Some uncomplemented subspaces of $C(X)$ of the type $C(X)$

by

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Introduction. In 1962, Amir [1] proved that $C[0,1]$ contains an uncomplemented subspace which is isometric to $C[0,1]$. In 1965, Arens [4] constructed a countable, closed subset $X$ of $[0,1]$ and a decomposition of $X$ such that the subspace of functions which are constant on each set of the decomposition is uncomplemented in $C(X)$. The crucial step in Arens’ construction depends upon the following theorem:

If $X$ is a compact set in a metric space $X$ and if the boundary of $X$ contains $n$ points, then each linear projection of $C(X)$ onto the subspace of functions constant on $X$ is of norm at least $3-2/n$.

Theorem 1.3. substantially generalizes this result and replaces the hypothesis of metrizability with normality and $T_1$. In Corollary 1.4, sufficient conditions on a decomposition $D$ are given in order that $C(X,D)$ will be uncomplemented in $C(X)$.

In Chapter 2, the main result is Theorem 2.9 which states that if $X$ is a $T_1$ space and if all successive derived sets $X^{(0)}, X^{(1)}, \ldots$ are non-empty, then there is a decomposition $D$ such that $C(X,D)$ is uncomplemented in $C(X)$. The following characterization is obtained if $X$ is a compact metric space: Some finite derived set of $X$ is empty if and only if for each Hausdorff decomposition $D$ of $X$, $C(X,D)$ is complemented in $C(X)$.

This answers a question raised by A. Pelczyński (cf. [17], p. 71).

The notation and terminology used herein follow Kelley’s General Topology, except for the following: a decomposition $D$ of a topological space $X$ is a disjoint collection of closed subsets of $X$ such that $X = \bigcup D$. The notation $C(X)$ is used to denote the space of bounded continuous scalar-valued functions in a topological space $X$ with $\|f\| = \sup_{x \in X} |f(x)|$.

A continuous function from a topological space $X$ onto a topological
space $Y$ is called an epimorphism. Two Banach spaces $X$ and $Y$ are isomorphic (isometric) if there exists a one-to-one linear epimorphism $T$ of $X$ onto $Y$ (with $\|T\| = \|[\bar{\cdot}]\|$). If $T$ is also multiplicative, we say that $X$ and $Y$ are algebraically isomorphic (algebraically isometric). A projection is a continuous linear operator from a Banach space $X$ into $X$ which is idempotent. A subspace $Y$ of $X$ is complemented in $X$ if there exists a projection from $X$ onto $Y$.

1. Lower bounds for norms of projections. The notation $X/D$ denotes the decomposition $D$ of a topological space $X$ with the quotient topology. If $D$ and $M$ are two decompositions of $X$, then $M$ implies $D$ if and only if for each $A$ in $M$ there is a $B$ in $D$ such that $A \subset B$. If $M$ implies $D$ and if we define $\lambda(B) = (A \in M : A \subset B)$ for each $B \in D$, then $\lambda(B) : B \in D$ is a decomposition of the quotient space $X/M$. This decomposition is denoted by $\lambda D$ and is called the quotient decomposition.

Suppose that $G$ is a family of disjoint subsets of a topological space $X$. If $B \subset X$, we say that $B$ is saturated (with respect to $G$) if and only if for each $A \in G$ either $A \subset B$ or $A \cap B = \emptyset$. A set $A$ in $G$ is called plural if it contains at least two members. An element $A$ of $G$ is a limit set if each neighborhood of $A$ intersects a plural set in $G \sim \{A\}$. Observe that a set in $G$ need not be plural in order to be a limit set. The notation $\bigcup G$ denotes $\{x : x \in A$ for some $A \in G\}$. The family $G^{(\alpha)}$ is the decomposition of $\bigcup G$ consisting of the plural limit sets of $G$ and remaining singleton sets. For an ordinal number $\alpha > 1$, the decomposition $G^{(\alpha)}$ of $\bigcup G$ is defined inductively. If $\alpha = \beta + 1$, then $G^{(\alpha)} = (G^{(\beta)})^{(\alpha)}$; if $\alpha$ is a limit ordinal, then $G^{(\alpha)}$ is the decomposition of $\bigcup G$ consisting of the plural sets in $\bigcup G^{(\beta)}$ and singleton sets. It is convenient to let $G^{(0)} = G$. If $G$ is a decomposition of $X$, $G^{(\alpha)}$ is called the $\alpha$ derived decomposition of $X$. These concepts are consistent with those introduced by R. Arens in [4].

A major difficulty with upper semicontinuous decompositions is that a plural set is replaced by it is a topological subspace. The resulting decompositions need not be upper semianominal. For example, for each $t$ in $\{0, 1, 1/2, 1/3, \ldots\}$ let $X_t$ denote the vertical segment $\{(t, y) : 0 \leq y \leq 1\}$ in the plane. Let $X = \bigcup X_t$. Let $D$ be the decomposition of $X$ whose elements are the sets $X_t$. Then $D$ is upper semianominal, but the decomposition obtained by replacing $X_t$ with its singleton subsots is not upper semianominal.

In order to avoid this difficulty, the notion of a contracting decomposition is introduced. Suppose that $X$ is a topological space, $D$ is a decomposition of $X$, and $A \subset D$. We say that $D$ is contracting at $A$ if and only if the decomposition of $X$ whose plural sets are the sets in $D \sim \{A\}$ is upper semianominal. Therefore, $D$ is contracting at $A$ if and only if $D$ is upper semianominal, and for each $x$ in $A$ and each neighborhood $U$ of $x$ there is a $(D \sim \{A\})$-saturated neighborhood $V$ of $x$ such that $V \subset U$. It is easy to see that $D$ is contracting at $A$ if and only if $D$ is upper semianominal and if $(A_{j})_{\alpha}$ is a family of sets of $D \sim \{A\}$ with $a_{j}$ and $b_{j}$ in $A_{j}$, then either (i) neither of the nets $(a_{j})$ and $(b_{j})$ converge to a point of $A$ or (ii) both converge to the same point. If $D$ is contracting at each of its sets, we say that $D$ is contracting. Thus, $D$ is contracting if and only if for each family $(A_{j})_{\alpha}$ of $D$-sets with $a_{j}$ and $b_{j}$ in $A_{j}$, $A_{j} \neq A$, either neither of the nets $(a_{j})$ and $(b_{j})$ converge to the same point. These definitions extend the definition of R. L. Moore (cf. [15], p. 285).

A basic property of contracting decompositions is stated in the following lemma:

**Lemma 1.1.** Let $D$ be an upper semianominal decomposition of a topological space $X$, let $D_{j} \subset D$, $D_{i} \subset D$, $D_{i} \cap D_{j} = \emptyset$, and suppose that $D$ is contracting at each set in $D_{i} \cup D_{j}$. Then the decomposition $X$ whose plural sets are the plural in $D \sim D_{i}$ is upper semianominal and contracting at each set in $D_{i}$.

If $g$ is a continuous map from a topological space $X$ into a topological space $Y$, then the induced mapping $g^{*}$ defined by $g^{*}(f) = f \circ g$ is a continuous, multiplicative, linear operator from $C(Y)$ into $C(X)$ of norm one (cf. [11], Chap. 10, and [19], p. 331). In particular, if $Y$ is the decomposition space $X/D$ and $g$ is the quotient map, then $g^{*}$ is an algebraic isometry of $C(X/D)$ onto the subspace of $C(X)$ consisting of the functions which are constant on each set in $D$. This fact is an easy consequence of Theorem 9 of [13], p. 95, and is part of the folklore of quotient space theory. It is generally convenient to identify the space $C(X/D)$ with the subspace of functions in $C(X)$ which are constant on each set in $D$ without specific reference to the isomorphism $g^{*}$.

Our next lemma removes the restriction in Theorem 3.1 of [4] that the decomposition set $D$ must contain only one plural set and weakens the restriction that the space must be metrizable. Another generalization of this theorem is given in Theorem 1.3.

**Lemma 1.2.** Let $X$ be a $T_{1}$ space and let $D$ be an upper semianominal decomposition of $X$. Suppose there exists a plural set $Y$ in $D$ such that $D$ is contracting at $Y$ and the boundary $\partial Y$ of $Y$ contains at least $n$ points. If $P$ is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geq 2/\pi$.

Moreover, if $\varepsilon > 0$, if $U$ is a neighborhood of $Y$, and if $y_{1}, \ldots, y_{n}$, are distinct points in $\partial Y$, then there exists an $\varepsilon$ and a neighborhood $V$ of $y_{i}$ such that for each $i$ in $V \sim Y$ there exists $f \in C(X)$ with $f(U - Y) = 0$, $\|f\| = \|f(1 - 1)\|$, and $P(f) > 2/\pi - \varepsilon$.

**Proof.** If $n = 0$, then the conclusion is clearly true. Hence, we assume $n \geq 1$. Let $M$ be the decomposition of $X$ consisting of the plural
sets in $D \sim \{Y\}$ and let $p$ be the quotient map of $X$ onto $X/M$. Then $M$ is upper semicontinuous (because $D$ is contracting at $Y$) and $X/M$ is a $T_1$-space (see [14], p. 185, or [13], p. 134, Problem M).

Suppose $B = \{y_1, y_2, \ldots, y_n\}$ is a set of distinct points in $\partial Y$ and $U$ is a neighborhood of $Y$. By possibly passing to a smaller neighborhood, we may assume that $U$ is both open and $D$-saturated. Then $p(B)$ consists of $n$ distinct points of $\partial Y$. Note that $p(X)$ is the only plural set in the decomposition $D/M$ of $X$. Since $C(X) = C(X/M) \sim C(X)$, the restriction $P'$ of $P$ to $C(X/M)$ is a projection of $C(X/M)$ onto $C(X) = C(X/M)$. But $X/D$ is homeomorphic to $(X/M)/(D/M)$ (cf. [5], p. 40, or [9], p. 72); hence, it follows from our identifications that $P' = (p^*)_D P p^*$ and that $P'$ is a projection of $C(X/M)$ onto the functions in $C(X/M)$ which are constant on $p(Y)$.

Let $R$ be the restriction operator of $C(X/M)$ onto $C(y_B)$. Suppose $\varepsilon > 0$ and $\delta = \varepsilon/3$. There exists $g$ of norm $1$ in $C(X/M)$ and $y$ in $B$ such that

\[ |(P'g - g)(yy)| > |R(P' - I)| - \varepsilon, \]

where $I$ denotes the identity operator on $C(X/M)$. By the continuity of $g$, there is an open neighborhood $W$ in $C(X/M)$ such that $|(P'g - g)(x)| > |(P'g - g)(yy)| - \delta$ for each $x$ in $W$. If we define $V = p^{-1}(W) \cup U$, then $V$ is an open $M$-saturated $X$-neighborhood of $y$ in $X$. Let $t$ be an element of $V \sim Y$. Then $p(t)$ belongs to the open subset $p(U) \sim p(Y)$ of $X/M$. We may assume that $(P'g - g)(pt) > 0$. By the Urysohn-Tietze Extension Theorem, there exists $f' \in C(X/M)$ such that $f'$ and $g$ are equal on $p(U)$, $f'$ vanishes off $p(U)$, and $f'(pt) = f'(pt) = 1$. Since $f' - g$ is constant on $p(Y)$, we have $P'(f' - g) = f' - g$ and $(P' - I)f' = (P' - I)g$. Let $f = pf'$. Since $p(t)$ belongs to $W$,

\[ P'(f') = (P'f)(pt) = (P'f)(pt) = (P' - I)f = (P' - I)f, \]

\[ |(P'f - f)(pt)| = 1 + |(P'g - g)(pt)| > 1 + |(P'g - g)(yy)| - \delta > 1 + |R(P' - I)| - 2\delta. \]

Let $\Theta$ be the functional on $C(X/M)$ such that $\Theta(f)$ is the constant value of $P'f$ on $p(Y)$. Let $(U_1, U_2, \ldots, U_n)$ be a family of disjoint open sets in $X/M$ such that $p(y_i)$ belongs to $U_i$ for each $i$. There exists $q$ in $C(X/M)$ such that $|q| = 1$ and $\Theta(q) > |\Theta| - \delta$. For each $i$, there exists by the Urysohn-Tietze Extension Theorem an $f_i$ in $C(X/M)$ such that $f_i = q$ on the complement of $U_i$, $\|f_i\| = 1$, and $f_i(yy) = -1$. Suppose that $\Theta(f_i) < \|\Theta\| (1 - 2/\varepsilon) - \delta$ for each $i$. Put $h = \sum f_i(y)$. Since $h_f - f_i = 0$ on the complement of $U_i$, we have $\|h\| = \max |h_f| = 2$. Then we obtain the contradiction.

\[ 2\|\Theta\| > |\Theta| |\Theta| = \sum \Theta(f_i) > \sum |(\Theta - \delta - |\Theta| + 2/\varepsilon)| |\Theta| + \delta = 2|\Theta|, \]

Thus for some $i$, $\Theta(f_i) > |\Theta| (1 - 2/\varepsilon) - \delta$. Then

\[ |R(P' - I)| > |R(P' - I)f_i| = |(P'f_i - f_i)(yy)| = \Theta(f_i) + 1 > 1 + |\Theta| (1 - 2/\varepsilon) - \delta. \]

Since $|\Theta| = |RP'\Theta| > 1$, we obtain that

\[ P(0) > 1 + |R(P' - I)| - 2\delta > 2 + |\Theta| (1 - 2/\varepsilon) - 3\delta > 3 - 2/\varepsilon. \]

As $\varepsilon > 0$ is arbitrary, and $|\Theta| = 1$, it follows that $|P| > 3 - 2/\varepsilon$.

The next theorem shows that the existence of repeated limits of plural sets is fundamental in raising the norm of projections of $C(X)$ onto decomposition subspaces of $C(X)$. A convenient feature of this theorem is that it can be applied to a decomposition that has no zero derived decompositions. One natural choice for the sets $S_1, S_2, \ldots, S_n$ is to let $S$ be a family of plural sets of the decomposition such that $S^{0-1}$ is non-zero and to let $S_i = S^{i-1}$ for each $i$. Another natural choice is to let $\Theta$ be a family of plural sets such that $S^{0-1}$ contains a plural non-limit set and to define $S_i$ to be the family of non-limit sets in $S^{i-1}$ for each $i$. The decomposition is required to be contracting at each set in each $S_i$ so that each neighborhood of each limit point of plural sets in these sets not only intersects a plural set but also contains a plural set. (If $q$ denotes the quotient map of the decomposition, a point $x$ in $X$ is a limit point of plural sets if for each neighborhood $U$ of $x$, $U \sim q(s)$ intersects a plural set.) An example is given following the theorem to show that the contracting condition can not be dropped. A more general theorem for the case in which $X$ is compact has been obtained independently by S. Ditor in his dissertation [7].

It is convenient to introduce the following definition prior to stating the theorem:

Definition. Let $m_1, m_2, \ldots, m_n$ be positive integers. A decomposition $D$ of a topological space $X$ has property $L(m_1, m_2, \ldots, m_n)$ if and only if there exist non-empty collections $S_1, S_2, \ldots, S_n$ of plural $D$-sets such that

(a) $D$ is contracting at each set in $\cup S_i$;
(b) the boundary of each set in $S_i$ contains at least $m_i$ points;
(c) the boundary of each set in $S_i$ contains at least $m_{i+1}$ limit points of the sets in $S_i \sim \{A\}$. 

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In particular, if $D$ is a contracting decomposition such that $D^{(k)}$ is non-zero and each plural set $A$ in $D^{(k)}$ contains at least $k$ limit points of the plural sets in $D^{(k-1)} \sim \sim \{A\}$ for $i = 2, \ldots, n$, then $D$ has property $L_{n}(k_{1}, k_{2}, \ldots, k_{n})$. If the boundary of each non-limit set in $D$ contains at least $k$ points, then $D$ also has property $L_{n+1}(k_{1}, k_{2}, \ldots, k_{n})$.

**Theorem 1.3.** If $D$ is a decomposition of a $T_{2}$-space $X$ with property $L_{n}(m_{1}, m_{2}, \ldots, m_{n})$ and $P$ is a projection of $C(X)$ onto $C(X/D)$, then

$$|P| \geq 2n+1 - \sum_{i=1}^{n} 2/m_{i}.$$ 

**Proof.** For $n = 1$ this inequality is a consequence of Lemma 1.2. Let $P$ be a projection of $C(X)$ onto $C(X/D)$. We shall consider the following property:

($\ast$) If $g > 0$, $A \in S_{n}$, and $G$ is a neighborhood of $A$, there exists $t$ in $G \sim A$ and $f$ in $C(X)$ such that $\|f\| = 1$, $f(x) \sim G = 0$, and

$$Pf(t) > 2n+1 - \sum_{i=1}^{n} 2/m_{i} - \delta.$$ 

If $n = 1$, ($\ast$) follows from Lemma 1.2.

Next, suppose that the theorem and property ($\ast$) hold for some positive integer $n$. Let $S_{1}, S_{2}, \ldots, S_{n}$ be non-empty collections of plural $D$-sets that satisfy (a), (b), and (c). Let $\delta > 0$. Choose $\varepsilon > 0$ so that $\varepsilon < 2\delta$. Let $Y$ be a $D$-plural set in $S_{n}$, and let $M$ be the decomposition of $X$ consisting of the plural sets in $D \sim (Y)$. Since $D$ is contracting at $X$, it follows that $M$ is upper semicontinuous and $X/M$ is a $T_{2}$-space. Let $a_{1}, a_{2}, \ldots, a_{n}$ be distinct points in $Y$ which are limit points of the plural sets in $S_{n} \sim (Y)$. Let $g$ denote the quotient map of $M$. Then $q(a_{1}), q(a_{2}), \ldots, q(a_{n})$ are distinct boundary points of $q(X)$ in $X/M$. Since $D/M$ has only the one plural set $q(Y)$, it is contracting.

Because $P$ is a projection of $C(X)$ onto $C(X/D)$ and $C(X/M) \circ C(X/M) = C(X)$, the restriction $P'$ of $P$ to $C(X/M)$ is a projection of $C(X/M)$ onto $C(X/M)$ (i.e., the functions in $C(X/M)$ which are constant on $q(Y)$). Technically, $P' = (q^{\ast})^{-1} P g^{\ast}$. Let $G$ be an $X$-centered neighborhood of $Y$. By Lemma 1.2 there exists an index $i$ and an open $X/M$-neighborhood $U$ of $q(y_{i})$ such that for each $x$ in $U \sim q(Y)$ there exists an $f_{x}$ in $C(X/M)$ such that $f_{x}$ vanishes off of $q(Y)$. If $f_{x} = 1$, and

$$P f_{x}(a) > 3 - 2/m_{n+1} - \varepsilon.$$ 

We may assume that $U \subset D$. Since $q(y_{i})$ is the limit point of points in $q(S_{n})$, there is an $A$ in $S_{n}$ such that $q(A)$ belongs to $U$. Since $X/M$ is $T_{2}$, there is a closed neighborhood $V$ of $q(A)$ contained in $U \sim q(Y)$. If $V' = \sim V$, then $V'$ is an $X$-neighborhood of $A$. By property ($\ast$) there

is a point $t$ in $V' \sim A$ and $g$ in $C(X)$ such that $g(x) \sim V' = 0$, $\|g\| = 1$ and

$$P g(t) > 2n+1 - \sum_{i=1}^{n} 2/m_{i} - \varepsilon.$$ 

Let $K = \sim q(Y)$ and observe that $K$ is a closed set in $X$ which does not intersect $V$. Since $X/M$ is $T_{1}$, there exists by the Urysohn-Tietze Extension Theorem a $h$ on $X/M$ such that $|h| = 1$, $h|F = 0$, and $h|V = 0$. Therefore, $h - f_{x}$ vanishes on $q(Y)$ and

$$P(h - f_{x})(q(t)) = \sum_{i=1}^{n} 2/m_{i} - \varepsilon - f_{x}(q(t))$$

since $\|g\| = 0$. Observe that $q^{\ast} h$ belongs to $C(X)$ and $\|q^{\ast} h\| = 1$. In fact, $q^{\ast} h = q^{\ast} g = 0$ if $x$ belongs to $V'$ and $g(x) = 0$ if $x$ does not belong to $V'$. Therefore,

$$P q^{\ast} h(t) + P q^{\ast} g(t) = (q^{\ast})^{-1} P q^{\ast} h(q(t)) + P g(t)$$

$$= P h(t) + P g(t) > (2-2/m_{n+1} - \varepsilon) + (2n+1 - \varepsilon - \sum_{i=1}^{n} 2/m_{i})$$

$$= 2(n+1) + 1 - \sum_{i=1}^{n} 2/m_{i} - \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, we have that

$$|P| \geq 2(n+1) + 1 - \sum_{i=1}^{n} 2/m_{i}.$$ 

It is easy to see that for $f = q^{\ast} h + g$, $t$ and $f$ satisfy $\ast$ for $n+1$. This completes the proof.

**Corollary 1.4.** If $D$ is a decomposition of a $T_{2}$-space $X$ with property $L_{n}(2, 2, \ldots, 2)$ for all $n$, then $C(X/D)$ is not complemented in $C(X)$.

The following example is due to J. Arens (cf. [1], p. 473):

**Example 1.5.** For each $i$ in $[0, 1]$, let $X_{i}$ denote the vertical segment $\{(t, y); 0 \leq y \leq 1\}$ in the plane. Let $X = \cup X_{i}$ and $D$ be the decomposition of $X$ whose elements are the sets $X_{i}$. Then $D$ is an upper semicontinuous decomposition of $X$. We use the families $S_{1}, S_{2}, \ldots, S_{n}$ of plural sets from $D$ by letting $S_{1} = D$ for each $i$, then each hypothesis of Theorem 1.3 with the exception of the contracting condition holds. However, $X/D$ is homeomorphic to the unit interval and there is a projection $P$ of $C(X)$ onto $C(X/D)$ with $|P| = 1$.

Let $X$ and $Y$ be Hausdorff spaces and $\psi$ be an epimorphism of $X$ onto $Y$. A continuous linear operator $\psi$ from $C(X)$ onto $C(Y)$ is a linear averaging operator for $\psi$ if and only if $\psi \psi^{\ast}$ is the identity on $C(Y)$.
is a simple relationship between linear averaging operators and projections (cf. [17], p. 16): ψ is a linear averaging operator with norm ≤ λ if and only if there is a projection with norm ≤ λ of C(X) onto its subspace ψ∗(C(X)). We let Dψ denote the decomposition of X consisting of the sets ψ∗(X) for ψ X. If ψ is closed or, equivalently, Dψ is upper semi-
continous, then X/Dψ is algebraically isometric to C(X). In [17], Pelczynski introduced the concept p(ψ) = inf {∥ψu∥: u is a linear averaging operator for ψ}. Therefore, p(ψ) = +∞ if and only if ψ does not admit a linear operator of averaging. By use of these notations, we can restate the two preceding results in terms of properties of continuous functions.

**Corollary 1.6.** Let ψ be an epimorphism of a T¬c-space X onto a Hausdorff space Y. If Dψ has property Ln(m1, m2, ..., mn), then

\[ p(ψ) ≥ 2n + 1 - \sum_{c1} 2/m1. \]

**Corollary 1.7.** Let ψ be an epimorphism of a T¬c-space X onto a Hausdorff space Y. If Dψ has property Ln(2, 2, ..., 2) for all n, then p(ψ) = +∞.

Suppose X is a topological space and D is a decomposition of X. We say that D is a metric decomposition of X if X/D is metrizable. Also, D is lower semi-continuous if the quotient map is open (cf. [14], p. 185, and [13], p. 97). In [4], B. Arens established results similar to Lemma 1.8 and Theorem 1.9 under the additional hypotheses that the decomposition is lower semi-continuous (line 2.06) and metric. He has commutated privately that upper semi-continuity should be included in the hypothesis of both results. The proof given here is similar to that given by Arens.

**Lemma 1.8.** Let X be a metric space and let D be an upper semi-
continous decomposition of X. If Pδ is a projection of C(X) onto C(X/Dδ), then there exists a projection P of C(X) onto C(X/Dδ) with \[\|P\| ≤ \|Pδ\| + 2.\]

The proof of this lemma is the same as the proof of statement 2.5 of Lemma 2.3 in [4], except that the definition of Q should be changed to

\[(Qf)(x) = \begin{cases} (Pδf)(z) & \text{for } x ∈ (Xz, Xz), \\
(Pδf)(z) & \text{for } x ∈ X. \end{cases}\]

The additional hypotheses of Lemma 3.5 are not needed.

**Theorem 1.9.** Let X be a metric space and let D be an upper semi-
continous decomposition of X with D0 = 0. Then there exists a projection P of C(X) onto C(X/D) with \[\|P\| ≤ 2n + 1.\]

Proof. Since D is upper semi-continous, the decompositions D0, D0, ..., Dn are also upper semi-continous. Also, Dn = 0; hence, it follows from Theorem 2.2 of [4], p. 471, or Theorem 8 of [30], p. 506, that there is a projection Pn of C(X) onto C(X/Dn) with \[\|Pn\| ≤ 2n + 1.\]

\[≤ 3. \text{Let } m \text{ be a positive integer less than } n \text{ and } P_m \text{ be a projection of } C(X) \text{ onto } C(X(D^{n-m})) \text{ with } \|P_m\| ≤ m + 1. \text{ Since } (D^{m+n}), (D^{n-m}), \text{ it follows from Lemma 1.8 that there exists a projection } P_{m+n} \text{ of } C(X) \text{ onto } C(X(D^{n-m})) \text{ with } \|P_{m+n}\| ≤ m + 1. \text{ Therefore, } \|P_{m+n}\| ≤ 2(m + 1) + 1 \text{ and the theorem is established.}

**Corollary 1.10.** Let ψ be a closed epimorphism of a metric space X onto a Hausdorff space Y. If Dψ = 0, then p(ψ) ≤ 2n + 1.

The following corollary is a immediate consequence of Corollaries 1.6 and 1.7.3.

**Corollary 1.11.** Let X be a metric space and let D be an upper semi-
continous decomposition of X such that each plural set A in X contains at least two limit points of the pointwise set D0 ≤ 1 \( A \) for each positive integer k. Then C(X/D) is isomorphically isomorphic to X in C(X) if and only if there exists a positive integer n such that Dn = 0.

2. Some uncomplemented subspaces of C(X). In this chapter we use the preceding results to determine a sufficient topological condition on X so that C(X) will have an uncomplemented subspace (Theorem 2.9). This result establishes a close relationship between uncomplemented subspaces of C(X) and the successive derived sets of X (cf. [14], p. 261, or [30], p. 64). We first restrict our attention to the derived sets of the subset D(X) consisting of the points of X which have a countable neighborhood base. If x is an ordinal number, then S(x) is denoted by S(x) and a point in S(x) is called an S-point.

The quotient space X/D inherits many of the properties of X. This is especially the case if D is upper semi-continous and each of its plural
sets is compact (e.g., see related discussion and problems in [5], [6], [8], [13], and [18]). In each result prior to Lemma 2.7, we also obtain Card \( (X) = \text{Card} \ (X/D) \) and for each ordinal number \( \lambda \), \( X^{(\lambda)} \neq \emptyset \) implies \( (X/D)^{(\lambda)} \neq \emptyset \) and \( S^t(X) \neq \emptyset \) implies \( S^t(X/D) \neq \emptyset \); moreover, if \( X \) is a compact metric space, then \( C(X) \) and \( C(X/D) \) are isomorphic by Corollary 3.7 of [17], p. 42. In this chapter, the hypothesis of each theorem and corollary is satisfied by the resulting space \( X/D \) whenever it is satisfied by \( X \) (except it may not be possible to use the same \( t \) for \( X/D \) in Theorem 2.1 and Corollary 2.4).

Let \( D \) and \( M \) be decompositions of \( X \). We say that \( M \) is a plural refinement of \( D \) if each plural set in \( D \) belongs to \( M \).

**Theorem 2.1.** Suppose that \( X \) is a compact space, \( n \) is a positive integer, and \( G \) is an open set in \( X \) with \( S^m(G) \neq \emptyset \). For each positive integer \( k \) and for each integer \( t \) with \( 1 \leq t \leq \text{Card} S^m(G) \), there exists a contracting decomposition \( D \) of \( X \) with the following properties:

1. Each plural set is contained in \( G \).
2. Each plural set consists of either \( t \) or \( k \) distinct points.
3. If \( M \) is an upper semicontinuous plural refinement of \( D \) which is contracting at each plural set of \( D \) and \( F \) is a projection of \( C(X) \) onto \( C(X/M) \), then

\[
\|F\| \geq 1 + 2n - \frac{2(n-1)}{k} - \frac{2}{t}.
\]

Before proving this theorem, we state three lemmas. The purpose of the first two lemmas is to simplify the task of showing that a decomposition is either contracting or upper semicontinuous. Most of the proof of the theorem is contained in the proof of the third lemma.

**Lemma 2.2.** Let \( X \) be a topological space and both \( D \) and \( M \) be decompositions of \( X \) such that

1. \( M \) is contracting (upper semicontinuous).
2. \( D \) implies \( M \).
3. For each \( A \in M \), the decomposition of \( X \) consisting of the plural sets of \( (F \times D; B = A) \) is contracting (upper semicontinuous).
4. Each limit set of \( M \) belongs to \( D \).

Then \( D \) is also contracting (respectively, upper semicontinuous).

**Lemma 2.3.** Let \( X \) be a topological space, \( D_1 \) and \( D_2 \) decompositions of \( X \), and \( D_1 \) a plural refinement of \( D_2 \). If \( D_2 \) is contracting (upper semicontinuous) and the decomposition of \( X \) consisting of the plural sets in \( D_1 \sim D_1 \) is contracting (respectively, upper semicontinuous), then \( D_1 \) is contracting (respectively, upper semicontinuous).

**Lemma 2.4.** Suppose \( X \) is a compact space, \( n \) is a positive integer and \( G \) is an open subset of \( X \) such that \( \text{Card} S^m(G) \geq 0 \). For each integer \( t \) with \( 1 \leq t \leq \text{Card} S^m(G) \) and for each positive integer \( k \), there is a contracting decomposition \( D \) of \( X \) such that

1. \( D^0 \) consists of singleton sets and a set \( F \) with \( t \) distinct points.
2. Each plural set in \( D \sim (F) \) consists of \( k \) points.
3. Each point in each plural set in \( D^0 \) is a limit point of the plural sets in \( D^0 \).
4. Each plural set in \( D \) is a subset of \( G \).

Proof. We inductively select non-empty families \( C_1, C_2, ..., C_n+1 \) and \( \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n+1} \) of subsets of \( G \) such that if \( 1 \leq m \leq n+1 \), then:

- Each set in \( C_m \) consists of \( k \) points from \( S^m(G) \) if \( m > 1 \).
- Each point of each set in \( C_{m+1} \) is a limit of the sets in \( C_m \).
- \( \mathcal{F}_m \) is a family of disjoint, closed subsets such that for each \( A \) in \( C_m \), there is a neighborhood \( U_A \) of \( A \) in \( \mathcal{F}_m \) which does not contain any other set in \( C_m \).
- If \( U \in \mathcal{F}_m \), then \( U \) does not intersect any set in \( C_j \) for \( 1 \leq j \leq m \).
- (e) The decomposition \( D_m \) of \( X \) consisting of the plural sets in \( m-1 \)

\[
\bigcup_{i=1}^{m-1} C_i \cup \mathcal{F}_m \text{ is contracting and each set in } \mathcal{F}_m \text{ is a non-limit set of } D_m.
\]

We select \( C_1 \) and \( \mathcal{F}_1 \) first. Let \( \{x_i\}_{i=1}^{m-1} \) be \( t \) distinct points of \( S^m(G) \) and define \( F = \{x_i\}_{i=1}^{m-1} \). Let \( C_1 = \{F\} \) and \( \mathcal{F}_1 = \{G\} \). It is easy to check that conditions (a) through (e) are satisfied for \( m = 1 \).

Next, suppose \( C_1, C_2, ..., C_n \) and \( \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n+1} \) have been selected and \( m < n \). Let \( A \) be a set in \( C_m \). We may suppose that \( A = \{a_1, a_2, ..., a_k\} \), where each \( a_i \) belongs to \( S^m(G), z = t \) if \( m = 1 \), and \( z = k \) if \( m > 1 \). There exists a neighborhood \( U_A \) of \( A \) in \( \mathcal{F}_m \) which does not intersect any other set in \( C_m \). We may select a family \( \{U_{a_i}\}_{i=1}^{m-1} \) of closed disjoint sets such that for each \( i \), \( U_i \) is a neighborhood of \( a_i \) and is contained in \( U_A \).

Let \( \{W_i\}_{i=1}^{m-1} \) be a closed neighborhood base for \( a_i \) such that \( V_i \subset U_i \) for each \( i \). Let \( \{a_i\}_{i=1}^{m-1} \) be a set of distinct \( S^m \) points such that \( a_i \neq a_j \) and \( a_i \) is an element of the interior of \( V_i \) for each \( i \). We may select a family \( \{W_i\}_{i=1}^{m-1} \) of disjoint closed sets such that \( W_i \) is a neighborhood of \( a_i \) and \( V_i \sim \{a_i\} \) is a neighborhood of \( W_i \).

Next, we define

\[
A_i = \{a_{i+k+1} \mid 1 \leq i \leq k \}
\]

for each \( 1 \leq i \leq z \) and \( j = 0, 1, 2, ... \),

\[
U_{i,j} = \bigcup_{i=1}^{m-1} W_{i+k+1}
\]

for each \( 1 \leq i \leq z \) and \( j = 0, 1, 2, ... \),

\[
C_i = (A_i \mid 1 \leq i \leq z \text{ and } j = 0, 1, 2, ...)
\]

and

\[
\mathcal{F}_m = (U_{i,j} \mid 1 \leq i \leq z \text{ and } j = 0, 1, 2, ...).
\]
Observe that each set in $C_\delta$ contains $k$ points from $B^{n-m}(\xi)$. Also, since $a_i$ converges to $a_i$ in $A$, each point in $A$ is a limit point of the sets in $C_\delta$. Therefore, if we define $C_{\delta+1} = \bigcup (C_d : A \in C_\delta)$, the hypotheses (a) and (b) are satisfied for $m+1$. If we also define $\psi_{\delta+1} = \bigcup (\psi_d : A \in C_\delta)$, it also follows that (c) and (d) are valid for $m+1$.

For each $A \in C_{\delta+1}$, we let $D_A$ be the decomposition of $X$ whose plural sets are the plural sets in $C_A$. It follows that each set in $C_A$ is a non-limit set in $D_A$. However, if $K \in C_A$ for some $A \in C_{\delta+1}$, $K \in C_A$, and $A \in C_\delta$ is a neighborhood of $K$. Since the only plural sets of $D_{\delta+1}$ that intersect $U_A$ are the plural sets in $U_A$, $K$ is a non-limit set of $D_{\delta+1}$. Therefore, each set in $C_{\delta+1}$ is a non-limit set of $D_{\delta+1}$.

The only limit sets in $D_{\delta+1}$ are the singleton sets $\{a\}$ for $a_i \in A$. Using this fact, it follows by a direct argument that $D_{\delta+1}$ is contracting. Let $M_{\delta+1}$ denote the decomposition of $X$ whose plural sets are the sets in $\bigcup (A)$. Since the decomposition of $X$ consisting of the one plural set $A$ is contracting, it follows from Lemma 3.3 that $M_{\delta+1}$ is contracting. Observe that $M_{\delta+1}$ is precisely the decomposition of $X$ whose plural sets are the plural sets of $D_{\delta+1}$, which are contained in $U_A$. Since (c) implies that each limit set of $D_{\delta}$ belongs to $D_{\delta+1}$, we have satisfied the hypothesis of Lemma 2.2 for $M = D_{\delta+1}$ and $D = D_{\delta+1}$. Thus $M_{\delta+1}$ is contracting and (c) is established for $m+1$.

By induction, it follows that $C_{\delta+1}, C_{\delta+2}, \ldots, C_{\delta+n}$ and $\psi_{\delta+1}, \psi_{\delta+2}, \ldots, \psi_{\delta+1}$ can be selected subject to the conditions (a)-(c). Let $D$ be the decomposition of $X$ consisting of the plural sets in $\bigcup \bigcup C_j$. It follows from Lemma 2.2 that since $D_{\delta+1}$ is contracting, $D$ is also contracting (i.e., let $D_{\delta+1} = M$ in this lemma).

By induction, we can show that for $0 \leq m \leq n$, $D^{(m)}$ satisfies the following properties:

(a) $\bigcup \bigcup C_j$ is the set of plural sets.

(b) $C_{\delta+n+1}$ is the set of non-limit plural sets.

From (a) we obtain that the set of plural sets of $D^{(m)}$ is $C_m$. However, $C_m = (P)$ by our construction, and $P$ consists of exactly $\delta$ points. This completes the proof of the lemma.

Proof of Theorem 2.1. By Lemma 2.4, there is a contracting decomposition $R$ of $X$ which satisfies conditions (1) through (4) of this lemma. Let $D = D^{(0)}$ and suppose $M$ is an upper semicontinuous plural refinement of $D$ which is contracting at each set in $D$. Let $S_i$ be the family of non-limit plural sets in $D^{(i-1)} \sim D^{(i)}$ for $1 \leq i \leq n$. It follows by Theorem 1.3 that if $P$ is a projection of $C(X)$ onto $C(X|M)$, then

$$|P| \geq 2n + 2|\delta| - \sum_{i=1}^{n} |P_i|,$$

This completes the proof.

A special case of Theorem 2.1 is stated in the following corollary:

**Corollary 2.5.** Suppose $X$ is a first-countable $T_0$-space with $\text{Card}(X) \geq 1$. For each $\epsilon > 0$, there is a contracting decomposition $D$ of $X$ with each plural set finite such that if $P$ is a projection of $C(X)$ onto $C(X|M)$, then $|P| \geq 2n + 1 - 2|\delta| - \epsilon$.

We shall see in Remark 2.10 that the requirement that $S^{(\delta)}(X)$ be non-empty in Theorem 2.6 can be replaced with the requirement that $X^{(\delta)}$ be non-empty for each positive integer $n$ if $X$ is first countable. Since the decomposition $D$ selected by this theorem is such that $S^{(\delta)}(X) \neq \emptyset$, it follows that if $X$ is a $T_0$-space with $S^{(\delta)}(X) \neq \emptyset$, then $X$ contains infinitely many uncomplemented subspaces of the type $C(Y)$.

**Theorem 2.6.** Suppose $X$ is a $T_0$-space such that $S^{(\delta)}(X) \neq \emptyset$. There is a contracting decomposition $D$ of $X$ such that $C(X|M)$ is not complemented in $C(X)$. If $k$ is an integer and $k \geq 2$, $D$ can be selected so that each plural set of $D$ contains exactly $k$ elements.

Proof. Let $\pi$ be an $\alpha$-point of $X$ and $(\{U_n\}_{n=1}^{\infty})$ an open neighborhood basis for $\pi$. By induction, we may select a sequence $(\pi_n)_{n=1}^{\infty}$ of distinct points such that for each $\pi, \pi_n$ is an $\alpha$-point, $\pi_n \neq \pi$, and $U_n$ is a neighborhood of $\pi_n$. Next, we inductively select subsets $V_1, V_2, \ldots, V_n, \ldots$ such that for each $n$

(a) $V_n$ is a closed neighborhood of $\pi_n$.

(b) $(V_n)_{n=1}^{\infty}$ is a family of disjoint closed sets.

(c) $V_n \subset U_n \sim \{\pi\}$.

Let $R$ be the decomposition of $X$ consisting of the plural sets $(V_n)_{n=1}^{\infty}$. This decomposition is contracting. Let $\delta$ be an integer greater than 1. By Theorem 2.1 there exists for each $n$ a contracting decomposition $M_n$ of $X$ with each plural set contained in $V_n$ such that if $M$ is a contracting, plural refinement of $M_n$ and $P$ is a continuous linear projection of $C(X)$ onto $C(X|M)$, then $|P| \geq n$. In fact, $M_n$ can be selected so that each plural set contains exactly $k$ elements. Let $D$ be the decomposition of $X$ consisting of the plural sets of $M_n$ for each $n$ and singleton sets. It follows by Lemma 2.2 that $D$ is contracting. Suppose $P$ is a continuous linear projection of $C(X)$ onto $C(X|M)$. Since $D$ is a contracting plural refinement of $M_n$ for each $n$, $|P| \geq n$ for each $n$. It follows from this contradiction that $C(X|M)$ is not complemented in $C(X)$.

Many interesting spaces, such as $\beta \mathcal{N}$ and $\beta \mathcal{N} \sim \mathcal{N}$, do not have non-trivial convergent sequences. It is known (cf. [32], p. 433) that no extreme disconnected space contains a non-trivial convergent sequence. (See [11] for additional information on $\beta \mathcal{N}$, $\beta \mathcal{N} \sim \mathcal{N}$, and extremely disconnected spaces.) Therefore, it is desirable to remove

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the dependence of the preceding theorems upon convergent sequences. The following lemma is our first step in this direction.

**Lemma 2.7.** Suppose $X$ is a $T_4$-space, $G$ is an open set in $X$, and $n$ is a non-negative integer. If $x \in G^{(n)}$, then there is an upper semicontinuous decomposition $D$ of $X$ with each plural set of $D$ contained in $G$ such that if $q$ is the induced quotient map, then $q(x)$ is an $S^n$-point of $q(G)$.

**Proof.** First, we establish the lemma for $n = 0$. We may select open neighborhoods $U_1, U_2, \ldots, U_n$ of $x$ such that $\text{Cl}(U_i) \subseteq G$ and, for each $U_i$, $\text{Cl}(U_{i+1}) \subseteq \text{Cl}(U_i)$. Define

$$K = \bigcap_{k=1}^{n} \text{Cl}(U_k).$$

Then $K$ is a closed subset of $G$ and $\{U_i\}_{i=1}^n$ is an open neighborhood base for $K$. Let $D$ be the decomposition of $X$ with the one plural set $K$ and $q$ the induced quotient map. Then $D$ is upper semicontinuous and $q(x)$ has a countable base. This establishes the lemma for $n = 0$.

Next, we suppose the lemma has been established for some $m < n$ and show that it is valid for $m + 1$. We inductively select $x_1, x_2, \ldots, x_{m+1}$ from $X$ and neighborhoods $U_1, U_2, \ldots, U_{m+1}$ of $x$ such that for each $k$,

(a) $x_k \in G^{(m)}$;
(b) $x_k \in U_k$ and $U_k$ is an open subset of $G$;
(c) $\text{Cl}(U_k) \subseteq U_{k-1} \sim \{x_{k-1}\}$ if $k > 1$.

Let $V_k = \bigcap_{l=1}^{k} \text{Cl}(U_k)$. Then $(V_k)_{k=1}^{m+1}$ is a family of disjoint open sets such that $x_k \in V_k$. Let $W_k$ be a closed neighborhood of $x_k$ contained in $V_k$. Define $F = \bigcap \text{Cl}(U_k)$ and let $M$ be the decomposition of $X$ consisting of the plural sets $(\{W_k\}_{k=1}^{m+1} \cup \{F\})$. Then $M$ is upper semicontinuous.

By inductive hypothesis, there is an upper semicontinuous decomposition $D_k$ of $X$ with each plural set of $D_k$ contained in $W_k$ such that if $q_k$ is the induced quotient map, then $q_k(x_k)$ is an $S^{m}$-point of $X/D_k$.

Let $D$ be the decomposition of $X$ whose plural sets are the sets in $F \cup (\bigcup_{k=1}^{m+1} D_k)$. By Lemma 2.2, it follows that $D$ is upper semicontinuous. If $q$ is the induced quotient map, then $q(x_k)$ is an $S^{m+1}$-point for each $k$. Since $q(F)$ has a countable base in $X/D$ and is the limit of a sequence of $S^n$-points, it is an $S^{m+1}$-point. This completes the proof.

The next theorem replaces the requirement that $S^n(X) \neq \emptyset$ in Theorem 2.1 with the requirement that $X^{\omega} \neq \emptyset$. As a result, it is also applicable to spaces in which sequences of distinct points do not converge. Although the decomposition $D$ of $X$ obtained from this theorem is upper semicontinuous, the plural sets are not necessarily finite. However, if $X$ is compact, the plural sets must be compact since they are closed. It is an immediate consequence of the Stone-Cech compactification theory that $C(X)$ is isometric to $C(X)$; hence, we may require that $X$ be compact without loss of generality. The construction of $D$ in this theorem ensures that $X/D$ has an $S^n$-point.

**Theorem 2.8.** Let $X$ be a $T_4$-space and $n$ a positive integer such that $X^{\omega} \neq \emptyset$. For each integer $t$ with $1 \leq t \leq \text{Card} X^{\omega}$ and for each $s > 0$ there is a decomposition $D$ of $X$ such that each projection of $C(X)$ onto $C(X/D)$ has norm at least $2n + 1 - 2t - s$.

**Proof.** Let $x_1, x_2, \ldots, x_t$ be distinct points in $X^{\omega}$. Let $\{U_i\}_{i=1}^t$ be a family of disjoint open sets such that $x_i \in U_i$ for each $i$. By Lemma 2.7, there exists an upper semicontinuous decomposition $H$ of $X$ with each plural set contained in $U_i$ such that if $q_i$ is the induced quotient map, then $q_i(x_i)$ is an $S^n$-point of $X/H$. Let $H$ be the decomposition of $X$ consisting of the plural sets in each $H_i$. Then $H$ is upper semicontinuous. If $q$ is the associated quotient map, then $q(x_1), q(x_2), \ldots, q(x_t)$ are distinct $S^n$-points of $X/H$.

Let $k$ be a positive integer such that $2(n-1)/k < \epsilon$. By Theorem 2.1, there is a contracting decomposition $K$ of $X/H$ such that if $P$ is a projection of $C(X/H)$ onto $C(X/H)/K$, then

$$|P| > 1 + 2n - \frac{2(n-1)}{k} - \frac{2}{t}.$$
(a) $x_n$ belongs to $X^{(0)}$;
(b) $\{V_i\}_{i=1}^\infty$ is a family of closed disjoint sets and $V_n$ is a neighborhood of $x_n$;
(c) if $G_n = \bigcup_{i=1}^n V_i$ contains an element of $X^{(0)}$, then $X \sim G_n$ contains an element of $X^{(0)}$.

The remainder of the proof consists of the consideration of two cases.

Case I. Suppose that $S = \{x_n\}_{n=1}^\infty$ is closed. Let

$$F = \operatorname{Cl}(\bigcap_{i=1}^\infty V_i) \sim \operatorname{Int}(\bigcap_{i=1}^\infty V_i),$$

where $\operatorname{Int}$ denotes the interior operator. Then $F$ is a closed set disjoint from $S$. By normality, there exists a closed neighborhood $V$ of $S$ which does not intersect $F$. Let $W_i = V_i \cap V$ for each $i$. Then $\bigcup_{i=1}^\infty W_i$ is closed for each subset $J$ of positive integers. If $M$ denotes the decomposition of $X$ consisting of the plural sets in $\{W_i\}_{i=1}^\infty$, then it follows from the preceding remark that $M$ is upper semicontinuous.

Since $W_i$ is a neighborhood of $x_i$ for each $i \in X^{(0)}$, it follows by Lemma 2.7 that $W_i$ is a neighborhood of each plural set in $K_i$ and if $K_i$ is the induced quotient map, then $k_i(x_i)$ is an $S^0$-point of $X/K_i$. Let $K$ be the decomposition of $X$ consisting of each plural set in $K_i$ for each $i$. As $M$ is upper semicontinuous and each $K_i$ is upper semicontinuous, it follows from Lemma 2.2 that $K$ is upper semicontinuous. Let $k$ be the quotient map of $X$ onto $K$. Then $k(x_i)$ is an $S^0$-point of $X/K$. By Theorem 2.1, there is a contracting decomposition $H_0$ of $X/K$ consisting of each plural set in $H_i$ contained in $k(W_i)$ such that if $H$ is an upper semicontinuous plural refinement of $H_0$, which is contracting at each set in $H_i$ and $P$ is a continuous linear projection of $C(X/K)$ onto $C((X/K)/H_0)$, then $|P| \geq i$.

Let $L$ be the decomposition of $X/K$ whose plural sets are the sets $k(W_i)$ for $i = 1, 2, \ldots$. Since $M$ does not have any singular or plural limit sets, $L$ does not either. Therefore, $L$ is a contracting decomposition. Let $H$ be the decomposition of $X/K$ consisting of each plural set in $H_i$ for each $i$. Since $H$ is contracting and $H_i$ is contracting for each $i$, it follows from Lemma 2.3 that $H$ is also contracting.

Let $D = (\bigcup A : A \in H)$. Then $D$ is a decomposition of $X$ and $H = D/K$. Since $H$ and $K$ are both upper semicontinuous, it follows that $D$ is upper semicontinuous ([3], p. 128, Problem 5). Suppose that $P$ is a projection of $C(X)$ onto $C(X/D)$. Since $C(X) \subset C(X/K) \subset C((X/K)/(D/K))$ and $(X/K)/(D/K)$ is homeomorphic to $X/D$ by [3], p. 40, or [5], p. 72, we have that the restriction $P'$ of $P$ to $C(X/K)$ is a projection of $C(X/K)$ onto $C((X/K)/H)$. But $H$ is a contracting plural refinement of each $K_i$; hence $|P| \geq i$ for each $i$. This contradicts the assumption that $C(X/D)$ is not complemented in $C(X)$.

Case II. Suppose that $S = \{x_n\}_{n=1}^\infty$ is not closed. Let $F = \operatorname{Cl}(S) \sim S$. By normality, there is a closed neighborhood $W_i$ of $x_i$ contained in $\operatorname{Int}(V_i)$. Let

$$F_j = \operatorname{Cl}(\bigcup_{i=1}^\infty W_i) \cup F$$

for each positive integer $j$. Then $F_j$ is a closed set contained in the closed set $\bigcap_{i=1}^\infty (X \sim \operatorname{Int}(V_i))$ and $W_i \cap F_j = \emptyset$ for $i < j$.

Next, we inductively select $U_1, U_2, \ldots$, so that for each $n$,

(a) $U_n$ is an open neighborhood of $F_{n+1}$;
(b) $\operatorname{Cl}(U_n) \subset U_{n+1}$;
(c) $\operatorname{Cl}(U_n)$ and $\bigcup_{i=1}^n W_i$ are disjoint.

We define $F^* = \bigcap_{i=1}^\infty (\operatorname{Cl}(U_i)/)$ and observe that $\{U_i\}_{i=1}^\infty$ is a countable base for $F^*$. If $M$ denotes the decomposition of $X$ consisting of the plural sets in $\{P_i\} \cup \{W_i\}_{i=1}^\infty$, then $M$ is upper semicontinuous.

As $W_n$ is a neighborhood of $x_n \in X^{(0)}$, there is an upper semicontinuous decomposition $K_n$ of $X$ with each plural set contained in $W_n$ such that if $k_n$ is the associated quotient map, then $k_n(x_n)$ is an $S^0$-point of $X/K_n$. Let $U_n$ denote the decomposition of $X$ consisting of the plural sets in $K_n$ for each $n$ and the set $F^*$. According to Lemma 3.2, $K$ is upper semicontinuous. Let $k$ be the quotient map of $X$ onto $X/K$. Then $k(W_n)$ is a neighborhood of $k(x_n)$ and $k(x_n)$ is an $S^0$-point of $X/K$. Also, $k(F^*)$ has a countable base and is the limit of the sequence $(k(x_n))_{n=1}^\infty$. Thus, $q(F^*)$ is an $S^0$-point of $X/K$.

By Theorem 2.6, there is an upper semicontinuous decomposition $H$ of $X/K$ such that $C(X/K)/H)$ is not complemented in $C(X)$. Let $D = (\bigcup A : A \in H)$. Then $H = D/K$ and by [5], p. 128, Problem 5, $D$ is upper semicontinuous since both $H$ and $X$ are upper semicontinuous. Also, $(X/K)/(D/K)$ is homeomorphic to $X/D$; hence, $(X/K)$ is not complemented in $C(X)$. This completes the proof.

Remark 2.10. If $X$ is a $T_\rho$-space with a closed subset of $\{x_n\}_{n=1}^\infty$ of distinct points such that $x_n \in S(X)$ for each $n$, then in Case I of the preceding argument the decompositions $K_i$ and $K$ are unnecessary. In this case, for each integer $n \geq 2$, $D$ can be selected so that it is contracting and each plural set consists of $n$ points.

Amir [2, 3] and Poleszczuk [17], p. 56, have established the equivalence of conditions (1) through (4) of Theorem 2.11. Recall that according to Theorem 2.9, the condition that $X^{(0)} = \emptyset$ for all $n$ is sufficient to ensure...
that a $T_\lambda$-space $X$ has an upper semicontinuous decomposition $D$ such that $C(X \setminus D)$ is uncomplemented in $C(X)$. It follows from conditions (1) and (5) of Theorem 2.11 that if $X$ is a compact metric space, then this condition is also necessary. Equivalence (10) gives an affirmative answer to question 30 (b) of Pełczyński in [17], p. 74. He communicated privately that the statement of Problem 30 should be changed to read: "Are the conditions (9.13.1)-(9.13.3) equivalent to the negation of the following conditions?"

For an ordinal number $\xi$, we let $[\xi]$ denote the set $\{\eta \mid \eta$ is an ordinal number and $\eta < \xi\}$ with the order topology. Let $\lambda > 1$. A separable Banach space $X$ is a $\beta_\lambda \omega$ space if for each separable Banach space $Y$ and for each linear isometry $u : X \to Y$, there is a projection $P$ of $Y$ onto $u(X)$ with $\|P\| \leq \lambda$.

**Theorem 2.11.** Suppose $X$ is an infinite compact metric space. Then the following conditions are equivalent:

1. Some derived set of $X$ of finite order is empty;
2. $C(X)$ is isomorphic to the space $e$;
3. $X$ is homeomorphic to $[\xi]$ for some ordinal $\xi < \omega^*$;
4. $C(X)$ is a $\beta_\lambda \omega$ space for some $\lambda > 1$;
5. for each upper semicontinuous decomposition $D$ of $X$, $C(X \setminus D)$ is complemented in $C(X)$;
6. for each Hausdorff space $X$ and each epimorphism $f$ of $X$ onto $Y$, $f^* (C(Y))$ is complemented in $C(X)$;
7. for each compact metric space $X$ and each epimorphism $f$ of $X$ onto $Y$, $f^* (C(Y))$ is complemented in $C(X)$;
8. for each compact Hausdorff space $X$ and each epimorphism $f$ of $X$ onto $Y$, there exists a linear averaging operator for $f$.

Proof. The equivalence of conditions (1) through (4) is established by [2], [3], and [17], p. 55. We prove the following implications: (1) $\Rightarrow$ (5) and (7) $\Rightarrow$ (8) $\Rightarrow$ (1). The implication (6) $\Rightarrow$ (7) is trivial.

(1) $\Rightarrow$ (5). Suppose that $D$ is an upper semicontinuous decomposition of $X$. Since $X^{(n)} = \emptyset$ for some positive integer $n$, we obtain that $D^{(n)} = \emptyset$. Therefore, by Theorem 2.9, $C(X \setminus D)$ is complemented in $C(X)$.

Next, suppose that $f$ is a epimorphism of $X$ onto a Hausdorff space $Y$, Since $X$ is a compact, $f$ is closed and $X$ has the quotient topology. Therefore, if $D = f^{-1}(Y)$, then $Y$ is homeomorphic to $X \setminus D$. Since $f$ is closed, $D$ is upper semicontinuous; hence, by (5), $f^* (C(Y))$ is complemented in $C(X)$.

(7) $\Rightarrow$ (8). Suppose that $f$ is a continuous function from $X$ onto a compact Hausdorff space $Y$. Since $X$ is a compact metric space and $f$ is a closed continuous map, it follows from Theorem 1 of [18] that $Y$ is a metric space. Therefore, by (7) there is a projection of $C(X)$ onto $f^* (C(Y))$. It follows that $f$ has a linear averaging operator.