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Fourier transforms of vector-valued functions and measures

by

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An answer to the question whether, for a given sequence a_n, $n = 0, \pm 1, \pm 2, \ldots$, of complex numbers, there exists a measure on $[0, 2\pi]$ (or on the unit circle) such that a_n are the Fourier–Stieltjes coefficients of this measure can be given in terms of Fejér's means

$$\sigma_N(t) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=-m}^{m} e^{int}a_n, \qquad N = 1, 2, \ldots$$

Namely, such a measure does exist if and only if

$$\int_0^{2\pi} |\sigma_N(t)|\,dt \leqslant k,$$

where k is independent of N. This condition means that the maps Φ_N from the space $C([0, 2\pi])$ of continuous functions on the unit circle into complex numbers field defined by

$$\Phi_N(\psi) = \int_0^{2\pi} \psi(t)\,\sigma_N(t)\,dt$$

map the unit ball of this space into a bounded set not depending upon N.

If a_n are now elements of a quasi-complete locally convex topological vector space X, the Fejér's means can be formed again and the maps Φ_N as well. These maps will take values in X. There exists an X-valued measure on the unit circle such that a_n are its Fourier coefficients if and only if the maps Φ_N carry the unit ball of $C([0, 2\pi])$ into a weakly compact set common for all $N = 1, 2, \ldots$

Such result and a similar one concerning characterization of Fourier transforms of X-valued integrable functions is stated for any locally compact Abelian group.

0. Introduction. Let G be a locally compact Abelian group and let Γ be its dual group. $C_0(G)$ and $C_{00}(G)$ denote the space of all complex continuous functions on G vanishing at infinity and with compact supports, respectively. They are equipped with sup-norm. $\mathscr{B}(G)$ will stand for the σ-ring of all Borel sets in G (i.e. the σ-ring generated by compact sets; terminology as in [5]). $\int \ldots dg$ and $\int \ldots d\gamma$ denote integrations with respect to (a choice of) Haar measures on G and Γ, respectively.

Let I be an index-set directed by a relation $\leqslant$ and, for every $\iota \in I$, let u_ι and ω_ι be functions on G and Γ, respectively, with the properties:

(i) $u_\iota \geqslant 0$, $\int u_\iota(g)\,dg = 1$.

(ii) ω_ι is continuous and vanishes outside of a compact set, and $\omega_\iota(\Gamma) \subset [0,1]$.

(iii) For every neighbourhood V of 0 in G and $\varepsilon > 0$, there is $\iota_0 \in I$ such that, for every ι with $\iota_0 \leqslant \iota$, $\int_V u_\iota(g)\,dg > 1-\varepsilon$ and $g \notin V$ implies $u_\iota(g) < \varepsilon$.

(iv) $\lim_\iota \omega_\iota(\gamma) = 1$ uniformly with respect to γ on compact subsets of Γ.

(v) $\omega_\iota(\gamma) = \int_G u_\iota(g)(-g,\gamma)\,dg$, $\gamma \in \Gamma$; $u_\iota(g) = \int_\Gamma \omega_\iota(\gamma)(g,\gamma)\,d\gamma$, $g \in G$.

The existence of such a kernel was proved in [6] and [8] generalizing the Riesz' summation of Fourier transforms on real-line and Fejér's means on the unit circle, respectively. If G is metrizable then the net of natural numbers with its usual order can be chosen for I, i.e. $\{u_\iota\}$ and $\{\omega_\iota\}$ become ordinary sequences.

For $\psi \in L^1(\Gamma)$ we write $\hat{\psi}(g) = \int_\Gamma (-g,\gamma)\psi(\gamma)\,d\gamma$, $g \in G$.

Let X be a quasi-complete locally convex topological vector space and X' its dual. The weak topology of X means the $\sigma(X, X')$ topology on X.

Let Y be a locally convex topological vector space and let, for every $\iota \in I$, Φ_ι: $Y \to X$ be a linear map. The (set of) maps Φ_ι, $\iota \in I$, will be called (weakly) equi-compact if there is a neighbourhood W of zero in Y and a (weakly) compact set $C \subset X$ such that $\Phi_\iota(W) \subset C$, for every $\iota \in I$.

Let f: $\Gamma \to X$ be a bounded weakly continuous function. We put

$$F_\iota(g) = \int_\Gamma (g,\gamma)\,\omega_\iota(\gamma) f(\gamma)\,d\gamma, \qquad g \in G, \tag{1}$$

for every $\iota \in I$.

THEOREM 1. *There exists a (regular) measure μ: $\mathscr{B}(G) \to X$ such that $f = \hat{\mu}$, i.e.*

$$f(\gamma) = \int_G (-g,\gamma)\,\mu(dg), \qquad \gamma \in \Gamma, \tag{2}$$

if and only if, for every $\iota \in I$, the function F_ι is integrable (with respect to Haar measure) and the maps Φ_ι: $C_0(G) \to X$, $\iota \in I$, defined by

$$\Phi_\iota(\psi) = \int_G \psi(g) F_\iota(g)\,dg, \qquad \psi \in C_0(G),\ \iota \in I,$$

are weakly equi-compact.

THEOREM 2. *Let F: $G \to X$ be an integrable function. Let Φ_ι, $\iota \in I$, be defined as in Theorem* 1, *and let*

$$\Phi(\psi) = \int_G \psi(g) F(g)\,dg, \qquad \psi \in C_0(G).$$

Then $f = \hat{F}$, i.e.

$$f(\gamma) = \int_G (-g,\gamma) F(g)\,dg, \qquad \gamma \in \Gamma, \tag{3}$$

if and only if

$$\lim_\iota \Phi_\iota(\psi) = \Phi(\psi)$$

uniformly with respect to $\psi \in C_0(G)$, $\|\psi\| \leqslant 1$.

In the case X is the complex numbers field, these theorems were stated in [10] (under some restrictions) and in [8]. In this case the weak equi-compactness condition of Φ_ι, $\iota \in I$, reduces to the requirement that $\int |F_\iota(g)|\,dg \leqslant k$ for a constant k independent of ι and the uniform convergence condition of Theorem 2 means that $F_\iota \to F$ in L^1-norm on G. Since $L^1(G)$ is a complete space if X is a complex number field, Theorem 2 can be stated in a stronger form. There exists $F \in L^1(G)$ such that $f = \hat{F}$ if and only if $\{F_\iota\}$ is Cauchy in $L^1(G)$. Situation is similar in the case in which X is a Banach space and μ is of finite total variation or F is Bochner integrable as it has been shown in [3] (under some restriction on G).

1. Lemmas on vector integration. If P is a set and $\mathscr{S}$ a σ-ring of its subsets, by a measure μ: $\mathscr{S} \to X$ is meant a function on $\mathscr{S}$ with values in X which is σ-additive (no matter in which topology compatible with the dual-pair (X, X') since, by Orlicz–Pettis lemma, conditions of σ-additivity in all these topologies are equivalent).

LEMMA 1. *The range $R(\mu) = \{\mu(M)\colon M \in \mathscr{S}\}$ of any measure μ: $\mathscr{S} \to X$ is relatively weakly compact set in X.*

Proof. See [9].

If ν is a complex-valued measure on $\mathscr{S}$, then there is a set $M_\nu \in \mathscr{S}$ such that $\nu(M) = \nu(M \cap M_\nu)$, for all $M \in \mathscr{S}$ (see e.g. [5], 17 (3)). If ψ is a function on P (not necessary $\mathscr{S}$-measurable) such that ψ is ν-integrable on every such M_ν (in the usual sense e.g. as in [5]), we say that ψ is integrable and put $\int \psi\,d\nu = \int_P \psi\,d\nu = \int_{M_\nu} \psi\,d\nu$. It is easy to show (and well-known) that this definition is unambiguous.

Given a complex-valued function ψ on P and a measure $\mu\colon \mathscr{S}\to X$, we say that ψ is μ-integrable, if it is integrable with respect to every measure $M\to\langle\mu(M), x'\rangle$, $x'\in X'$, in the sense of the previous remark and if, for every $M\in\mathscr{S}$, there is an element $x_M\in X$ such that

$$(4)\qquad \langle x_M, x'\rangle = \int\limits_M \psi(p)\langle\mu(dp), x'\rangle, \qquad x'\in X'.$$

If ψ is integrable, the net $\{x_M\}_{M\in\mathscr{S}}$, where $\mathscr{S}$ is considered to be directed by inclusion, is weakly Cauchy, since, for every $x'\in X'$, there is $M_{x'}\in\mathscr{S}$ such that $\langle x_M, x'\rangle = \langle x_{M_{x'}}, x'\rangle$, for every $M\in\mathscr{S}$ such that $M_{x'}\subset M$. The correspondence $M\to x_M$, $M\in\mathscr{S}$, is a measure (it is obviously weakly σ-additive), hence, by Lemma 1 the set $\{x_M\colon M\in\mathscr{S}\}$ is relatively weakly compact. It follows that the net $\{x_M\}_{M\in\mathscr{S}}$ is weakly convergent. Its weak imit $x\in X$ has the property that

$$\langle x, x'\rangle = \int \psi(p)\langle\mu(dp), x'\rangle, \qquad x'\in X'.$$

We write

$$x = \int\psi\,d\mu = \int\psi(p)\,\mu(dp).$$

Any bounded function on P such that its restriction to every set $M\in\mathscr{S}$ is $\mathscr{S}$-measurable provides an example of a μ-integrable function. In fact, on every $M\in\mathscr{S}$, ψ is a uniform limit of $\mathscr{S}$-measurable finite-valued functions. Using the quasi-completeness of X we easily deduce the existence of $x_M\in\mathscr{S}$ such that (4) holds for every $M\in\mathscr{S}$.

If follows in particular that, for every bounded continuous function ψ on G and a measure $\mu\colon \mathscr{B}(G)\to X$, the integral $\int\psi\,d\mu$ has a meaning and is an element in X. Hence the Fourier–Stieltjes transform of any measure $\mu\colon\mathscr{B}(G)\to X$ can be defined by the equation (2).

Denote by $\mathscr{B}_0(G)$ the σ-ring of all Baire sets in G, i.e. the σ-ring generated by all compact G_δ sets in G.

A measure $\mu\colon\mathscr{B}(G)\to X$ is said to be *regular* if, for every $x'\in X'$, the complex measure $M\to\langle\mu(M), x'\rangle$, $M\in\mathscr{B}(G)$, is regular (i.e. its variation is regular in the sense of [5]).

Since any complex measure on $\mathscr{B}_0(G)$ is regular ([5]; 52 G), the concept of regularity of an X-valued measure on $\mathscr{B}_0(G)$ does not give anything new or, alternatively, every measure on $\mathscr{B}_0(G)$ is regular.

LEMMA 2. *Given any measure $\mu_0\colon\mathscr{B}_0(G)\to X$, there exists a unique regular measure $\mu\colon\mathscr{B}(G)\to X$ which is an extension of μ_0.*

Proof. It is proved in [4], Theorem 5, that there exists a unique regular extension μ of μ_0 onto $\mathscr{B}(G)$ with values in the completion of X. From the quasi-completeness of X and from Lemma 1 it follows easily that the values of μ lie in X, in fact.

The following proposition is virtually known. Because of the lack of a reference in a convenient form we include the proof. For the case G is compact and X is a Banach space it was first proved in [1].

PROPOSITION 1. *Let $\Phi\colon C_{00}(G)\to X$ be a linear map. Then there exists a regular measure $\mu\colon\mathscr{B}(G)\to X$ such that*

$$(5)\qquad \Phi(\psi) = \int\psi\,d\mu, \qquad \psi\in C_{00}(G)$$

if and only if Φ is weakly compact, i.e. the set

$$R_1(\Phi) = \{\Phi(\psi)\colon \psi\in C_{00}(G), \|\psi\|\leqslant 1\}$$

is relatively weakly compact in X.

Proof. If there is a measure $\mu\colon\mathscr{B}(G)\to X$ such that (5) holds, then $\Phi(\psi)\in\overline{\mathrm{co}}\,R(\mu)$, for every $\psi\in C_{00}(G)$, $\|\psi\|\leqslant 1$, where $\overline{\mathrm{co}}\,R(\mu)$ stands for the symmetric closed convex hull of the range of μ. Hence, by Lemma 1 and by the Krein theorem (e.g. [9]), the set $R_1(\Phi)$ is relatively weakly compact in X.

Suppose now that $R_1(\Phi)$ is relatively weakly compact. Its closure $R_1(\Phi)^-$ is symmetric convex weakly compact.

By the Riesz–Kakutani theorem, for every $x'\in X'$ there exists a unique regular complex measure $\nu_{x'}$ on $\mathscr{B}(G)$ such that

$$(6)\qquad \langle\Phi(\psi), x'\rangle = \int\psi\,d\nu_{x'}, \qquad \psi\in C_{00}(G).$$

Consider the system $\mathscr{S}$ of all sets $M\in\mathscr{B}(G)$ for which there exists an element $\mu(M)\in R_1(\Phi)^-$ with $\langle\mu(M), x'\rangle = \nu_{x'}(M)$, $x'\in X'$.

(a) If M is a set which can be expressed as a finite disjoint union of sets of the form $C_1 - C_2$ where C_1 and C_2 are compact G_δ, then its characteristic function χ_M is a pointwise limit of a bounded sequence $\{\psi_n\}$ of functions in $C_{00}(G)$. Since, by the Lebesgue Dominated Convergence Theorem,

$$\nu_{x'}(M) = \int\chi_M\,d\nu_{x'} = \lim_n\int\psi_n\,d\nu_{x'} = \lim_n\langle\Phi(\psi_n), x'\rangle, \qquad x'\in X',$$

and $\Phi(\psi_n)\in R_1(\Phi)^-$, the weak limit of $\{\Phi(\psi_n)\}$ exists. Hence, $M\in\mathscr{S}$ with $\mu(M)\in R_1(\Phi)^-$.

(b) Using the weak compactness of $R_1(\Phi)^-$ it is easy to show that $\mathscr{S}$ contains the limit of every monotonic sequence of sets in $\mathscr{S}$. Hence $\mathscr{B}_0(G)\subset\mathscr{S}$ (see [5]; 6 B).

(c) Lemma 2 together with the uniqueness of regular measures $\nu_{x'}$, $x'\in X'$, such that (6) holds, gives that $\mathscr{B}(G)\subset\mathscr{S}$.

The result readily follows.

Let now again P be a set and $\mathscr{S}$ a σ-ring of its subsets. Let λ be non-negative extended real-valued measure on $\mathscr{S}$ and let $F\colon P\to X$

be a function. We say that F is *λ-integrable* if, for every $M \in \mathcal{S}$, there exists an element $x_M \in X$ such that

$$\langle x_M, x' \rangle = \int_M \langle F(p), x' \rangle \lambda(dp), \quad x' \in X'.$$

If F is integrable, then the assignment $M \to x_M$ is a σ-additive function, hence Lemma 1 applies. Using the fact that $\{x_M\}_{M \in \mathcal{S}}$ is a weakly Cauchy net (if $\mathcal{S}$ is ordered by inclusion) and that its values belong to a weakly compact set we deduce that the weak $\lim_M x_M = x$ exists and belongs to X. We write

$$x = \int F d\lambda = \int F(p)\lambda(dp).$$

If ψ is a complex-valued function and $F\colon P \to X$ a λ-integrable function then about the existence of $\int \psi(p)F(p)\lambda(dp)$ can be decided using the properties of integrals with respect to vector-valued measures. We put

$$\nu(M) = \int_M F(p)\lambda(dp).$$

If ψ is ν-integrable, then it can be easily shown that $\int \psi(p)F(p)\lambda(dp)$ exists and equals to $\int \psi(p)\nu(dp)$. Using this remark one easily shows that the Fourier transform of any integrable function $F\colon G \to X$ is by (3) well-defined.

Lemma 3. *Let $F\colon G \to X$ be a function such that, for every $\psi \in C_{00}(G)$, there exists an element $x_\psi \in X$ with*

$$\langle x_\psi, x' \rangle = \int \psi(g) \langle F(g), x' \rangle dg, \quad x' \in X'.$$

Suppose that $\{x_\psi\colon \psi \in C_{00}(G), \|\psi\| \leqslant 1\}$ is a relatively weakly compact subset of X. Then F is integrable and

$$x_\psi = \int \psi(g)F(g)dg, \quad \psi \in C_{00}(G).$$

Proof. For $\psi \in C_{00}(G)$ define $\Phi(\psi) = x_\psi$. Application of the Proposition 1 gives the existence of a measure $\mu\colon \mathcal{B}(G) \to X$ such that (5) holds. For every $M \in \mathcal{B}(G)$,

$$\langle \mu(M), x' \rangle = \int_M \langle F(g), x' \rangle dg, \quad x' \in X',$$

hence F is integrable.

So far we made no use of the group structure of G. We used merely the fact that G is a locally compact space and all relevant result hold for any locally compact space. In the following lemma the group structure is important, however.

Lemma 4. *If $F\colon G \to X$ is integrable (with respect to Haar measure), then*

$$\lim_{h \to 0} \int_M \big(F(g-h) - F(g)\big) dg = 0$$

uniformly with respect to $M \in \mathcal{B}(G)$.

Proof. Denote

$$\nu(M) = \int_M F(g)dg, \quad \lambda(M) = \int_M dg$$

for $M \in \mathcal{B}(G)$.

Let U be a neighbourhood of 0 in X. Let U_1 be another neighbourhood such that $U_1 - U_1 + U_1 + U_1 - U_1 \subset U$. Let $C \subset G$ be a compact set such that $\nu(M) \in U_1$, for $M \in \mathcal{B}(G)$, $M \cap C = \emptyset$. Let $\delta > 0$ be such that $\nu(M) \in U_1$ if $\lambda(M) < \delta$ (see [7], Theorem 2.5, for the case X is a Banach space; general case is similar). Let V be a neighbourhood of 0 in G such that

$$\int |\chi_C(g-h) - \chi_C(g)| dg < \delta$$

for all $h \in V$.

Then, for all $M \in \mathcal{B}(G)$ and $h \in V$,

$$\lambda\big((M \cap C - h) \cap (M \cap C)'\big) < \delta,$$

$$\lambda\big((M \cap C) \cap (M \cap C - h)'\big) < \delta, \quad \lambda\big((M \cap C' - h) \cap C\big) < \delta,$$

$$\nu\big((M \cap C' - h) \cap C'\big) \in U_1, \quad \nu(M \cap C') \in U_1.$$

$$\int_M \big(F(g-h) - F(g)\big) dg = \nu(M-h) - \nu(M)$$
$$= \nu\big((M \cap C - h) \cap (M \cap C)'\big) - \nu\big((M \cap C) \cap (M \cap C - h)'\big)$$
$$+ \nu\big((M \cap C' - h) \cap C'\big) + \nu\big((M \cap C' - h) \cap C\big) - \nu(M \cap C') \in U.$$

Corollary. *If $F\colon G \to X$ is integrable, then*

$$\lim_{h \to 0} \int \psi(g) \big(F(g-h) - F(g)\big) dg = 0$$

uniformly with respect to $\psi \in C_0(G)$, $\|\psi\| \leqslant 1$.

In the next section we will have to prove some equalities of the form

$$\int_P \Big(\int_Q f(p, q)\nu(dq)\Big) \mu(dp) = \int_Q \Big(\int_P f(p, q)\mu(dp)\Big) \nu(dq)$$

where either one of the measures μ or ν or f is X-valued. Provided both integrals exist this equality is equivalent to

$$\Big\langle \int_P \Big(\int_Q f d\nu\Big) d\mu, x' \Big\rangle = \Big\langle \int_Q \Big(\int_P f d\mu\Big) d\nu, x' \Big\rangle, \quad x' \in X'.$$

In virtue of the definitions of integrals these equalities reduce to the equalities between scalar-valued integrals.

Hence we will interchange the order of iterated integrals freely if it is simple to decide about their existence and if the resulting scalar integrals are interchangeable (in virtue of Fubini's theorem, say).

2. Proofs of theorems. If $F\colon G\to X$ is integrable, define

$$(u_\iota * F)(g) = \int u_\iota(h)F(g-h)\,dh, \quad g\in G,\ \iota\in I.$$

If $\mu\colon \mathscr{B}(G)\to X$ is a regular measure, define

$$(u_\iota * \mu)(g) = \int u_\iota(g-h)\mu(dh), \quad g\in G,\ \iota\in I.$$

LEMMA 5. *$\{u_\iota\}_{\iota\in I}$ is an approximate identity for the space of X-valued integrable functions, i.e.*

$$\lim_\iota \int \psi(g)\big((u_\iota*F)(g)-F(g)\big)\,dg = 0$$

uniformly with respect to $\psi\in C_0(G)$, $\|\psi\|\leqslant 1$, for every integrable $F\colon G\to X$.

Proof. Let U be a closed convex symmetric neighbourhood of 0 in X. Let ν have the same meaning as in the proof of Lemma 4 and let C be a weakly compact (hence bounded) set in X such that $\nu(M)\in C$, for all $M\in\mathscr{B}(G)$. Let $\varepsilon>0$ be such that $2\varepsilon(C-C)\subset U$. Let V be a neighbourhood of 0 in G such that, for every $\psi\in C_0(G)$, $\|\psi\|\leqslant 1$,

$$\int \psi(g)\big(F(g-h)-F(g)\big)\,dg \in \tfrac{1}{2}U,$$

for every $h\in V$. V exists in virtue of the Corollary to the Lemma 4. Let $\iota_0\in I$ be such that $\int_V u_\iota(g)\,dg > 1-\varepsilon$ for $\iota\in I$, $\iota_0\leqslant\iota$. Let $\iota\in I$, $\iota_0\leqslant\iota$ and $\psi\in C_0(G)$, $\|\psi\|\leqslant 1$ be arbitrary. Since

$$\int\psi(g)\big(F(g-h)-F(g)\big)\,dg \in \overline{\mathrm{co}}\,(C-C)\subset \frac{1}{2\varepsilon}U,$$

for every $h\in G$, we have

$$\begin{aligned}\int \psi(g)\big((u_\iota*F)(g)-F(g)\big)\,dg &= \int\psi(g)\Big(\int u_\iota(h)\big(F(g-h)-F(g)\big)\,dh\Big)\,dg\\ &= \int u_\iota(h)\Big(\int\psi(g)\big(F(g-h)-F(g)\big)\,dg\Big)\,dh\\ &= \int_V u_\iota(h)\Big(\int\psi(g)\big(F(g-h)-F(g)\big)\,dg\Big)\,dh+\\ &\quad+\int_{G\cap V'} u_\iota(h)\Big(\int\psi(g)\big(F(g-h)-F(g)\big)\,dg\Big)\,dh\\ &\in \tfrac{1}{2}U+\tfrac{1}{2}U\subset U.\end{aligned}$$

LEMMA 6. *If $f=\hat{\mu}$, for a measure $\mu\colon\mathscr{B}(G)\to X$, then $F_\iota = u_\iota*\mu$, $\iota\in I$. If $f=\hat{F}$, for an integrable $F\colon G\to X$, then $F_\iota=u_\iota*F$, $\iota\in I$.*

Proof. By (1) and (2), given $g\in G$ and $\iota\in I$,

$$\begin{aligned}F_\iota(g) &= \int_\Gamma (g,\gamma)\omega_\iota(\gamma)\Big(\int_G(-h,\gamma)\mu(dh)\Big)\,d\gamma\\ &= \int_G\Big(\int_\Gamma(g-h,\gamma)\omega_\iota(\gamma)\,d\gamma\Big)\mu(dh) = \int_G u_\iota(g-h)\mu(dh).\end{aligned}$$

The second part is similar or a consequence of the first one.

LEMMA 7. *If $f=\hat{\mu}$, for a measure $\mu\colon\mathscr{B}(G)\to X$, then the functions F_ι, $\iota\in I$, are integrable and the operators Φ_ι, $\iota\in I$, are weakly equi-compact.*

Proof. Lemma 6 gives that F_ι is continuous since u_ι is uniformly continuous, for every $\iota\in I$. Let $\psi\in C_{00}(G)$ and $\iota\in I$ be arbitrary. For every $\varphi\in C_{00}(G)$, there exists $x_\varphi\in X$ such that

$$\langle x_\varphi, x'\rangle = \int\varphi(g)\psi(g)\langle F_\iota(g),x'\rangle\,dg, \quad x'\in X',$$

(see [2], III.3.2 Proposition); moreover if $\|\varphi\|\leqslant 1$, then x_φ belongs to the closed convex hull of the range of ψF_ι which is a compact set. Lemma 3 implies that ψF_ι is integrable.

Since

$$\int\psi(g)F_\iota(g)\,dg = \int\psi(g)\Big(\int u_\iota(g-h)\mu(dh)\Big)\,dg = \int\Big(\int\psi(g)u_\iota(g-h)\,dg\Big)\mu(dh)$$

and

$$\Big|\int\psi(g)u_\iota(g-h)\,dg\Big|\leqslant\|\psi\|\int u_\iota(g-h)\,dg = \|\psi\|,$$

we have

$$\int\psi(g)F_\iota(g)\,dg\in\|\psi\|\,\overline{\mathrm{co}}\,R(\mu).$$

Lemma 1 together with the Krein theorem implies the weak equi-compactness of restrictions of Φ_ι, $\iota\in I$, to $C_{00}(G)$. Lemma 3 implies the integrability of F_ι. Since $C_0(G)$ is the uniform closure of $C_{00}(G)$, the weak equi-compactness of Φ_ι, $\iota\in I$, on $C_0(G)$ follows easily.

LEMMA 8. *Let Φ_ι, $\iota\in I$, be equi-bounded. Then, for every $x'\in X'$, there exists a unique regular complex measure $\nu_{x'}$ on $\mathscr{B}(G)$ such that*

$$\lim_\iota\langle\Phi_\iota(\psi),x'\rangle = \int\psi(g)\nu_{x'}(dg), \tag{7}$$

for every $\psi\in C_0(G)$. Moreover,

$$\langle f(\gamma),x'\rangle = \int_G(-g,\gamma)\nu_{x'}(dg), \quad x'\in X'. \tag{8}$$

Proof ([8]). For every $\varphi\in L^1(\Gamma)$, put

$$J_{x'}(\hat{\varphi}) = \int_\Gamma\varphi(\gamma)\langle f(\gamma),x'\rangle\,d\gamma, \quad x'\in X'.$$

By the inversion theorem for Fourier transforms,

$$\int_G (-g, \gamma)\langle F_\iota(g), x'\rangle dg = \omega_\iota(\gamma)\langle f(\gamma), x'\rangle,$$

for all $\iota \in I$, $x' \in X'$. By Fubini's theorem

$$\langle \Phi_\iota(\hat\varphi), x'\rangle = \int_G \hat\varphi(g)\langle F_\iota(g), x'\rangle dg = \int_\Gamma \varphi(\gamma)\omega_\iota(\gamma)\langle f(\gamma), x'\rangle d\gamma.$$

Since $\omega_\iota(\gamma)\langle f(\gamma), x'\rangle \to \langle f(\gamma), x'\rangle$ uniformly on compact sets (and $\varphi(\gamma)d\gamma$ is a complex regular measure),

$$\lim_\iota \langle \Phi_\iota(\hat\varphi), x'\rangle = J_{x'}(\hat\varphi),$$

for every $\varphi \in L^1(\Gamma)$, $x' \in X'$.

Equi-boundedness implies the existence of a constant $k_{x'}$ such that

$$|\langle \Phi_\iota(\psi), x'\rangle| \leqslant k_{x'}\|\psi\|, \quad \text{for all } \psi \in C_0(G).$$

Moreover functions $\hat\varphi$, for all $\varphi \in L^1(\Gamma)$, lie densely in $C_0(G)$. Hence $\lim_\iota \langle \Phi_\iota(\psi), x'\rangle$ exists, for all $\psi \in C_0(G)$. Denote this limit by $J_{x'}(\psi)$ without ambiguity. We have

$$|J_{x'}(\psi)| = k_{x'}\|\psi\|, \quad \psi \in C_0(G).$$

The Riesz–Kakutani representation theorem gives the existence of a unique regular complex measure $\nu_{x'}$ such that (7) holds.

By Fubini's theorem now

$$\int_\Gamma \varphi(\gamma)\langle f(\gamma), x'\rangle d\gamma = \int_G \hat\varphi(g)\nu_{x'}(dg) = \int_\Gamma \varphi(\gamma)\Big(\int_G (-g, \gamma)\nu_{x'}(dg)\Big) d\gamma,$$

for every $\varphi \in L^1(\Gamma)$, hence (8) follows.

LEMMA 9. *If Φ_ι, $\iota \in I$, are weakly equi-compact then there exists a unique regular measure $\mu\colon \mathscr{B}(G) \to X$ such that*

$$\lim_\iota \langle \Phi_\iota(\psi), x'\rangle = \int \psi(g)\langle \mu(dg), x'\rangle, \quad x' \in X',$$

for every $\psi \in C_0(G)$ and

$$\langle f(\gamma), x'\rangle = \int_G (-g, \gamma)\langle \mu(dg), x'\rangle, \quad x' \in X'.$$

Proof. Let C be a symmetric convex compact set in X such that $\Phi_\iota(\psi) \in C$, for every $\iota \in I$ and $\psi \in C_0(G)$, $\|\psi\| \leqslant 1$. By Lemma 8, $\lim_\iota \langle \Phi_\iota(\psi), x'\rangle$ exists for every $\psi \in C_0(G)$ and $x' \in X'$. Since $\Phi_\iota(\psi) \in \|\psi\| C$, for all $\iota \in I$, there is $\Phi(\psi) \in X$ such that $\lim_\iota \langle \Phi_\iota(\psi), x'\rangle = \langle \Phi(\psi), x'\rangle$, $x' \in X'$, for every $\psi \in C_0(G)$.

Φ is a linear map from $C_0(G)$ to X and $\Phi(\psi) \in C$, for $\psi \in C_0(G)$, $\|\psi\| \leqslant 1$. By the Proposition 1, there exists a regular measure $\mu\colon \mathscr{B}(G) \to X$ such that

$$\langle \Phi(\psi), x'\rangle = \int_G \psi(g)\langle \mu(dg), x'\rangle, \quad x' \in X'.$$

Uniqueness of the measure in the Riesz–Kakutani representation theorem gives that $\langle \mu(M), x'\rangle = \nu_{x'}(M)$, $M \in \mathscr{B}(G)$, $x' \in X'$, where $\nu_{x'}$ is the measure satisfying (7) and, therefore, (8) as well. Hence the result.

Proof of Theorem 1 follows immediately from Lemma 7 and Lemma 9.

Proof of Theorem 2. If $f = \hat F$ for an integrable $F\colon G \to X$, then, by Lemma 6 and Lemma 5, $\lim_\iota \Phi_\iota(\psi) = \Phi(\psi)$ uniformly on the set $\{\psi\colon \|\psi\| \leqslant 1, \psi \in C_0(G)\}$.

If $\lim_\iota \Phi_\iota(\psi) = \Phi(\psi)$ uniformly on this set then $\lim_\iota \langle \Phi_\iota(\psi), x'\rangle = \langle \Phi(\psi), x'\rangle$ uniformly on $\{\psi\colon \psi \in C_0(G), \|\psi\| \leqslant 1\}$, for every $x' \in X'$. It follows that

$$\lim_\iota \int |\langle F_\iota(g), x'\rangle - \langle F(g), x'\rangle| dg = 0.$$

Since $\omega_\iota(\gamma)\langle f(\gamma), x'\rangle$ is the Fourier transform of $\langle F_\iota(g), x'\rangle$, by the inversion theorem for Fourier transforms, we have $\omega_\iota(\gamma)\langle f(\gamma), x'\rangle \to \langle \hat F(\gamma), x'\rangle$ uniformly on Γ, for every $x' \in X'$. Since $\omega_\iota(\gamma) \to 1$, for every $\gamma \in \Gamma$, the relation $\langle f(\gamma), x'\rangle = \langle \hat F(\gamma), x'\rangle$, $x' \in X'$, follows. Hence $\hat F = f$.

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On relatively disjoint families of measures, with some applications to Banach space theory

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INTRODUCTION

We give here details of the results announced in [21], and also extend these results to situations involving sets of arbitrary cardinality. Thus in [21] we proved that every injective Banach space of infinite dimension contains an isomorph of l^∞; here we prove that if an injective Banach space contains an isomorph of $c_0(\Gamma)$ for some set Γ, it contains an isomorph of $l^\infty(\Gamma)$ (Corollary 1.5 below). From this we deduce easily the result of Amir [1] that l^∞/c_0 is not injective, and assuming the continuum hypothesis, that if K is a closed subset of βN such that $C(K)$ is injective, then K is Stonian (Corollary 1.6). (βN denotes the Stone-Čech compactification of N, the discrete set of positive integers).

These results are consequences of the key Proposition 1.2 which asserts that if T: $l^\infty(\Gamma) \to B$ is an operator such that $T|c_0(\Gamma)$ is an isomorphism (i.e. $T|c_0(\Gamma)$ is one-one with closed range), then there is a $\Gamma' \subset \Gamma$ with $\operatorname{card}\Gamma' = \operatorname{card}\Gamma$ such that $T|l^\infty(\Gamma')$ is an isomorphism. (Throughout, "operator" [resp. "projection"] refers to a "bounded linear operator" [resp. "bounded linear projection"]. Throughout the introduction, B and X denote Banach spaces and Γ and Λ denote infinite sets). Proposition 1.2 in turn yields the considerably stronger Theorem 1.3, which implies immediately that if X is complemented in X^{**} and X contains an isomorph of $c_0(\Gamma)$, then X contains an isomorph of $l^\infty(\Gamma)$. (We regard X as being canonically imbedded in X^{**}.) Theorem 1.3 can also be used to prove a result concerning extensions of isomorphisms of subspaces of $l^\infty(\Gamma)$ into injective Banach spaces, thus generalizing a result in [13]. (Cf. Corollary 1.7 and the Theorem following it.)

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