even for such simple groups as the groups of motions (cf. [1]). The aim of this note is to propose a property which resembles symmetry of a Banach *-algebra and which, on one hand, is much easier to prove for $L_1(G)$ for a large and natural (cf. [2], [5], [8], [9], and [14]) class of locally compact groups $G$ and, on the other hand, implies the positiveness of multiplicative functionals on a *-subalgebra of $L_1(G)$ in which the functions with compact support are dense.

This will lead to two theorems in section 4, one of which asserts that if $G$ is $[FC^-]$, then the set of extreme positive-definite, normalized class functions is equal to the set of the multiplicative functionals on the center of $L_1(G)$. Under a more restrictive assumption a similar result has been recently obtained on another way by H. Kaniuth. The other implies e. g. that the spherical functions on a group which is an extension of a nilpotent group by a compact group are positive-definite.

The paper is organized as follows. Section 1 is devoted to a theorem on Banach *-algebras and the crucial property (A) which implies that multiplicative functionals are positive. In section 2 we turn to group algebras and we find a condition (C) on a group which implies that the *-subalgebras of functions with compact support satisfy (A). Section 3 is devoted to locally compact groups for which (C) is satisfied. Finally, in section 4, we give applications of theorems of the preceeding sections to the proof of the facts mentioned above.

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**1. Banach *-algebras.** Let $\boldsymbol{A}$ be a Banach *-algebra with the unit $e$ and the cone of positive elements

$$P = \{\Sigma_i x_i^* x_i \colon x_i \in \boldsymbol{A}\}^-.$$

Let $S$ denote the set of positive normalized functionals on $(\boldsymbol{A}, P, e)$ equipped with the *-weak topology and let $\partial S$ denote the set of extreme points of $S$.

The following theorem is well-known and easy to prove (cf. e. g. [6] and [10]):

THEOREM 1.1 *If $A$ is a commutative Banach *-algebra, then*

$$\partial S = \boldsymbol{A}^\wedge \cap S,$$

*where $\boldsymbol{A}^\wedge$ denotes the set of homomorphisms of $\boldsymbol{A}$ onto complex numbers.*

We impose a condition on a Banach *-algebra which implies that every multiplicative functional is positive, i. e. that $\partial S = \boldsymbol{A}^\wedge$.

THEOROM 1.2. *Let $\boldsymbol{A}_0$ be a *-subalgebra of a Banach *-algebra $\boldsymbol{A}$ containing the unit element and not necessarily closed in $\boldsymbol{A}$. Suppose that for a *-representation $T$ of $\boldsymbol{A}$ into the bounded operators in a Hilbert space we have*

(A) $$\lambda(x) = \|T_x\| = \lim_{n\to\infty} (x^n)^{1/n} = \nu(x) \quad \text{for } x^* = x \text{ in } \boldsymbol{A}_0.$$

*Then any multiplicative functional on $\overline{\boldsymbol{A}}_0^{\|\cdot\|}$ takes real non-negative values on $\boldsymbol{A}_0 \cap P$.*

Proof. Let $y \in \boldsymbol{A}_0 \cap P$. Then there is a sequence $\{x_n\}$ of elements of the form

$$x_n = \Sigma_i x_{in}^* x_{in}, \quad x_{in} \in \boldsymbol{A},$$

convergent to $y$ in the norm. Then, clearly, $\{x_n\}$ is convergent to $y$ in the norm $\lambda$, since $\lambda(x) \leqslant \|x\|$ for $x^* = x$ in $\boldsymbol{A}$. Let $\overline{\boldsymbol{A}}^\lambda$ denote the completion of $\boldsymbol{A}$ in the norm $\lambda$. Then $\overline{\boldsymbol{A}}^\lambda$ is a $C^*$-algebra and, consequently, $Sp_{\overline{\boldsymbol{A}}^\lambda} y$ is real, non-negative and, as such, it does not separate the plane. Therefore (cf. e. g. [11], p. 36), if $\boldsymbol{A}_y$ is the *-algebra generated by $y$ and the unit, then

$$Sp_{\overline{\boldsymbol{A}}_y^\lambda} y = Sp_{\overline{\boldsymbol{A}}^\lambda} y \geqslant 0.$$



But $\boldsymbol{A}_y$ is commutative and so $\nu$ is subadditive on $\boldsymbol{A}_y$ and, since $\boldsymbol{A}_y \subset \boldsymbol{A}_0$, by assumption, $\nu(x) = \lambda(x)$ for $x^* = x$ in $\boldsymbol{A}_y$. Moreover, since $\nu(x) = \nu(x^*)$ and $\lambda(x) = \lambda(x^*)$, we have

$$\nu(x+iz) \leqslant \nu(x)+\nu(z) = \lambda(x)+\lambda(z) \leqslant 2\lambda(x+iz) \leqslant 4\nu(x+iz),$$

which shows that the norms $\nu$ and $\lambda$ are equivalent on $\boldsymbol{A}_y$. Consequently

$$\overline{\boldsymbol{A}}_y^\lambda = \overline{\boldsymbol{A}}_y^\nu,$$

whence we infer that $Sp_{\overline{\boldsymbol{A}}_y^\nu} y$ is real non-negative.

If $s$ is a multiplicative functional on $\overline{\boldsymbol{A}}_0^{\|\cdot\|}$, then, of course, it is multiplicative on $\overline{\boldsymbol{A}}_y^{\|\cdot\|}$ and, as such, it is continuous with respect to $\nu$. Hence $s$ defines a homomorphism of $\overline{\boldsymbol{A}}_y^\nu$ into complex numbers and, by Gelfand's theorem,

$$\langle y, s\rangle \in Sp_{\overline{\boldsymbol{A}}_y^\nu} \geqslant 0,$$

which completes the proof.

**2. A condition on a locally compact group.** Let $G$ be a locally compact group and let $|M|$ denote the right invariant Haar measure of the set $M$. Let $A^n = \{a_1, \ldots, a_n \colon a_j \in A\}$.

CONDITION (C). *For any compact set $A$ in $G$ and any $c > 1$*

$$|A^n| = o(c^n) \quad \text{as } n \to \infty.$$

We note first that if $G$ satisfies (C), then $G$ is unimodular, (cf. [2]). In fact, if for a $g$ in $G$ and a compact set $A$ of positive measure such that $g \in A$ we have $|Ag| = \Delta(g)|A|$, then $|A^n| \geqslant |Ag^{n-1}| = |A|\Delta(g)^{n-1}$, which shows that $G$ does not satisfy (C), if only $\Delta(g) > 1$ for a $g$ in $G$.

Let $\boldsymbol{A}$ denote the $L_1(G)$ algebra with the unit element $e$ adjoined to it, if $G$ is not discrete. Let, further,

$$\boldsymbol{A}_0 = \{\alpha e + x \colon \alpha \in C,\ x \text{ in } L_1(G) \cap L_2(G) \text{ and compact support}\}.$$

Finally, let $x \to L_x$ be the left regular representation of $\boldsymbol{A}$ on $L_2(G)$ and let $\lambda(x) = \|L_x\|$.

THEOREM 2.1. *If $G$ satisfies condition (C), then*

(2.1) $$\nu(x) = \lambda(x)$$

*for $x^* = x$ in $\boldsymbol{A}_0$.*

Proof. We note first that in case $G$ is not discrete it is sufficient to prove (2.1) for $x^* = x$ in $\boldsymbol{A}_1 = \boldsymbol{A}_0 \cap L_1(G)$ only. In fact, $\overline{\boldsymbol{A}}_1^\lambda$ is a $C^*$-algebra without unit and so

$$\lambda(x) = \max\{Sp_{\overline{\boldsymbol{A}}_1^\lambda} x\} = \max\{Sp_{\overline{\boldsymbol{A}}_1^\nu} x\}.$$

Thus

$$\lambda(\alpha e+x) = \max\{Sp_{\bar{A}_0^{\lambda}} \alpha e+x\} = \max\{\alpha+Sp_{\bar{A}_1^{\nu}} x\}$$

$$= \sup\{\alpha+x^{\wedge}(M)\colon\ M \in \mathfrak{M}_x\} = \nu(\alpha e+x),$$

where $\mathfrak{M}_x$ is the space of maximal ideals of the *-algebra generated by $x$. We then fix an element $x = x^*$ with compact support $A$ in $L_1(G)$. Then, for $n \geqslant 2$,

$$\|x^n\| = \int_{A^n} |x^n| \leqslant |A^n|^{1/2}\cdot \|x^n\|_2$$

$$\leqslant |A^n|^{1/2}\cdot \lambda^{n-1}(x)\ \|x\|_2.$$

Consequently,

$$\nu(x) \leqslant \lim_{n\to\infty}\big(\|x\|_2^{1/n}(|A^n|^{1/2})^{1/n}\cdot \lambda^{(n-1/n)}(x)\big) = \lambda(x).$$

This, in view of the fact that $\nu(x) \geqslant \lambda(x)$ is always valid for $x = x^*$, completes the proof.

COROLLARY 2.1. *Let $B$ be a Banach *-subalgebra containing the unit of the group algebra $\boldsymbol{A}$ of a group which satisfies condition* (C). *If the elements $\alpha e+x$ with* supp $x$ *compact form a dense subset of $B$, then any multiplicative functional on $B$ is positive.*

**3. Groups which satisfy condition** (C). Condition (C) has been already invastigated by severial authors, especially for discrete groups, cf. [2], [5], [6], [7], and [14]. The following simple fact (noticed by Emerson and Greenleaf) provides an easy way of proving that Abelian and nilpotent groups satisfy (C).

THEOREM 3.1. *Suppose that for every finite subset $F$ of a locally compact group $G$ we have*

$$\operatorname{card}(F^n) = o(c^n) \quad \textit{for every } c > 1.$$

*Then $G$ satisfies condition* (C).

In fact, if $A$ is an open set with compact closure, then there is a finite set $F$ such that $AA \subset \overline{AA} \subset FA$. Hence $A^n \subset F^{n-1}A$ and so $|A^n| \leqslant \operatorname{card}(F^{n-1})\,|A|$.

COROLLARY 3.1. *If $G$ is a locally compact nilpotent group, then $G$ satisfies condition* (C).

A locally compact group $G$ is called an $[FC^-]$-*group*, if for every $g$ in $G$ the set $g^G = \{h^{-1}gh\colon h \in G\}$ has compact closure.

It follows from the work of Gressev and Moskowitz [3] that $[FC^-]$-groups satisfy condition (C). It is also an immediate consequence of the structure theorem of $[FC^-]$-groups due to Robertson [12] which depends on [3] and the following trivial theorem:

THEOREM 3.2. *If $G$ is an extension of a compact group by a group which satisfies condition* (C), *then $G$ satisfies condition* (C).



THEOREM 3.3 (L. Robertson). *$G$ is an $[FC^-]$-group, if and only if $G$ contains a normal compact subgroup $K$ such that $G/K = V\times D$, where $V$ is a vector group and $D$ is discrete and $[FC^-]$.*

THEOREM 3.4. *Let $G$ be a separable* (1) *locally compact group, $H$ a normal subgroup of $G$ such that $G/H = K$ is compact and $H$ satisfies condition* (C). *Then $G$ satisfies condition* (C).

Proof. We start with a few well-known and easy to prove facts.

Let $\pi$ be the natural projection $G \to G/H = K$.

(i) There is a compact set $C$ in $G$ such that $\pi(C) = K$ and the unit element of $G$ belongs to $C$.

(ii) There exists a Borel one-to-one function $\theta\colon K \to C$ such that $\theta(K)$ is Borel and $\pi_0\theta(k) = k$ for $k$ in $K$.

(iii) If $M$ is a Borel subset of $H$, then $\theta(K)M$ is Borel in $G$ and

$$|\theta(K)M|_G = |M|_H,$$

where $|\cdot|_G$ and $|\cdot|_H$ denote the Haar measure in $G$ and $H$, respectively.

(iv) For any compact subset $A$ of $H$ and a compact set $C$ in $G$

$$A^C = \bigcup_{g\in C} g^{-1}Ag \subset H$$

is compact.

Now let $A$ be a compact set in $G$. We have

$$A = A \cap \bigcup_{k\in K}\theta(k)H = \bigcup_{k\in K}\theta(k)(H \cap \theta(k)^{-1}A) \tag{3.1}$$

$$\subset \bigcup_{k\in K}\theta(k)(H \cap C^{-1}A) = \theta(K)A_0,$$

where $A_0 = H \cap C^{-1}A$. Clearly, $A_0$ is compact. We have

$$\theta(k_1k_2)^{-1}\theta(k_1)\theta(k_2) \in C^{-1}CC \cap H \subset B,$$

where $B$ is a compact subset of $H$ containing $C^{-1}CC \cap H$ and the unit of the group. Consequently,

$$\theta(k_1)\theta(k_2) = \theta(k_1k_2)b \quad \text{with } b \text{ in } B. \tag{3.2}$$

By (3.1), we have

$$A^n \subset \big(\theta(K)A_0\big)^n.$$

Thus, if $g$ belonges to $A^n$, then for some $a_1, \ldots, a_n \in A_0$, $k_1, \ldots, k_n \in K$ we have

$$g = \theta(k_1)a_1\ldots\theta(k_n)a_n = \theta(k_1)\ldots\theta(k_n)a_1^{\theta(k_1)\ldots\theta(k_n)}\ldots a_{n-1}^{\theta(k_n)}a_n. \tag{3.3}$$

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(1) This is assumed for simplicity sake. Since only subsets of compact sets play role in the proof, it is sufficient to know that $G$ is first countable and the general case reduces to this because any locally compact group contains a compact normal subgroup $N$ such that $G/N$ is first countable.

In virtue of (3.2), we define $b_1, \ldots, b_{n-1}$ in $B$ by

$$\theta(k_j)\,\theta(k_{j+1}\ldots k_n) = \theta(k_j \ldots k_n)\,b_j, \quad j = 1, \ldots, n-1.$$

Then

$$\theta(k_j) \ldots \theta(k_n) = \theta(k_j \ldots k_n)\,b_j b_{j+1} \ldots b_{n-1}.$$

If

$$a_j' = a_j^{\theta(k_{j+1}\ldots k_n)}, \quad j = 1, \ldots, n-1,$$

then $a_j' \in A_0^C$. By (3.3), we have

$$\begin{aligned} g &= \theta(k_1 \ldots k_n)\,b_1 \ldots b_{n-1}\,a_1^{\theta(k_2\ldots k_n)b_2\ldots b_{n-1}} \ldots a_{n-1}^{\theta(k_n)} a_n \\ &= \theta(k_1 \ldots k_n)\,b_1 \ldots b_{n-1}\,a_1'^{b_2\ldots b_{n-1}} \ldots a_{n-2}'^{b_{n-1}} a_{n-1}' a_n \\ &= \theta(k_1 \ldots k_n)\,b_1 a_1' b_2 a_2' \ldots b_{n-1} a_{n-1}' a_n \in \theta(K)(BA_0^C)^n. \end{aligned}$$

Consequently,

$$A^n \subset \theta(K)(BA_0^C)^n$$

and

$$|A^n|_G \leqslant |(BA_0^C)^n|_H = o(c^n) \quad \text{for any } c > 1,$$

because $BA_0^C$ is compact and $H$ satisfies condition (C).

**4. Applications.** **$[FC^-]$-groups.** Let $G$ be an $[FC^-]$-group. For a function $x$ on $G$ we denote by $x^g$ the function

$$x^g(h) = x(g^{-1}hg), \quad h \in G.$$

Let $\boldsymbol{A}$ be the group algebra of $G$ and let $\mathfrak{z}$ be the center of $\boldsymbol{A}$. It is easy to verify that $\mathfrak{z}$ consists of the elements $ae + x$ of $\boldsymbol{A}$ such that $x^g = x$ for all $g$ in $G$.

Lemma 4.1. *A linear functional $t$ on $\boldsymbol{A}$ satisfies*

(4.1) $$\langle xy, t\rangle = \langle yx, t\rangle \quad \textit{for all } x, y \textit{ in } \boldsymbol{A}$$

*if and only if for $x \in L_1(G)$ we have*

(4.2) $$\langle x, t\rangle = \int \tau(g)\,x(g)\,dg \quad \textit{and} \quad \tau^g = \tau \textit{ for all } g \in G.$$

Proof. Trivial.

Let $T$ denote the set of normalized positive functionals on $\boldsymbol{A}$ which satisfy (4.1). Then, clearly, the $\tau$ in (4.2) is a continuous positive-definite function constant on conjugacy classes. $T$ is a convex and $\omega^*$-compact subset of the dual space $\boldsymbol{A}'$ of $\boldsymbol{A}$. Let $\partial T$ denote the set of extreme points of $T$.

Theorem 4.1. *There is a linear homeomorphism $\pi^*$ of $\mathfrak{z}'$ into $\boldsymbol{A}'$ such that*

$$\pi^*(\mathfrak{z}^\wedge) = \partial T,$$

*where $\mathfrak{z}^\wedge$ denotes the subset of $\mathfrak{z}'$ consisting of multiplicative (non-zero) functionals on $\mathfrak{z}$.*

Proof. We define a projection

$$\pi\colon \boldsymbol{A} \to \mathfrak{z},$$

by "averaging over the conjugacy classes".

Let $m$ be an invariant mean on $L_\infty(G)$. It follows immediately from theorem 3.2 and [4] that such a mean exists; an independent proof of this fact can be found also in [7]. For an $x$ in $\boldsymbol{A}$ let $F$ be a linear functional on $\boldsymbol{A}'$ defined by

$$\langle F, f\rangle = m\langle x^g, f\rangle, \quad f \in \boldsymbol{A}'.$$

$F$ is continuous in *-weak topology of $\boldsymbol{A}'$ beacause for a fixed $x$ the set $\{x^g\colon g \in G\}$ is weakly conditionally compact and $\underset{g}{m}\varphi(g)$ is a limit of $\Sigma\alpha_i\varphi(g_i)$ with $\alpha_i \geqslant 0$ and $\Sigma\alpha_i = 1$. Consequently, $\langle F, f\rangle = \langle \pi(x), f\rangle$ for a $\pi(x) \in \boldsymbol{A}$. Clearly, $\pi$ is the required projection and, since $\langle \pi(x^*x), f\rangle = \underset{g}{m}\langle (x^g)^* x^g, f\rangle$, we have $\pi(x^*x) \in P$.

Let

$$\eta\colon \mathfrak{z} \to \boldsymbol{A}$$

be the natural embedding of $\mathfrak{z}$ into $\boldsymbol{A}$. Then $\pi \circ \eta = id_{\mathfrak{z}}$. If $\pi^*$ and $\eta^*$ are the adjoined mappings of $\pi$ and $\eta$, respectively, then

(4.3) $$\pi^*\colon \mathfrak{z}' \to \boldsymbol{A}'$$

is a monomorphism and

$$\eta^*\colon \boldsymbol{A}' \to \mathfrak{z}'$$

is the restriction of a functional on $\boldsymbol{A}$ to $\mathfrak{z}$. We have

(4.4) $$t \in \pi^*(\mathfrak{z}') \textit{ if and only if } t \textit{ satisfies } (4.1).$$

In fact, if $t = \pi^* s$ and $x \in L_1(G)$ we have

$$\langle t^g, x\rangle = \langle t, x^{g^{-1}}\rangle = \langle s, \pi x^{g^{-1}}\rangle = \langle s, \pi x\rangle = \langle t, x\rangle.$$

On the other hand, if $\langle t^g, x\rangle = \langle t, x\rangle$ for a $t$ in $\boldsymbol{A}'$ and $x$ in $L_1(G)$, then

$$\langle \pi^*\eta^* t, x\rangle = \langle \eta^* t, \pi x\rangle = \langle t, \pi x\rangle = \langle t, \underset{g}{m} x^g\rangle = \underset{g}{m}\langle t^{g^{-1}} x\rangle = \langle t, x\rangle,$$

whence $t = \pi^*(\eta^* t)$.

By (4.3) we see that $\pi^*$ maps the set $S$ of positive normalized functionals on $\mathfrak{z}$ onto $T$. Since $\pi^*$ is linear and one-to-one, $\pi^*(\partial S) = \partial T$.

By Theorem 1.1, $\partial S = \eta^* \partial T$ consists of functionals which are multiplicative on $\mathfrak{z}$ and positive. But, clearly, the set

$$\mathfrak{z}_0 = \{ae + x \in \mathfrak{z}:\ a \in \boldsymbol{C},\ \operatorname{supp} x \text{ is compact}\}$$

is dense in $\mathfrak{z}$, so, by Corollary 2.1, any multiplicative functional on $\mathfrak{z}$ is positive, that is $\mathfrak{z}^\wedge = \partial S$, which completes the proof of theorem 4.1.

*Spherical functions.* Let $G$ be a locally compact group, $K$ a compact subgroup of $G$. Let $L_1(K\diagdown G/K)$ be the Banach *-algebra of $L_1$ functions bi-invariant with respect to $K$, i. e. $x \in L_1(K\diagdown G/K)$ implies

$$x(k'gk'') = x(g) \quad \text{for } k', k'' \in K \text{ and } g \in G.$$

By a *K-spherical bounded function* on $G$ we mean a continuous bounded bi-invariant function $\varphi$ on $G$ such that

$$x\varphi = a_x\varphi, \quad \text{where } a_x \in \boldsymbol{C}.$$

Clearly,

$$x \to a_x$$

is an $L_1$-continuous multiplicative functional on $L_1(K\diagdown G/K)$. A spherical function is called *normalized* if $\varphi(e) = 1$, where $e$ denotes the unit element of the group. For a bounded normalized spherical function we have

$$a_x = \int x(t)\varphi(t^{-1})\,dt = \langle x, \varphi^\sim\rangle.$$

THEOREM 4.2. *If $G$ satisfies condition* (C), *then any spherical bounded normalized function is positive-definite.*

Proof. If

$$\boldsymbol{A}_0 = \{ae + x:\ x \in L_1(K\diagdown G/K) \cap L_2(K\diagdown G/K),\ \operatorname{supp} x \text{ compact}\},$$

then for any bounded normalized spherical function $\varphi$ the functional

$$ae + x \to a + \langle x, \varphi\rangle$$

is a multiplicative (continuous) functional on $\overline{\boldsymbol{A}}_0$. By Corollary 2.1, such a functional is positive on $\overline{\boldsymbol{A}}_0 = L_1(K\diagdown G/K)$. We define a positive projection

$$\pi\colon \boldsymbol{A} \to \overline{\boldsymbol{A}}_0$$

by

$$\pi(ae + x) = ae + \int_K x(k'gk'')\,dk'dk''.$$

An argument very similar to the one used in the proof of Theorem 4.1 shows that $\varphi = \pi^*\varphi$ for any bounded normalized spherical function, which proves that $\varphi$ is positive-definite.



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