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A duality in the theory of group representations

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INTRODUCTION

The development of the decomposition theory of group representations has caused the evolution of theorems on the duality in the theory of induced representations.

It was shown by Mackey [12]-[14] in 1952-60 that the notions of the intertwining operator and the direct integral can be used succesfully to obtain far generalizations of the classical Frobenius reciprocity theorem, referring to unitary representations of non-compact groups. The investigations of Gelfand and Piateckiĭ-Šapiro [4], [2] led in 1959 to the formulation of the famous duality theorem in the theory of automorphic functions. The form and the proof of the theorem exhibited a great utility of the notion of Gelfand triplet, which at the same time enabled K. Maurin to obtain the nuclear spectral theorem and related decomposition theorems. The labour of K. Maurin and L [18]-[20] accomplished the results of Gelfand and Piateckiĭ-Šapiro. They proved the duality theorem for arbitrary locally compact group G and Γ, without any assumption on G/Γ [20].

But it is suggested by the results of Bruhat [1] that the decomposition theory is more like the source of useful notions than a tool in the duality theory.

Bruhat created the theory of differentiable induced representations, which are beyond the scope of the decomposition theory, and he found a duality theorem using the notion of invariant bilinear form instead of that of the intertwining operator.

It was our aim to obtain a result similar to the theorem of Gelfand and Piateckiĭ-Šapiro but going for a large class of (as well as non-unitary) representations, namely **J**-representations.

The main result formulated in Theorem 3.6 concerns a Yamabe group G with a large subgroup and an arbitrary subgroup Γ.

Given a completely irreducible **J**-representation (U, J) of G we construct a Gelfand triplet $\Phi \subset J \subset \overline{\Phi}'$. We consider the space of nuclear operators intertwining for the differentiable induced representation $U^{\varkappa}$ and U ($\varkappa$ is an arbitrary homomorphism of Γ in C^1). We state the existence of a linear bijection of this space onto the space of common eigenvectors with eigenvalue $\varkappa$ of the contragradient representation ${}^tU|\Gamma$ in $\overline{\Phi}'$.

Since the non-unitary characters are admitted, even in the case of unitary representation U the theorem essentially generalizes the result of Gelfand and Piateckiĭ-Šapiro.

We begin with section 1 which comprises a summary of definitions and fundamental facts on the theory of representations in vector spaces. The second part contains more detailed description of compact group representations in spaces with indefinite metric.

Part 3 is devoted to prove the main theorem. The next sections are intended for showing its utility in various questions of the theory of representations. In section 4 our theorem in conjunction with K. Maurin's nuclear spectral theorem succeeds in immediate obtaining the fundamental Mackey's theorems on irreducible representations of a semidirect product of a compact and Abelian group. By terms in part 6 we prove a theorem on the form of an irreducible representation of a semisimple group in Pontryagin space. In section 5 we apply the main theorem to the theory of conical distributions and representations, introduced by Helgason [10] as dual objects to spherical functions and representations of a semisimple group with finite center. The theorem gives a possibility of such an extension of the definition of the conical representation, with that several Helgason's results, concerning principal series of representations may be generalized to the general case.

1. NOTATION AND FUNDAMENTAL CONCEPTS

Let G be a locally compact unimodular group with the Haar measure dg, and X a locally convex quasicomplete vector space.

Definition 1.1. By a *representation* of G in X we shall mean a homomorphism of G into the group of all invertible continuous operators on X with requirements that

(CI) the map $G \ni g \to V_g x \in X$ is continuous for each $x \in X$,

(CII) for every compact set $K \subset G$ the family of operators $\{V_k \mid k \in K\}$ is equicontinuous.

Condition (CI) implies (CII) in the case where X is a barreled space.

If Y is a homogeneous G-space, μ a Borel measure on G and ν a Borel measure on Y, then the *convolution* $\mu * \nu$ is defined by the formula

$$\int_Y f(y)\, d\mu*\nu(y) = \int_Y \int_G f(g\cdot y)\, d\mu(g)\, d\nu(y)$$

for $f \in C_0(G)$.

Let us assume that there exists a G-invariant measure dm on Y. Integrable functions f and φ on G and Y, respectively, will be identified with measure fdg and φdm. Then the *convolution of functions* is given by the formula

$$f*\varphi(y) = \int_G f(g)\varphi(g^{-1}\cdot y)\, dg.$$

Let $M(G)$ denote the convolution algebra of finite measures on G with compact supports. The topology in $M(G)$ is that of the inductive limit of the subspaces

$$M_K = \Big\{\mu \in M(G) \mid \operatorname{supp}\mu \subset K,\ \|\mu\| = \int d|\mu|\Big\}.$$

Each representation (X, V) of G can be extended to a representation of the algebra $M(G)$ by the formula

$$M(G) \ni \mu \to \int_G V_g\, d\mu(g) =: V_\mu \in L(X).$$

Condition (CII) guarantees the correctness of the definition. The map $\mu \to V_\mu$ is continuous if we regard $L(X)$ provided with the topology of the uniform convergence on bounded sets.

If μ is absolutely continuous with respect to dg with Radon-Nikodym derivative f, we shall write V_f instead of V_μ.

Given a representation (V, X) of G we define the *contragredient representation* $({}^tV, X')$ as

$$\langle x, {}^tV_g x'\rangle := \langle V_g^{-1}x, x'\rangle$$

for any $x \in X$ and $x' \in X'$.

The contragredient representation satisfies condition (CII) although in general it does not satisfy (CI).

Example. Put $X = L^1(R^1)$ and $X' = L^\infty(R^1)$. The additive group R^1 acts in X and X' by translations. Hence the map $R^1 \ni h \to L_h f \in L^\infty(R^1)$ is continuous if and only if f is uniformly continuous on R^1.

We shall be concerned with three kinds of irreducibility of representations.

Definition 1.2. The representation (V, X) of G is called *algebraically irreducible* if

(I1) X contains no non-trivial subspace invariant under V_g, $g \in G$;

topologically irreducible if

(I2) X contains no non-trivial invariant closed subspace;

completely irreducible if

(I3) every continuous operator on X is a strong limit of operators V_μ, $\mu \in M(G)$.

It is obvious that in the general case (I1) implies (I2). By the Burnside theorem all conditions are equivalent for finite-dimensional representations. Topological irreducibility is equivalent to complete irreducibility in the case where V is a unitary representation in a Hilbert space. This is also true for representations in a Hilbert space, whenever the set $\{V_\mu\}$ is invariant under the map $V \to V^*$ (von Neumann [22]).

Definition 1.3. A compact group $K \subset G$ is said to be *large* in G if every irreducible representation of K is contained at most finite many times in a completely irreducible representation of G.

Compact and Abelian groups are the trivial examples of groups with a large subgroup (namely $K = \{e\}$). The most important ones were considered by Harish-Chandra and Godement.

Lemma 1.4 (Harish-Chandra [6] and Godement [5]). *Any maximal compact subgroup of a connected semisimple group with finite center is large.*

Lemma 1.5 (Godement [5]). *Let G be a locally compact group of the form $G = K \cdot A$, where K is a compact and A is an Abelian subgroup of G. Then K is large in G.*

Let χ denote a character of the compact subgroup $K \subset G$. To each χ we associate a measure μ_χ given by the formula

$$\mu_\chi(f) := \int_K \chi(k) f(k)\, dk$$

and a projector

$$P_\chi := n_\chi^{-1} V(\mu_\chi) \in L(X),$$

where n_χ is the dimension of the irreducible representation of K corresponding to χ.

P_χ is the continuous projection of X onto the closed subspace invariant under V_k, $k \in K$. If X is the Hilbert space, $P_\chi X$ is the orthogonal sum of irreducible mutually equivalent representations associated to χ.

Differentiable induced representations. Throughout what follows G will denote a second countable Yamabe group, i.e. a projective limit of Lie groups,

$$G = \lim_{\leftarrow i} G/N_i = \lim_{\leftarrow i} G_i,$$

where N_i denotes a normal compact subgroup of G such that $G_i = G/N_i$ is a Lie group.

Yamabe [24] found that every connected locally compact group is a Yamabe group.

The class of regular functions with compact supports on G is defined as an inductive limit of Schwartz spaces $\mathcal{D}(G_i)$,

$$\mathcal{D}(G) := \lim_{\to i} \mathcal{D}(G_i),$$

where the injections $j_i\colon \mathcal{D}(G_i) \to \mathcal{D}(G_{i+1})$ are given by

$$j_i f(gN_{i+1}) := f(gN_i).$$

We shall often identify the space of functions on the group G_i with the space of functions on G invariant under translations from N_i.

$\mathcal{D}(G, X)$ will denote the class of regular functions with values in the topological vector space X and compact supports.

Let Γ denote a closed subgroup of G with left Haar measure $d\gamma$ and modular function Δ.

Lemma 1.6 (Mackey [13]). *There exist a continuous positive function ϱ on G for which*

$$\varrho(g\gamma) = \Delta(\gamma)\varrho(g)$$

and a measure dm on G/Γ such that

$$\int_G f(g)\varrho(g)\, dg = \int_{G/\Gamma} dm \int_\Gamma f(g\gamma)\, d\gamma \quad \text{for any } f \in C_0(G).$$

Let (V, X) be a representation of the subgroup Γ.

Denote by $\mathcal{D}^V$ the class of functions on G with values in X and satisfying the following conditions:

1° for any $f \in \mathcal{D}^V$ and $\varphi \in \mathcal{D}(G)$ the product φf belongs to $\mathcal{D}(G, X)$;

2° the support of f is compact modulo Γ (i.e. the set $\operatorname{supp} f \cdot \Gamma$ is compact in G/Γ);

3° $f(g\gamma) = \Delta(\gamma) V_\gamma f(g)$.

Let us define the mapping $\beta\colon \mathcal{D}(G, X) \to \mathcal{D}^V$ as follows:

$$\beta f(g) := \int_\Gamma \Delta(\gamma) V_\gamma^{-1} f(g\gamma)\, d\gamma.$$

Lemma 1.7 (Bruhat [1]). *The map β is onto $\mathcal{D}^V$ and commutes with left regular representation of G.*

In view of the lemma it is possible to introduce in $\mathscr{D}^V$ the topology of the quotient space $\mathscr{D}(G, X)/\mathrm{Ker}\,\beta$.

Definition 1.8. The left regular representation of G in $\mathscr{D}^V$ will be called the *differential representation induced by* (Γ, V, X) and denoted by U^Γ.

Let (V, X) and (U, Y) be the representations of G. The operator $\tau \in L(X, Y)$ satisfying

$$\tau \circ V_g = U_g \circ \tau$$

is called the *intertwining operator* for V and U.

2. GROUP REPRESENTATIONS IN SPACES WITH INDEFINITE METRIC

Definition 2.1. By ***J**-space* we mean a triplet $\{J, (\cdot|\cdot), I\}$ where $\{J, (\cdot|\cdot)\}$ is a Hilbert space, and I is an invertible linear operator on J.

An operator $H \in L(J_1, J_2)$ is called a *homomorphism* of J_1 in J_2 if

$$H^* I_2 H = I_1. \tag{2.1}$$

The operator I is uniquely determined by the formula

$$[x|y]: = (Ix|y)$$

which in general is indefinite.

Condition (2.1) is then equivalent to

$$[Hx|Hy] = [x|y]. \tag{2.2}$$

A representation (V, J) of G is called ***J**-representation* if the operators V_g are isomorphisms of **J**-structure.

Definition 2.2. The representations (V, J_1) and (U, J_2) are called *equivalent* if there exists an isomorphism $H: J_1 \to J_2$ intertwining for V and U.

If we change the inner product $(\cdot|\cdot)$ on J to equivalent one, we clearly obtain the equivalent **J**-representation with the isomorphism $H: = Id_J$.

Let us recall that **J**-spaces in contrast with Hilbert ones may contain closed subspaces without orthogonal complements (orthogonality is understood here and in the following with respect to the indefinite scalar product $[\cdot|\cdot]$).

Owing to this, the problem of analyzing a general **J**-representation in terms of irreducible ones presents unsurmountable obstacles. Theorem 2.3 shows that in the theory of compact group representations this difficulties do not appear.

THEOREM 2.3. *Let (V, J) be a **J**-representation of a compact group K. Then there exists an orthogonal sum decomposition of J,*

$$J = J_+ \oplus J_-,$$

where $J_\pm$ are closed, invariant under V_k, $k \in K$, subspaces, such that the form $[\cdot|\cdot]$ restricted to J_+ (J_-) is positive (negative) definite and induces in J_+ (J_-) the norm equivalent to $(\cdot|\cdot)^{1/2}$.

COROLLARY 2.4. *Theorem 2.3 yields the decomposition of the representation (V, J) into irreducible components.*

Let us notice that $(V, J_+, [\cdot|\cdot])$ and $(V, J_-, [\cdot|\cdot])$ are in essence unitary representations in Hilbert spaces, hence splits into orthogonal sums of irreducible finite-dimensional unitary representations:

$$\begin{gathered} J_\pm = \bigoplus_{i=1}^{\infty} J_\pm^i, \\ J = \Big(\bigoplus_{i=1}^{\infty} J_+^i\Big) \oplus \Big(\bigoplus_{k=1}^{\infty} J_-^k\Big), \qquad V = \Big(\bigoplus_{i=1}^{\infty} V_+^i\Big) \oplus \Big(\bigoplus_{k=1}^{\infty} V_-^k\Big). \end{gathered} \tag{2.3}$$

Proof of theorem 2.3. Let us change the inner product on J to equivalent one, being invariant by $V(K)$:

$$\{x|y\}: = \int_K (V_k x | V_k y)\, dk.$$

Now, we define $I \in L(J)$ by the formula

$$\{Ix|y\}: = [x|y].$$

The spectral theorem yields

$$I = \int_R \lambda\, dE(\lambda),$$

the operator I being hermitian. But we also have

$$\{V_k I x|y\} = \{Ix|V_k^{-1}y\} = [x|V_k^{-1}y] = [V_k x|y] = \{I V_k x|y\}$$

for all $x, y \in J$.

Hence V_k commutes with I as well as with all projectors $E(\lambda)$. The spectrum of I is separated from zero, I being non-singular. Therefore the operator

$$I_-: = \int_{\lambda<0} \lambda\, dE(\lambda)$$

is non-singular in the space $J_-: = E(0)J$, as well as the operator

$$I_+: = \int_{\lambda>0} \lambda\, dE(\lambda)$$

in the space $J_+: = \big(Id_J - E(0)\big)J$.

$J_{\pm}$ are clearly invariant under $V(K)$ and the proof follows.

Occasionally, we obtain

Lemma 2.5. *The class of irreducible representations of K in J-spaces is identical with the class of unitary irreducible representations in Hilbert spaces.*

Remark. The spaces $F_\chi := P_\chi J$ are mutually orthogonal in J and invariant by $V(K)$. In particular, this means that any triplet $\{F_\chi, I, V\}$ is a factor representation in the J-space $\{F_\chi, (\cdot|\cdot), I\}$.

3. THE DUALITY THEOREM

Let (U, J) be a topologically irreducible representation of G in the J-space J. Suppose that every irreducible representation of a compact subgroup $K \subset G$ is contained in $U|K$ with finite multiplicity.

We begin with the construction of a Gelfand triplet $\Phi \subset J \subset \Phi'$. Let us select any $x_0 \in F_\chi$ for certain χ. Consider a mapping $\alpha\colon \mathscr{D}(G) \to J$ defined by

$$\alpha(f) := U(f)x_0, \quad f \in \mathscr{D}(G). \tag{3.1}$$

The operator α intertwins the left regular representation of G in $\mathscr{D}(G)$ and the representation (U, J):

$$U_g \alpha(f) = U(L_g f), \tag{3.2}$$

$$P_\chi \alpha(f) = \alpha(P_\chi f). \tag{3.3}$$

The last equality implies that $\alpha(\mathscr{D}(G))$ is invariant under all projectors P_χ. We denote by Φ the range of α provided with the quotient topology of $\mathscr{D}(G)/\mathrm{Ker}\,\alpha$.

Although this construction involves a choice of χ and x_0, the space Φ does not depend upon it. Indeed, we have $P_\chi \Phi = F_\chi$ for any χ, since $P_\chi \Phi$ is dense in the finite-dimensional space F_χ. Hence by (3.3) all F_χ are contained in Φ, and the construction with $x_1 \in F_{\chi_1}$ leads to Φ_1, which contains $x_0 \in F_\chi$. This yields $\Phi_1 \supset \Phi$ as well as $\Phi_1 \subset \Phi$. In virtue of the above remarks the representation of the convolution algebra $\mathscr{D}(G)$ given by $f \to U(f)$ is algebraically irreducible in Φ.

Now we use the bilinear invariant form $[\cdot|\cdot]$ to define a continuous imbedding of the space J into $\overline{\Phi}'$ — the space of all antilinear continuous functionals on Φ with the topology of weak convergence:

$$\eta\colon J \ni x \to \eta x = [\cdot|x].$$

The map η is the intertwining operator for the representation U restricted to Φ acd ${}^tU$. We verify that

$$\langle \varphi, {}^tU_g \eta x\rangle = \langle U_g^{-1}\varphi, \eta x\rangle = [U_g^{-1}\varphi|x] = [\varphi|U_g x] = \langle \varphi, \eta U_g x\rangle$$

for $x \in J$ and $\varphi \in \Phi$.

Since Φ is isomorphic to $\mathscr{D}(G)/\mathrm{Ker}\,\alpha$, the space $\overline{\Phi}'$ can be identified with a closed subspace of $\mathscr{D}'(G)$. Namely:

$$\overline{\Phi}' \cong \{\overline{T} \mid T \in \mathscr{D}'(G),\ T|\mathrm{Ker}\,\alpha = 0\}.$$

Both spaces are nuclear, whence quasireflexive. Since the form $[\cdot|\cdot]$ is non-degenerate on Φ, the subset $\eta\Phi$ of $\overline{\Phi}'$ is total, whence dense in $\overline{\Phi}'$ by quasireflexivity of Φ.

The construction of the Gelfand triplet $\Phi \subset J \subset \overline{\Phi}'$ is finished.

Let us now fix $\omega_0 \in \overline{\Phi}'$ and consider a mapping $\sigma\colon \mathscr{D}(G) \to \overline{\Phi}'$ defined by

$$\langle \varphi, \sigma f\rangle := \langle U(f^*)\varphi, \omega_0\rangle, \quad \text{where } f^*(g) := \bar{f}(g^{-1}).$$

Let us define:

$$(X\mathscr{D})_0 := \{f \in \mathscr{D}(G) \mid P_\chi f = f \text{ for some } \chi\}$$

$$X\mathscr{D} = \text{linear envelope of } (X\mathscr{D})_0.$$

Lemma 3.1. *There exists a $\tau\colon X\mathscr{D} \to \Phi$ which makes the following diagram commutative:*

$$\begin{array}{ccccc} \mathscr{D}(G) & \xrightarrow{\alpha} & \Phi \subset J & \xrightarrow{\eta} & \overline{\Phi}' \\ & & \tau\uparrow & & \uparrow\sigma \\ & & X\mathscr{D} & \xrightarrow{\text{injection}} & \mathscr{D}(G) \end{array}$$

Proof. Let $P_\chi f = f$, $P_{\chi_1} f_1 = f_1$, $\varphi = \alpha(f_1)$;

$$\langle \varphi, \sigma f\rangle = \langle U(f^*)U(f_1)x_0, \omega_0\rangle = \langle U(f^* * f_1)x_0, \omega_0\rangle.$$

But $f^* * f_1 = (P_\chi f)^* * (P_{\chi_1} f_1) = f^* * (P^*_{\chi * \chi_1} f_1)$. Since $\chi \neq \chi_1$ implies $\chi^* * \chi_1 = 0$, the functional σf vanishes on the subspace $F = \bigoplus_{\chi_1 \neq \chi} F_{\chi_1}$ of Φ.

Let us now notice that $\dim\{\omega \in \overline{\Phi}' \mid \langle P_{\chi_1}\Phi, \omega\rangle = 0 \text{ for } \chi_1 \neq \chi\} = \dim\{\Phi/\text{closure of } F\} = \dim F_\chi$.

But the linear subspace $\eta(F_\chi) \subset \overline{\Phi}'$ consists just of the functionals vanishing on F. Since $\dim \eta(F_\chi)$ equals $\dim F_\chi$, η being monomorphism, the range of $P_\chi \mathscr{D}(G)$ under σ is identical with $\eta(F_\chi)$.

Hence the map $\tau := \eta^{-1}\circ\sigma$ is well defined on $X\mathscr{D}$ and the proof follows.

Let us remark that if $f \in X\mathscr{D}$, then $\sigma L_g f$ belongs to $\eta\Phi$ and $\tau L_g f$ is well defined. Indeed:

$$\langle\varphi, \sigma L_g f\rangle = \langle U^*(L_g f)\varphi, \omega_0\rangle = \langle \big(U_g U(f)\big)^*\varphi, \omega_0\rangle$$
$$= \langle U(f^*)U_g^{-1}\varphi, \omega_0\rangle = \langle U_g^{-1}\varphi, \sigma f\rangle = [\varphi, U_g\tau f].$$

Hence $\tau(L_g f) = U_g\tau f$.

This equality defines τ on the linear span of the subset

$$\{f \in \mathscr{D}(G)\colon f = f_1 * f_2, \quad f_1 \in \mathscr{D}(G), \quad f_2 \in X\mathscr{D}(G)\}.$$

Let us define a hermitian form on $(X\mathscr{D})\times(X\mathscr{D})$ as follows:

$$B(f, f_1) = [\tau(f)\,|\,\tau(f_1)] = \langle\tau(f), \sigma f_1\rangle = \langle U(f^*)\tau(f), \omega_0\rangle.$$

With the last expression $B(\cdot,\cdot)$ is evidently partially continuous with respect to the topology in $X\mathscr{D}$ induced by $\mathscr{D}(G)$. But B is also invariant:

$$B(L_g f, L_g f_1) = [U_g\tau(f)\,|\,U_g\tau(f_1)] = B(f, f_1).$$

Therefore any extension of B to $\mathscr{D}(G)\times X\mathscr{D}$ satisfies

$$B(f, f_1) = B(f, P_\chi f_1) = B(P_\chi f, f_1) \tag{3.4}$$

where $f_1 = P_\chi f_1$ and $f \in \mathscr{D}(G)$. Since the right-hand side of (3.4) is well defined for an arbitrary $f \in \mathscr{D}(G)$, the above equality uniquelly determines this extension.

Unfortunately it is not in general continuous with respect to the second variable. We are forced to introduce a linear dense submanifold continuously imbedded into $\mathscr{D}(G)$, which is more convenient domain of B. To this end we need the following

LEMMA 3.2. *The linear submanifold $X\mathscr{D}$ is dense in $\mathscr{D}(G)$.*

Proof. Let us recall that $\mathscr{D}(G) = \lim\limits_{\to i}\mathscr{D}(G_i)$. Suppose that $T \in \mathscr{D}'(G)$ vanishes on $X\mathscr{D}$. Consider the restriction of T to any $\mathscr{D}(G_i)$. Select a compact set K_1 such that $K\cdot K_1\cdot N_i = K_1$.

The topology in the subspace $\mathscr{D}(K_1)\cap\mathscr{D}(G_i)\subset\mathscr{D}(G)$ of functions from $\mathscr{D}(G_i)$ with supports in K_1 is given by the family of seminorms $\|\cdot\|_r$:

$$\|\varphi\|_r^2 := \sum_{|\alpha|\leqslant r}\int_G \overline{D^\alpha\varphi}\, D^\alpha\varphi\, dg,$$

where D^α are left invariant differential operators on G.

There exists an r such that $|T(\varphi)| \leqslant c\|\varphi\|_r$ for $\varphi \in \mathscr{D}(K_1)$. Hence T is a continuous functional on the prehilbert space $\mathscr{D}(K_1)$ equipped with the inner product

$$(\varphi\,|\,\psi) := \sum_{|\alpha|\leqslant r}\int_G \overline{D^\alpha\varphi}\, D^\alpha\psi\, dg.$$

The group K operates in $\mathscr{D}(K_1)$ by left translations and the representation is unitary with respect to $(\cdot|\cdot)$.

The space $\mathscr{D}(K_1)\cap X\mathscr{D}$ is the direct sum of all factor subrepresentations contained in $\mathscr{D}(K_1)$ and so is dense in $\big(\mathscr{D}(K_1), (\cdot|\cdot)\big)$. Hence T vanishes on $\mathscr{D}(K_1)$ for any K_1, consequently on $\mathscr{D}(G_i) = \lim\limits_{\to K_1}\mathscr{D}(K_1)$, and on the whole $\mathscr{D}(G)$.

Applying the Hahn-Banach theorem we are completing the proof.

DEFINITION 3.3. Let $\tilde{\mathscr{D}}(G)$ be the linear space spanned by the set $\{f \in \mathscr{D}(G)\colon f = f_1 * f_2, \quad f_1 \in \mathscr{D}(G),\ f_2 \in X\mathscr{D}(G)\}$.

We equip $\tilde{\mathscr{D}}(G)$ with the finer topology absorbing all (bounded in $\mathscr{D}(G)$) sets of the form

$$\tilde{\mathscr{B}}_{\varphi,\mathscr{B}} = \{f \in \mathscr{D}(G)\colon f = f_1 * f_2,\ f_1 \in \mathscr{B},\ \varphi \in X\mathscr{D}(G)\}$$

where $\mathscr{B}$ ranges through bounded sets in $\mathscr{D}(G)$.

The immediate consequence of the definition is that the topology in $\tilde{\mathscr{D}}(G)$ is finer then that induced by $\mathscr{D}(G)$.

In Appendix we prove the following

LEMMA A.1. 1° *The form B on $X\mathscr{D}(G)\times X\mathscr{D}(G)$ is extendible to the continuous invariant form on $\tilde{\mathscr{D}}(G)\times\mathscr{D}(G)$.*

2° *Any continuous hermitian symmetric form on $\tilde{\mathscr{D}}(G)\times\mathscr{D}(G)$ is of the form $(\varphi,\psi)\to T(\varphi^* * \psi)$ where $T \in (\tilde{\mathscr{D}}(G))'$.*

PROPOSITION 3.4. *The map $\tau\colon \tilde{\mathscr{D}}(G)\to J$ is continuous.*

Proof. Theorem 2.3 implies a direct sum decomposition, $J = J_+ \oplus J_-$, such that the invariant form $[\cdot|\cdot]$ is not only definite on $J_\pm$ but on each of them it induces a norm equivalent to the previous one. Let us put

$$\mathscr{D}_0 = \{\varphi \in \tilde{\mathscr{D}}(G)\,|\,T(\varphi * \psi) = 0 \text{ for each } \psi \in \mathscr{D}(G)\},$$

and denote by π the natural homomorphism onto the quotient space $\tilde{\mathscr{D}}(G)/\mathscr{D}_0$. The kernel of τ is contained in $\mathscr{D}_0$, whence the map $\bar{\tau}\colon \pi(X\mathscr{D})\to J$ is well defined by $\tau(\pi\varphi) = \bar{\tau}\varphi$.

Let us write

$$X\mathscr{D}_\mp = \{\varphi \in \pi(X\mathscr{D})\,|\ [\bar{\tau}\varphi\,|\,J_\pm] = 0\}.$$

Clearly, $\pi(\tilde{\mathscr{D}}(G)) = \mathscr{D}_+ + \mathscr{D}_-$, where $\mathscr{D}_\mp$ denote the closure of $X\mathscr{D}_\pm$ in $\tilde{\mathscr{D}}(G)$. The maps $p_\pm$ are the projectors in $\pi(\mathscr{D}(G))$:

$$p_\pm\colon \pi\big(\tilde{\mathscr{D}}(G)\big)\ni\varphi\to\varphi_\pm \in \pi\big(\mathscr{D}(G)\big),$$

where $\varphi_+ \in \mathscr{D}_+$, $\varphi_- \in \mathscr{D}_-$, and $\varphi = \varphi_+ + \varphi_-$.

The map $p := p_+ - p_- = Id - 2p_-$ is an isomorphism of $\pi(\tilde{\mathscr{D}}(G))$ and by the general closed graph theorem it is continuous. This implies that $p_\pm$ are both continuous. Now we see that the mapping $\varphi\to p_\pm\varphi \to T\big((p_\pm\varphi)^* * (p_\pm\varphi)\big)$ is continuous. But the ranges of $X\mathscr{D}$ under $\bar{\tau}\circ p_\pm$

are contained in $J_\pm$, respectively, where the norms

$$T\big((p_\pm\varphi)^**(p_\pm\varphi)\big)^{1/2} = [\bar{\tau}\circ p_\pm\varphi\,|\,\bar{\tau}\circ p_\pm\varphi]^{1/2}$$

are equivalent to the primary ones. Hence the maps $\pi(X\mathscr{D})\ni\varphi\to\bar{\tau}p_\pm\varphi$ as well as $\tau = \bar{\tau}\circ p_+\circ\pi+\bar{\tau}\circ p_-\circ\pi$ are all continuous.

The proof results now from Lemma 3.2.

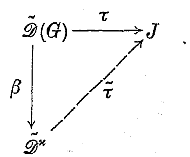
THEOREM 3.5. *Under the above assumptions on G and (U, J) let Γ be a closed subgroup of G. Let $\omega_0\in\bar{\Phi}'$ be an eigenvector of the contragradient representations ${}^tU|\Gamma$ in $\bar{\Phi}'$ with the eigenvalue $\varkappa$:*

$$(3.5)\qquad {}^tU_\gamma\omega_0 = \varkappa(\gamma)\,\omega_0.$$

Then there exists a non-zero operator $\tilde{\tau}\in L(\tilde{\mathscr{D}}^\varkappa, J)$ intertwining for $U^\varkappa$ and U.

By $\tilde{\mathscr{D}}^\varkappa$ we denote the space $\beta(\tilde{\mathscr{D}}(G))$ with projective topology transported by β.

Proof. We have to complete the diagram



where τ is the intertwining operator in Proposition 3.4 and β is the linear homomorphism in Lemma 1.7. Owing to (3.5) we have for any $f\in\tilde{\mathscr{D}}(G)$ and $\varphi\in\Phi$:

$$[\varphi\,|\,\tau(R_\gamma f)] = \langle\varphi, \sigma R_\gamma f\rangle = \langle U^*(R_\gamma f)\varphi, \omega_0\rangle = \langle U_\gamma^{-1}U(f^*)\varphi, \omega_0\rangle$$
$$= \langle U(f^*)\varphi, {}^tU_\gamma\omega_0\rangle = \varkappa(\gamma)\langle\varphi, \sigma f\rangle.$$

Thus

$$(3.6)\qquad \tau(R_\gamma f) = \varkappa(\gamma)\tau(f).$$

The form $B(f_1, f_2) = [\tau f_1\,|\,\tau f_2]$ then satisfies

$$B(R_\gamma f_1, R_\delta f_2) = \bar{\varkappa}(\gamma)\varkappa(\delta)B(f_1, f_2).$$

Recall that B is represented by the functional T:

$$B(f_1, f_2) = T(f_1^* * f_2).$$

The identity $(R_\gamma f_1)^* * R_\delta f_2 = L_\gamma R_\delta f^* * f_2$ yields

$$T(L_\gamma R_\delta f^* * f_2) = \bar{\varkappa}(\gamma)\varkappa(\delta)T(f_1^* * f_2).$$

Since an arbitrary $f\in\tilde{\mathscr{D}}(G)$ is of the form $f = f_1^* * f_2$, we conclude that

$$(3.7)\qquad T(L_\gamma R_\delta f) = \bar{\varkappa}(\gamma)\varkappa(\delta)T(f).$$

Now we are going to show that $\beta(f) = 0$ implies $\tau(f) = 0$ Using Lemma 1.6 we obtain the following expression for $f_1^* * f_2$:

$$f_1^* * f_2(y) = \int_G \bar{f}_1(g)f_2(gy)\,dg = \int_{G/\Gamma} dm(x)\int_\Gamma \bar{f}_1(x\gamma)\,\varrho^{-1}(x\gamma)f_2(x\gamma y)\,d\gamma$$
$$= \int_{G/\Gamma} dm(x)\,\varrho^{-1}(x)\int_\Gamma \Delta^{-1}(\gamma)\bar{f}_1(x\gamma)f_2(x\gamma y)\,d\gamma.$$

Hence for every $f_2\in\tilde{\mathscr{D}}(G)$:

$$T(f_1^* * f_2) = \int_{G/\Gamma} dm(x)\,\varrho^{-1}(x)\int_\Gamma \Delta^{-1}(\gamma)\bar{f}_1(x\gamma)\,T(L_\gamma^{-1}R_x^{-1}f_2)\,d\gamma$$
$$= \int_{G/\Gamma} dm(x)\,\varrho^{-1}(x)\int_\Gamma \bar{\varkappa}(\gamma^{-1})\,\Delta(\gamma^{-1})\bar{f}_1(x\gamma)\,T(L_x^{-1}f_2)\,d\gamma$$
$$= \int_{G/\Gamma} dm(x)\,T(L_x^{-1}f_2)\,\varrho^{-1}(x)\,\overline{\beta f_1(x)}.$$

The assumption $\beta f_1 = 0$ yields $[\tau f_1\,|\,\tau f_2] = 0$ for every f_2, whence $\tau(f_1) = 0$. Putting $\tilde{\tau}(\beta f) := \tau f$ we obtain the desirable operator.

We sum our results up in the following

THEOREM 3.6. *Let G be a Yamabe group with a large subgroup, and (U, J) a completely irreducible **J**-representation of G.*

Then there exists a linear bijection of the set of common eigenvectors $\omega_0\in\bar{\Phi}'$ of all operators ${}^tU_\gamma$, $\gamma\in\Gamma$, with the eigenvalue $\varkappa(\gamma)$ onto the set of all operators intertwining for the induced representation $(U^\varkappa, \tilde{\mathscr{D}}^\varkappa)$ and (U, J).

Proof. In virtue of Theorem 3.5 to each $\omega_0\in\bar{\Phi}'$ which satisfies ${}^tU_\gamma\omega_0 = \varkappa(\gamma)\,\omega_0$, $\gamma\in\Gamma$, there is associated a continuous operator $\tilde{\tau}$ intertwining for $U^\varkappa$ and U. The construction of $\tilde{\tau}$ guarantees that the mapping $\omega_0\to\tilde{\tau}$ is monomorphic.

It remains only to prove that the mapping is also surjective. Let us assume that $\tilde{\tau}\in L(\tilde{\mathscr{D}}^\varkappa, J)$ is the intertwining operator for $U^\varkappa$ and U, and define a functional $\omega_0\in\Phi'$ as follows

$$\omega_0\big(U(\varphi)x_0\big) = [x_0\,|\,\tilde{\tau}(\beta\varphi^*)],\qquad \varphi\in\tilde{\mathscr{D}}(G),$$

which evidently satisfies the eigenvalue equation (3.5).

The construction in the proof of Theorem 3.5 shows that $\tilde{\tau}$ is just the intertwining operator associated to ω_0 in virtue of this theorem. The proof is complete.

It would be useful to establish connections between the eigenvector $\omega_0 \in \bar{\Phi}'$ and the bilinear form or distribution T which appear in the proofs of the above theorems. The following proposition is concerned with this question:

PROPOSITION 3.7. *Let $\tilde{\tau}$ be an intertwining operator from $\mathscr{D}^\varkappa(G)$ into J. Then there exists a unique linear continuous functional $\tilde{T}$ on $\mathscr{D}^\varkappa$ such that*

$$\tilde{T}(\beta f) = T(f) = \overline{\langle \tilde{\tau} f, \omega_0 \rangle}. \tag{3.8}$$

If $\tilde{\tau} \in L(\tilde{\mathscr{D}}^\varkappa, J)$ then $\tilde{T} \in (\tilde{\mathscr{D}}^\varkappa)'$.

$\tilde{T}$ satisfies the following condition:

$$\tilde{T}(L_\gamma f) = T(f)\overline{\varkappa}(\gamma), \qquad \gamma \in \Gamma, f \in \mathscr{D}^\varkappa.$$

Proof. Let us define a bilinear form on $\mathscr{D}^\varkappa \times \mathscr{D}^\varkappa$ as follows:

$$\tilde{B}(\beta f_1, \beta f_2) = [\tilde{\tau}\beta f_1 | \tilde{\tau}\beta f_2] = [\tau f_1 | \tau f_2] = T(f_1^* * f_2).$$

Let $\{\varphi^n\}$ denote approximative unity on G. Then the formula

$$\beta f \to \tilde{B}(\beta\varphi_n, \beta f) = T(\varphi_n^* * f) = T * \varphi_n^*(f)$$

defines a sequence of functionals on $\mathscr{D}^\varkappa$, which is convergent, since the right-hand side tends to $T(f)$. The limit $\tilde{T}$ of this sequence satisfies the equation $\tilde{T}(\beta f) = T(f)$ and is continuous, the topology in $\mathscr{D}^\varkappa$ being quotient with respect to β.

On the other hand, for $f \in X\mathscr{D}$ we have

$$T(\varphi_n^* * f) = [\tau f | \tau\varphi_n] = \langle \tau f, \sigma\varphi_n \rangle = \langle U(\varphi_n^*)\tau(f), \omega_0 \rangle$$
$$= \langle \tau(\varphi_n * f), \omega_0 \rangle.$$

Hence $T * \varphi_n^*(f)$ tends to $\overline{\langle \tau(f), \omega_0 \rangle} = T(f)$. Since the right-hand side is well defined for each $f \in \mathscr{D}(G)$, formula (3.8) is valid.

In case $\tilde{\tau} \in L(\tilde{\mathscr{D}}^\varkappa, J)$ the proof is analogous.

Remark. If the functional $\tilde{T}$ is given, the bilinear form $\tilde{B}$ may be reproduced by the formula

$$\tilde{B}(\beta f_1, f_2) = \int_G \bar{f}_1(g) \tilde{T}(L_g^{-1} f_2)\, dg \qquad (f_1 \in \mathscr{D}(G), f_2 \in \mathscr{D}^\varkappa). \tag{3.9}$$

Throughout what follows $(\tilde{\mathscr{D}}^\varkappa, \tilde{T})$ will denote the left regular representation of G in the space $\tilde{\mathscr{D}}^\varkappa/\mathrm{Ker}\,\tilde{\tau}$ with the invariant scalar product determined by formula (3.9).

Differential properties of the distribution T. In this part we assume additionally that G is a connected group. Let M be a linear continuous operator in $\mathscr{D}(G)$ such that its restriction to any $\mathscr{D}(G_i)$ belongs to the right invariant universal enveloping algebra of the Lie group G_i. The set of all such operators will be called the *right invariant universal enveloping algebra* of the group G and denoted by $\mathscr{E}(G)$. The center of $\mathscr{E}(G)$ will be denoted by $Z(G)$. The elements of the center commutes with right as well as left translations on G. By M^+ we denote the formal adjoint of M.

THEOREM 3.8 (cf. Maurin [16]). *The distribution T defined by* (3.8) *is the common eigendistribution of all central elements of $\mathscr{E}(G)$.*

Proof. To any $M \in Z(G)$ we associate a linear operator $dU(M)$ defined on the subset $\tau(\mathscr{D}(G))$ of J:

$$dU(M)\tau(\varphi) := \tau(M\varphi). \tag{3.10}$$

We verify that the definition is proper, i.e. from $\tau(\varphi) = 0$ it results $\tau(M\varphi) = 0$.

Suppose that for any $\psi \in \mathscr{D}(G)$ we have $[\tau(\psi) | \tau(\varphi)] = 0$. Then

$$[(\tau(\psi) | \tau(M\varphi))] = T(\psi^* * M\varphi) = T((M^+\psi)^* * \varphi)$$
$$= [\tau(M^+\psi) | \tau(\varphi)] = 0,$$

whence $\tau(M\varphi) = 0$.

Now our purpose is to show that the subspace $\Phi \subset \tau(\mathscr{D}(G))$ is invariant under $dU(M)$. Clearly, the operator $dU(M)$ commutes with the representation of G in $\tau(\mathscr{D}(G))$:

$$U_g dU(M)\tau(\varphi) = \tau(L_g M\varphi) = \tau(ML_g\varphi) = dU(M) U_g \tau(\varphi).$$

Hence

$$P_\chi dU(M) = dU(M) P_\chi \tag{3.11}$$

and

$$dU(M) U(\varphi) = U(\varphi) dU(M). \tag{3.12}$$

Recall that any $\varphi \in \Phi$ is of the form $U(f)x_0$, where $P_\chi x_0 = x_0$ for some χ. This yields

$$dU(M) U(f) x_0 = U(f) dU(M) P_\chi x_0 = U(f) P_\chi dU(M) x_0 = U(f) x_1,$$

where $x_1 = P_\chi dU(M) x_0 \in F_\chi$. Thus $dU(M)\Phi \subset \Phi$.

It follows by (3.11) that the finite-dimensional subspace $P_\chi \Phi = F_\chi$ is invariant under $dU(M)$. Then there exists an eigenvector $y \in F_\chi$ of $dU(M)$ such that $dU(M)y = \lambda_0 y$.

Owing to (3.12) the set $\Omega := \{y \in \Phi \mid dU(M)y = \lambda_0 y\}$ is invariant under $U(f), f \in \mathscr{D}(G)$. But the representation of the convolution algebra $\mathscr{D}(G) \ni f \to U(f)$ in Φ is algebraically irreducible, hence $\Omega = \Phi$.

According to Proposition 3.7 and the above remarks we obtain

$$T(Mf) = \overline{\langle \tau Mf, \omega_0 \rangle} = \overline{\langle dU(M)\tau(f), \omega_0 \rangle} = \lambda_0 \overline{\langle \tau(f), \omega_0 \rangle} = \lambda_0 T(f)$$

at least for $f \in X\mathscr{D}$, i.e. on the dense subset of $\mathscr{D}(G)$. Hence $MT = \lambda T$, $\lambda \in C^1$, for $M \in Z(G)$, which was to be proved.

Lifting the operators $M \in Z(G)$ to the space $\mathscr{D}^{\times}$ we obtain

COROLLARY 3.1. *The functional $\tilde{T}$ in Proposition 3.7 is an eigenunctional of all central elements of $\mathscr{E}(G)$.*

4. MACKEY'S THEOREMS ON IRREDUCIBLE REPRESENTATIONS OF SEMI-DIRECT PRODUCT OF GROUPS

In this part we are concerned with a connected group G being the semidirect product of a compact group K and an Abelian locally compact group A. Recall that by Godement's theorem K is the large subgroup of $G = K \times_t A$.

Let (U, H) be an arbitrary irreducible representation of G in a Hilbert space H. Let $\Phi \subset H \subset \overline{\Phi}'$ be the Gelfand triplet defined in the preceding section. Consider the restriction of (U, H) to the Abelian group A. The nuclear spectral theorem proved by Maurin [15] states that there exists a positive Borel measure μ on A such that the scalar product in H is represented by the integral

$$(4.1) \qquad (\varphi \mid \psi) = \int_{\hat{A}} \overline{\langle \varphi, e_\chi \rangle} \langle \psi, e_\chi \rangle d\mu(\chi),$$

where μ-almost all e_χ belong to Φ' and

$$\langle U_a \varphi, e_\chi \rangle = \chi(a) \langle \varphi, e_\chi \rangle, \quad a \in A.$$

Let us fix an arbitrary $e_\chi \in \Phi'$ and define $\bar{e}_0 \in \overline{\Phi}'$ as follows:

$$\langle \varphi, \bar{e}_0 \rangle = \overline{\langle \varphi, e_\chi \rangle}.$$

All assumptions of Theorem 3.6 are satisfied with $\omega_0 = \bar{e}_0$ and we state that a continuous intertwining operator maps $\tilde{\mathscr{D}}^{\times}$ in H. Write

$$K' = \{k \in K \mid \chi(t(k)a) = \chi(a)\}.$$

The space $\tilde{\mathscr{D}}^{\chi}$ is invariant under right translations from K'.

Let r be a matrix element of an irreducible representation of K'. The space of functions spanned by all left translations of r is denoted by H_r. The left regular (irreducible) representation of K' in H_r will be denoted by U_r. Let us define:

$$\mathscr{D}_r^\chi = \{f \in \tilde{\mathscr{D}}^\chi \mid f(g) = \int_{K'} \varphi(gk') r(k'^{-1}) dk' \text{ for some } \varphi \in \mathscr{D}^\chi\}.$$

Owing to Lemma 3.2 and well known connections between matrix elements and characters it is obvious that the linear manifold $\overset{+}{_r}\mathscr{D}_r^\chi$ is dense in $\tilde{\mathscr{D}}^\chi$ (the index r runs over the set of all matrix elements of all irreducible representations of K'). Hence there exists an r such that $\tilde{\tau}(\mathscr{D}_r^\chi)$ is dense in H. This means, in particular, that the representation $(L_g, \mathscr{D}_r^\chi, T)$ is extendible to a unitary irreducible representation, which is equivalent to (U, H).

There is also a natural intertwining monomorphism which maps $\mathscr{D}_r^\chi$ into the space of induced representation $U^{U_r \times \chi}$. Namely, to each $f \in \mathscr{D}_r^\chi$ there is associated a function $\tilde{f}$ with values in H_r defined by the formula: $G \ni g \to f_g$, where $f_g(k') = f(gk')$. But it is well known that the representation $U^{U_r \times \chi}$ is irreducible (Mackey [13]). Hence $\mathscr{D}_r^\chi$ is also extendible to the irreducible representation $U^{U_r \times \chi}$. This fact immediately leads to the equivalence of the representations (U, H) and $U^{U_r \times \chi}$,

We have obtained

THEOREM 4.1 (Mackey [13]). *Every unitary irreducible representation of the semidirect product $K \times_t A$ is unitarily equivalent to a representation induced by a unitary irreducible representation of the subgroup $K' \times_t A$, where $K' \subset K$.*

We shall also verify another Mackey's theorem:

THEOREM 4.2 (Mackey [13]). *The measure μ in (4.1) is concentrated on the orbit of χ in $\hat{A}$ under the action of K.*

Proof. Let $\bar{e}_1 \in \overline{\Phi}'$ correspond to the character χ_1 and to the intertwining operator

$$\tilde{\tau}_1 \colon \mathscr{D}^{\chi_1} \to H.$$

(It is seen from the proof above that $\tilde{\tau}_1$ extends to the whole $\mathscr{D}^{\chi_1}$.) We introduce a bilinear form on $\mathscr{D}^\chi \times \mathscr{D}^{\chi_1}$ as $\tilde{S}(\varphi, \psi) = [\tilde{\tau}\varphi \mid \tilde{\tau}_1 \psi]$ and a distribution on $\mathscr{D}(G)$ by the formula $S(\varphi^* * \psi) = \tilde{S}(\beta\varphi, \beta_1\psi)$.

By arguments used in the proof of Proposition 3.7, we state that S can be lifted to the space $\mathscr{D}^\chi$ by the formula $\tilde{S}(\beta f) = S(f) = \langle \tau f, e_1 \rangle$. Thus for $g \in A$ we have

$$\tilde{S}(\beta L_g f) = \langle L_g \tau f, e_1 \rangle = \bar{\chi}_1(g) \tilde{S}(\beta f).$$

On the other hand,

$$\tilde{S}(\beta L_g f) = \tilde{S}_x\Big(\int_A \chi(a^{-1}) R_a f(g^{-1}x)\,da\Big)$$
$$= \tilde{S}_x\Big(\int_A \chi(a^{-1}) R_a f(xx^{-1}g^{-1}x)\,da\Big)$$
$$= \tilde{S}_x\Big(\chi(x^{-1}g^{-1}x)\int_A \chi(a^{-1}) R_a f(x)\,da\Big)$$
$$= \tilde{S}_x\big(\overline{\chi}(x^{-1}gx)\beta f(x)\big) = \tilde{S}_x\big(\overline{\chi}_x(g)\beta f(x)\big).$$

We have the following conclusion:

$$S_x\big((\chi_1(g)-\chi_x(g))f(x)\big) = 0 \quad \text{for any } f\in\mathcal{D}(G) \text{ and } g\in A.$$

It shows that the distribution S is concentrated on the subset of K consisting of $k\in K$ such that $\chi_k(a) := \chi(t(k)a) = \chi_1(a)$.

Since S is non-zero distribution, we have $\chi_k = \chi_1$ for some $k\in K$. Thus the set $\hat{A}\setminus\{\chi_k \mid k\in K\}$ consists of elements for which $e_\chi \notin \Phi'$ and is μ-measure null, as Maurin's theorem states.

5. A DUALITY FOR NON-COMPACT SYMMETRIC SPACES AND RELATED GROUP REPRESENTATIONS

The first part of this section comprises a summary of definitions and facts on the dual object of Riemannian globally symmetric space as well as on generalized Radon transform. The method used, notation and most of cited results originate from Helgason's papers [8] and [9].

Let S be a Riemannian globally symmetric space of non-compact type and let G denote the largest connected group of isometries of S in its usual compact open topology. G is well known to be semisimple Lie group without compact normal non-trivial subgroups.

For fixed $o\in S$ denote by K the isotropy subgroup of G at o. Let $\mathfrak{k}_0$ and $\mathfrak{g}_0$ denote the corresponding Lie algebras of K and G; let $\mathfrak{p}_0$ denote the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Killing form B of $\mathfrak{g}_0$. Then B is positive definite on $\mathfrak{p}_0$ and the Riemannian structure on S can be chosen so that $\mathfrak{p}_0$ is isometric to the tangent space S_0.

Every symmetric space contains flat totally geodesic submanifold E of the dimension $l>0$. Assume that $o\in E$ and denote by $\mathfrak{h}_{\mathfrak{p}_0}$ the subspace of vectors $X\in\mathfrak{p}_0$ for which $\operatorname{Exp} d\pi(X)\in E$, where π denotes the map $G\ni g\to g\cdot o\in S$. Let A be the analytic subgroup of G corresponding to $\mathfrak{h}_{\mathfrak{p}_0}$.

An element $H\in\mathfrak{h}_{\mathfrak{p}_0}$ is said to be *regular* if the kernel of $\operatorname{Ad} H$ in $\mathfrak{p}_0$ equals $\mathfrak{h}_{\mathfrak{p}_0}$. A linear functional α on $\mathfrak{h}_{\mathfrak{p}_0}$ is called a *root* if the subspace $\mathfrak{g}_\alpha = \{X\in\mathfrak{g}_0 \mid \operatorname{Ad} H(X) = \alpha(H)X\}$ is non-trivial. Let $\mathfrak{h}'_{\mathfrak{p}_0}$ denote the part

of $\mathfrak{h}_{\mathfrak{p}_0}$, where all roots are different from zero. The connected components of regular elements in $\mathfrak{h}'_{\mathfrak{p}_0}$ are called the *Weyl chambers*. Select any Weyl chamber C and order the set of roots putting $\alpha>\beta$ if $\alpha(H)>\beta(H)$ for $H\subset C$. Consider the subalgebra $\mathfrak{n}_0 = \sum_{\alpha>0}\mathfrak{g}_\alpha$ of $\mathfrak{g}_0$ and corresponding subgroup N of G. The Iwasava theorem asserts that the map $K\times A\times N \ni (k,a,n)\to kan\in G$ is an analytic diffeomorphism. Let M and M' denote the centralizer and normalizer of A in K. The finite group $W = M'/M$ is called the *Weyl group*. The natural action of $w\in W$ in A and $\mathfrak{h}_{\mathfrak{p}_0}$ will be denoted by $w(a)$ and $w(H)$, respectively.

The homogeneous space G/MN is called the *dual space* of S and is denoted by $\hat{S}$. The space is in one-to-one correspondence with the set of orbits of subgroups gNg^{-1} in the symmetric space. The orbits are called *horocycles*.

The following results make clear the structure of S and $\hat{S}$. First of it is the well known "polar coordinate decomposition" of the symmetris space.

LEMMA 5.1 (Cartan (see [7])). *The mapping*

$$(K/M)\times A\ni(kM,a)\to kaK\in S$$

is differentiable and surjective. The mapping restricted to $(K/M)\times\operatorname{Exp}(\mathfrak{h}'_{\mathfrak{p}_0})$ *is a covering of order of the group* W.

For $\hat{S}$ one has the corresponding result:

LEMMA 5.2. *The mapping* $(kM,a)\to kaMN$ *is a diffeomorphism of* $(K/M)\times A$ *onto* $\hat{S}$.

The following lemmas yield a description of the orbit spaces of K in S and of MN in $\hat{S}$.

LEMMA 5.3. *The following relations are natural identifications*:

1° $K\setminus G/K = A/W$;

2° $MN\setminus G/MN = A\times W$.

Remark. One has a natural action of the Weyl group W in the space of all roots: $\alpha\to\alpha\circ w$, where $\alpha\circ w(H) = \alpha(w(H))$. There exists unique element $w^*\in W$ which maps all positive roots into negative ones. The orbit of the subgroup MNA in $\hat{S}$ of the element $w^*MN\in\hat{S}$ is open in $\hat{S}$, all the others orbits have lower dimension.

The Radon transform maps the space $\mathcal{D}(S)$ into $\mathcal{D}(\hat{S})$:

$$\hat{f}(gMN) = \int_N f(gnK)\,dn.$$

Let us introduce a map $\varphi\to\check{\varphi}$ defined for $\varphi\in\mathcal{D}(\hat{S})$:

$$\check{\varphi}(gK) = \int_K \varphi(gkMN)\,dk.$$

The problem arises to relate f and $\hat{f}\check{}$ on S.

The simple measure theoretic computation yields

$$\int_S f(x)\check{\varphi}(x)\,ds(x) = \int_{\hat{S}} f(\xi)\varphi(\xi)\,d\hat{s}(\xi), \tag{5.1}$$

where ds and $d\hat{s}$ denote the G-invariant measures on S and $\hat{S}$, suitably normalized.

Lemma 5.4. 1° *For certain integro-differential G-invariant operator Λ on $\hat{S}$, Λ^* its adjoint we have*

$$f = (\Lambda\Lambda^*\hat{f})\check{}\,, \tag{5.2}$$

$$\int_S |f(x)|^2\,ds(x) = \int_{\hat{S}} |\Lambda\hat{f}(\xi)|^2\,d\hat{s}(\xi). \tag{5.3}$$

2° *If G is a complex group we have*

$$f = \Box\big((\hat{f})\check{}\big),$$

where $\Box$ is a certain G-invariant differential operator on S.

Invariant differential operators on S and $\hat{S}$. The natural action of the group G in a homogeneous space Y is denoted by $y \to g\cdot y$, $g \in G$; the corresponding action in the space of functions on Y is denoted by $l(g)\colon l(g)f := f\circ g^{-1}$.

A differential operator on Y is called *invariant* if it commutes with $l(g)$ for all $g \in G$. The space of all invariant differential operators on Y is denoted by $\boldsymbol{D}(Y)$. Let $\pi\colon G \to G/K$ and $\hat{\pi}\colon G \to G/MN$ denote the natural projections of G on S and $\hat{S}$, respectively.

A convenient description of the space $\boldsymbol{D}(S)$ was given by Harish-Chandra:

Lemma 5.5 (Helgason [7]). *Let $I(\mathfrak{h}_{\mathfrak{p}_0})$ denote the algebra of all polynomials on $\mathfrak{h}_{\mathfrak{p}_0}$ which are invariant under the Weyl group.*

Then there exists an isomorphism Γ of the algebra $\boldsymbol{D}(S)$ onto the algebra $I(\mathfrak{h}_{\mathfrak{p}_0})$.

Similar results for $\hat{S}$ were obtained by Helgason:

Lemma 5.6 (Helgason [9]). *There exists an isomorphism $\hat{\Gamma}$ of the algebra $\boldsymbol{D}(\hat{S})$ onto the algebra $S(\mathfrak{h}_{\mathfrak{p}_0})$ of all polynomials on $\mathfrak{h}_{\mathfrak{p}_0}$.*

If $D \to \hat{D}$ denotes the monomorphism of $\boldsymbol{D}(S)$ into $\boldsymbol{D}(\hat{S})$ given by $\hat{\Gamma}(\hat{D}) = \Gamma(D)$, then $(Df)\hat{} = \hat{D}\hat{f}$, $f \in \mathcal{D}(S)$.

Let us describe the explicit form of the isomorphism $\hat{\Gamma}$ in a way convenient for us. Select a base $\{H_i\}_1^n$ in the space $\mathfrak{h}_{\mathfrak{p}_0}$ and write

$$a(t_1, \ldots, t_n) := \operatorname{Exp}\sum_{i=1}^n t_i H_i,$$

where $t_i \in R^1$. Every $P \in S(\mathfrak{h}_{\mathfrak{p}_0})$ determines an operator $D \in \boldsymbol{D}(\hat{S})$ by the formula

$$D_P f(g\cdot\hat{o}) = P(\partial_1, \ldots, \partial_n) f\big(ga(t_1, \ldots, t_n)\cdot\hat{o}\big), \tag{5.4}$$

where ∂_i stands for $\partial/\partial t_i|_{t_i=0}$.

One proves that the map $P \to D_P$ is a monomorphism.

$\hat{\Gamma}$ is just given by $\hat{\Gamma}(D_P) = P$.

The following lemma results now from Lemma 5.6 by simple computation:

Lemma 5.7. 1° *For each $D \in \boldsymbol{D}(A)$ there exists a unique $D^A \in \boldsymbol{D}(\hat{S})$ such that for $f \in \mathcal{D}(G)$*

$$D^A\beta f = \beta D(f), \tag{5.5}$$

where

$$\beta f\big(\hat{\pi}(g)\big) = \int_M\int_N f(gmn)\,dm\,dn.$$

2° *The mapping $D \to D^A$ is onto $\boldsymbol{D}(\hat{S})$.*

Spherical and conical representations. An irreducible representation (U, J) of G is said to be *spherical* if there exists a non-zero vector in J fixed by $U|K$. The reason for introducing this term is that the distribution $\tilde{T}$ associated to (U, J) by Lemma 3.7 appears to be spherical function on S, i.e. a K-invariant common eigenfunction of all $D \in \boldsymbol{D}(S)$. This function is well known to be uniquely determined by the representation.

There is no need to consider "spherical distributions" on S; since the algebra $\boldsymbol{D}(S)$ contains at least one elliptic element, namely Laplace-Beltrami operator, those distributions will be functions.

In order to introduce on the space $\hat{S}$ the objects analogous to spherical functions on S it is necessary to use the distribution space.

Definition 5.8. A distribution on $\hat{S}$ is called *conical* if it is MN-invariant eigendistribution of each $D \in \boldsymbol{D}(\hat{S})$.

The notion of conical representation was introduced by Helgason in the following way: a representation (U, X) of G is called *conical* if in the space of the representation there exists a non-zero vector fixed under $U|MN$.

As far as we are concerned, this definition is not satisfactory. In the case $G = SL(2, R^1)$ one can easily assert that the only conical (in the meaning given by Helgason) representations are non-unitary, although the set of spherical unitary representations is very large.

One can try to remove this asymetry by changing the definition.

Definition 5.9. An irreducible representation (U, J) is called *conical* if

1° there exists in $\overline{\Phi}'$ a non-zero functional fixed under ${}^tU|MN$,

2° the functional $\tilde{T}$ associated to the representation is conical.

Further discussion of the definition we leave to Appendix.

In the case of finite-dimensional representations Helgason has proved

Lemma 5.10. 1° *There exists a bijection of the set of conical regular distributions onto the set of conical finite-dimensional representations.*

2° *Representation is spherical if and only if it is conical.*

This result is suggested by the following facts concerning finite-dimensional representations. By the well-known Lie theorem in the space of the representation (U, X) there exists a non-zero eigenvector of all U_g, $g \in MNA$. To the representation there is associated a character $\varkappa$ of the subgroup AN by the formula

$$U_g e = \varkappa(g) e, \quad g \in AN, e \in X.$$

Since N is the commutator subgroup of AN, the vector e is fixed by $U|N$. The character $\varkappa$ is called the *highest weight* of the representation (U, X). On the other hand, by Harish-Chandra formula, every spherical function is of the form

$$\Phi_\nu(g) = \int_K \exp\nu\big(\log\big(H(gk)\big)\big)\,dk,$$

where ν is a linear functional on $\mathfrak{h}_{\mathfrak{p}_0}$ and the map $H: G \to A$ is defined by $H(kan) = a$.

The correspondence between spherical and conical representations stated in Lemma 5.10.2° results from the correspondence between highest weights of representations and characters of the subgroup AN defined by $g \to \exp\nu\big(\log\big(H(g)\big)\big)$. Our aim will be to extend this relations to a large class of infinite-dimensional representations.

We begin with the following

Proposition 5.11. *Let (U, J) be a conical representation. Then the functional $\omega_0 \in \overline{\Phi}'$ fixed under ${}^tU|MN$ is an eigenvector of all ${}^tU_a$, $a \in A$.*

Proof. The notation being as in section 3, let us define a function Ψ on G by $\Psi(g) = \langle U_g x_0, \omega_0\rangle$. Clearly, Ψ is continuous on G and satisfies the equation

$$\int_K \Psi(gk)\chi(k)\,dk = n_\chi \Psi(g), \tag{5.6}$$

where χ is the character of factor representation which contains x_0. The function Ψ is closely related to the distribution T and the conical distribution $\tilde{T}$. Namely, if $f_0 \in \mathscr{D}(G)$ is such that $\tau f_0 = x_0$, then

$$\langle \tau(f*f_0), \overline{\omega}_0\rangle = \langle U(f)x_0, \omega_0\rangle = \int_G f(g)\overline{\Psi}(g)\,dg$$
$$= \tilde{T}\big(\beta(f*f_0)\big) = T(f*f_0) \quad \text{for all } f \in \mathscr{D}(G).$$

Hence $\overline{\Psi}(g) = T(L_g f_0)$.

Since T is the eigendistribution of all central elements of the algebra $\mathscr{E}(G)$, the function Ψ treated as a distribution on G satisfies the eigenvalue equations:

$$\int_G \Delta f(g)\overline{\Psi}(g)\,dg = T\big((\Delta f)*f_0\big) = T\big(\Delta(f*f_0)\big)$$
$$= \lambda(\Delta)T(f*f_0) = \lambda(\Delta)\int_G f(g)\overline{\Psi}(g)\,dg$$

for any $f \in \mathscr{D}(G)$ and $\Delta \in Z(G)$.

In particular, for the Casimir operator, which is of the form $\Delta = \Delta_1 + \Delta_2$, where Δ_1 is a central symmetric element of $\mathscr{E}(K)$ and Δ_2 is elliptic on G, we obtain

$$\int_G \Delta_1 f(g)\overline{\Psi}(g)\,dg = n_\chi \int_G \Delta_1 f(g) \int_K \overline{\Psi}(gk)\overline{\chi}(k)\,dk\,dg$$
$$= n_\chi \int_G\int_K \Delta_1 f(gk^{-1})\overline{\chi}(k)\,dk\,\overline{\Psi}(g)\,dg$$
$$= n_\chi \int_G\int_K f(gk)\,\Delta_1\overline{\chi}(k)\,dk\,\overline{\Psi}(g)\,dg$$
$$= \lambda_1 \int_G f(g)\overline{\Psi}(g)\,dg,$$

where we have made use of (5.6) and the fact that characters are common eigenfunctions of central elements of $\mathscr{E}(K)$.

Finally, we obtain

$$\int_G \Delta_2 f(g)\overline{\Psi}(g)\,dg = -\int_G \Delta_1 f(g)\overline{\Psi}(g)\,dg + \int_G \Delta f(g)\overline{\Psi}(g)\,dg$$
$$= \lambda_1 \int_G f(g)\overline{\Psi}(g)\,dg + \lambda(\Delta)\int_G f(g)\overline{\Psi}(g)\,dg$$

and $\overline{\Psi}$ appears to be the eigendistribution of the elliptic operator Δ_2. Being so, the function Ψ is harmonic on G.

Now, let $D \in \boldsymbol{D}(A)$. Using the hermitian symmetry of T we find

$$D\Psi(e) = P(\partial_1, \ldots, \partial_n)T(L_{a(t_1,\ldots,t_n)}f_0) = P(\partial_1, \ldots, \partial_n)T(R^{-1}_{a(t_1,\ldots,t_n)}f_0^*)$$
$$= T(D^+ f_0^*) = \tilde{T}\big((D^A)^+\beta(f_0^*)\big) = \nu(D)\tilde{T}\big(\beta(f_0^*)\big) = \nu(D)\Psi(e)$$

since T is the conical distribution.

Note, that the function on G given by the formula

$$\Psi_1(nak) = \exp\nu(\log a)\Psi(k) \tag{5.7}$$

satisfies the equation $D\Psi_1(e) = D\Psi(e)$ for any $D \in \boldsymbol{D}(G)$. Both being harmonic, Ψ_1 and Ψ are identical. Thus

$$\langle U_g x_0, {}^tU_a\omega_0\rangle = L_a\Psi(g) = \exp\nu(\log a^{-1})\Psi(g)$$
$$= \exp\big(-\nu(\log a)\big)\langle U_g x_0, \omega_0\rangle.$$

Since the set $\{U_g x_0 |\ g \in G\}$ is total in Φ, we find

$$ {}^tU_a \omega_0 = \exp\big(-\nu(\log a)\big)\,\omega_0 $$

and the proof follows.

Remark. In virtue of Proposition 5.11 it is easily seen that the linear space of functionals in $\bar{\Phi}'$ invariant under ${}^tU|\ MN$ is of finite dimension. It suffices to observe that the space of functions satisfying both (5.6) and (5.7) is finite-dimensional and the map $\omega_0 \to \Psi$ is monomorphic. This space is 1-dimensional provided that this representation is both conical and spherical.

THEOREM 5.12. *To every conical representation (U, J) there is associated a character $\varkappa$ of the subgroup $MAN \subset G$ and a operator $\tilde{\tau}\colon \tilde{\mathscr{D}}^{\varkappa} \to J$ intertwining for $U^{\varkappa}$ and U. The conical distribution corresponding to the representation satisfies the equation*

$$ \tilde{T}\big(l(man)f\big) = \bar{\varkappa}(a)\,\tilde{T}(f). $$

The proof immediately follows from Theorem 3.6 and Proposition 5.11.

In the case of classical groups, the family of conical distributions (and representations) is contained in Gelfand's list of irreducible representations (Gelfand and Naĭmark [3]). In the general case decisive results were obtained by Helgason [11].

Now, our aim is to characterize the role played by the Radon transform in the theory of representations.

In the sequel we shall assume that the representation (U, J) is both spherical and conical. Since the transform $f \to \hat{f}$ commutes with the action of G in S and $\hat{S}$, it is natural to expect that it is the intertwining operator for spherical and conical representations associated to a functional $\nu \in \mathfrak{h}^*_{\mathfrak{p}_0}$.

First we must overcome some difficulties connected with the fact that the Radon transform is not a surjective map.

The notation being as in section 3, let us consider two operators intertwining for the regular representation in $\mathscr{D}(S)$ and (U, J), namely α and γ:

$\alpha f = U(f)x_0$, where $x_0 \in \Phi$ and is fixed by $U|K$;

$\gamma f = \tilde{\tau}\hat{f}$, where $\tilde{\tau}$ is the intertwining operator associated to the MN-invariant functional ω_0.

As it will appear, α is proportional to γ. In order to prove this let us verify the following lemma:

LEMMA 5.13. *There exists a K-invariant function $\varphi_0 \in \mathscr{D}(G)$ such that $\gamma\varphi_0 \neq 0$ as well as $\alpha\varphi_0 \neq 0$.*

Proof. Assume the opposite, i.e. $\alpha\varphi \neq 0$ implies $[\gamma\varphi|\gamma\varphi] = 0$ for each K-invariant $\varphi \in \mathscr{D}(S)$. The last theorem states that there exists $\beta_\nu\colon \mathscr{D}(\hat{S}) \to \mathscr{D}^\nu$ and a distribution $\tilde{T}_\nu$ such that

$$ \tilde{T}(\psi, \psi) = \tilde{T}_\nu(\beta_\nu\psi, \beta_\nu\psi) = [\tilde{\tau}\psi|\tilde{\tau}\psi] $$

and

$$ \beta_\nu\psi(g) = \int_A \psi\circ\hat{\pi}(ga)\exp[\nu+2\varrho(\log a)]\,da $$

(where ϱ is defined by the formula $dg = \exp 2\varrho(\log a)\,dk\,da\,dn$; see e.g. Helgason [7]).

If ψ is invariant under translations from K we have

$$ \beta_\nu\psi(k'a'n') = \int_A \psi\circ\hat{\pi}(a'a)\exp[\nu+2\varrho(\log a)]\,da $$

$$ = \exp[-\nu-2\varrho(\log a')]\int_A \psi\circ\hat{\pi}(a)\exp[\nu+2\varrho(\log a)]\,da. $$

Thus for an arbitrary K-invariant $\psi \in \mathscr{D}(\hat{S})$ the value $\beta_\nu\psi$ is proportional to

$$ \psi_0(g) = \exp\big[-\nu-2\varrho\big(\log H(g)\big)\big] $$

with the coefficient

$$ \int_A \psi\circ\hat{\pi}(a)\exp[\nu+2\varrho(\log a)]\,da = \int_{\hat{S}} \psi(\xi)\exp\nu\big(H(\xi)\big)\,ds, $$

where $H(ka\cdot\hat{o})\colon = a$, and $d\hat{s}$ is as before the G-invariant measure on $\hat{S}$.

If $\psi = \hat{\varphi}$, we have in virtue of (5.1)

$$ \int_{\hat{S}} \hat{\varphi}(\xi)\exp\nu\big(\log H(\xi)\big)\,d\hat{s}(\xi) = \int_S \varphi(x)\,\Phi_\nu(x)\,ds(x), $$

where

$$ \Phi_\nu(g\cdot o) = \int_K \exp\nu\big(\log H(gk)\big)\,dk. $$

Finally,

$$ [\gamma\varphi|\gamma\varphi] = [\tilde{\tau}\hat{\varphi}|\tilde{\tau}\hat{\varphi}] = \tilde{T}_\nu(\beta_\nu\hat{\varphi}, \beta_\nu\hat{\varphi}) $$

$$ = \tilde{T}_\nu(\psi_0, \psi_0) = \Big|\int_S \varphi(x)\,\Phi_\nu(x)\,ds(x)\Big|^2, $$

where Φ_ν is just the spherical function related to (U, J).

The K-invariant space spanned by x_0 is unique (Godement [5]), whence makes the factor representation F_1, therefore, by Theorem 2.3, $[x_0|x_0] \neq 0$, otherwise $\tilde{T}_\nu(\psi_0, \psi_0) \neq 0$. This contradicts the assumption $[\gamma\varphi|\gamma\varphi] = 0$ for each K-invariant $\varphi \in \mathscr{D}(S)$, Φ_ν being non-zero.

Now, it becomes clear that conditions $\gamma\varphi_0 \neq 0$ and $a\varphi_0 \neq 0$ can be satisfied by the same $\varphi_0 \in \mathcal{D}(S)$, which was to be proved.

THEOREM 5.14. *The Radon transform intertwins the representations* $(\mathcal{D}(S), \Phi_\nu)$ *and* $(\mathcal{D}(\hat{S}), \tilde{T})$ *unitarily with respect to the scalar products induced by* Φ_ν *and* $\tilde{T}$ *(provided that* $\tilde{T}$ *is suitably normalized).*

Proof. Let $\varphi_0 \in \tilde{\mathcal{D}}(S)$ be such that $\gamma\varphi_0 \neq 0$ and $l(k)\varphi_0 = \varphi_0$. The existence of such φ_0 we have just proved in Lemma 5.13. Since the K-invariant subspace in J is unique, it follows that $a(\varphi_0)$ is proportional to $\gamma(\varphi_0)$. We find:

$$a(\varphi * \varphi_0) = U(\varphi)a(\varphi_0) = \lambda U(\varphi)\gamma(\varphi_0) = \lambda\gamma(\varphi * \varphi_0).$$

But we know that $a(\tilde{\mathcal{D}}(S)) = \Phi$, and the set $a\{\varphi * \varphi_0 \mid \varphi \in \mathcal{D}(G)\}$ is invariant under $U(\varphi), \varphi \in \mathcal{D}(G)$ subspace of Φ, being so, equals Φ by algebraical irreducibility of the representation of the convolution algebra $\mathcal{D}(G)$ in Φ.

The immediate consequence is that $a(\varphi) = \lambda\gamma(\varphi)$ for any $\varphi \in \tilde{\mathcal{D}}(S)$. Now it follows that

$$\Phi_\nu(\varphi, \psi) = [a\varphi \mid a\psi] = |\lambda|^2[\gamma\varphi \mid \gamma\psi] = |\lambda|^2 T(\hat{\varphi}, \psi\hat{\ }) \tag{5.8.}$$

which was to be proved.

Formula (5.8.) may be rewritten in the form (according to (3.9))

$$\int_S \bar{\varphi}(x)\Phi_\nu \times \psi(x)\,ds(x) = \int_{\hat{S}} \bar{\hat{\varphi}}(\xi)\tilde{T} \times \hat{\psi}(\xi)\,d\hat{s}(\xi),$$

where $T \times \psi(\pi g)$ stands for $T(l(g)\psi)$; $T \in \mathcal{D}'(H)$, $\psi \in \mathcal{D}(H)$ and $\pi: G \to G/\Gamma = H$.

By (5.1) we have

$$\int_{\hat{S}} \bar{\hat{\varphi}}(\xi)\tilde{T} \times \hat{\psi}(\xi)\,d\hat{s}(\xi) = \int_S \bar{\varphi}(x)(\tilde{T} \times \hat{\psi})\check{\ }(x)\,ds(x).$$

Concluding, we have $\Phi_\nu \times \psi = (\tilde{T} \times \hat{\psi})\check{\ }$. This equality suggests the following

THEOREM 5.15. *The transform* $f \to \check{f}$ *maps the set of functions*

$$\{\tilde{T} \times \psi \mid \psi \in \mathcal{D}(\hat{S})\}$$

into the set of solutions of the equations

$$Df = \nu(D)f, \quad D \in \boldsymbol{D}(S), \tag{5.9}$$

where ν *is defined by the formula*

$$D\Phi_\nu = \nu(D)\Phi_\nu.$$

Proof. Every $x \in J$ can be approximated in J by elements of the form $x_n = \tilde{\tau}(\hat{\varphi}_n)$. Assuming that $x = \tilde{\tau}f$, $f \in \mathcal{D}(\hat{S})$, we have for $\varphi \in \tilde{\mathcal{D}}(S)$

$$[\tilde{\tau}\hat{\varphi} \mid \tilde{\tau}(\hat{\varphi}_n)] = \int_{\hat{S}} \hat{\varphi}(\xi)(\tilde{T} \times \hat{\varphi}_n)(\xi)\,d\hat{s}(\xi)$$

tending to

$$[\tilde{\tau}\hat{\varphi} \mid \tilde{\tau}(f)] = \int_{\hat{S}} \hat{\varphi}(\xi)(\tilde{T} \times f)(\xi) f\,d\hat{s}(\xi).$$

Thus $\tilde{T} \times \hat{\varphi}_n$ treated as elements of $\mathcal{D}'(\hat{S})$ tend to $\tilde{T} \times f$. On the other hand, we have

$$\int_S \Phi_\nu \times \varphi_n(x)\varphi(x)\,ds(x) = \int_S (\tilde{T} \times \hat{\varphi}_n)\check{\ }(x)\varphi(x)\,ds(x)$$
$$= \int_{\hat{S}} (T \times \hat{\varphi}_n)(\xi)\hat{\varphi}(\xi)\,d\hat{s}(\xi)$$

and tends to

$$\int_{\hat{S}} (\tilde{T} \times f)(\xi)\hat{\varphi}(\xi)\,d\hat{s}(\xi) = \int_S (\tilde{T} \times f)\check{\ }(x)\varphi(x)\,ds(x).$$

Finally, $\Phi_\nu \times \varphi_n$ tends in $\mathcal{D}'(S)$ to $(\tilde{T} \times f)\check{\ }$.

Since each of the elements of the sequence $\Phi_\nu \times \varphi_n$ satisfies all equations (5.9), the same is true for $(\tilde{T} \times f)\check{\ }$, which completes the proof.

Theorems 5.14 and 5.15. were proved by Helgason for special class of conical distributions, namely for those associated to representatiosn of the principal series.

6. IRREDUCIBLE REPRESENTATIONS OF CONNECTED SEMISIMPLE GROUP IN PONTRYAGIN SPACES

In the course of our consideration we did not meet yet any simple example of (U, J) representation, which is not a unitary representation in Hilbert space. Now we are going to describe a class of such representation. Let us begin with $G = SL(2, R^1)$.

It is easily seen that any conical function on S is in this case given by the formula

$$\tilde{T}_r\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \hat{o}\right) = |\beta|^r, \quad r \in C^1.$$

For real r the function $\tilde{T}_r$ induces $\boldsymbol{J}$-representations. For $-1 < r < 0$ we obtain unitary infinite-dimensional representations in Hilbert spaces, so called *representations of supplementary series*. To $\tilde{T}_n$, n-positive even integer, there correspond finite-dimensional irreducible representations. At last, to $n < r < n+1$ there correspond infinite-dimensional representations in spaces with indefinite metric.

J-spaces which we here obtain are the so called *Pontryagin spaces.* In the direct sum decomposition stated in Theorem 2.3 only the space J_- is infinite-dimensional. The dimension of J_+ is finite and constant on the interval $]n, n-1[$. *J*-spaces with $\dim J_+ = \varkappa > 0$ are called $\Pi_\varkappa$*-spaces.* Similar facts hold for $G = SL(n, R^1)$ and $G = SL(n, C^1)$. In all cases the distribution T related to the representations in $\Pi_\varkappa$ are given by continuous functions on G.

The question of complete description of the class of all irreducible representations of a semisimple group in Pontryagin spaces is much simpler than the same question referring to representations in Hilbert spaces.

This fact results from the fundamental theorem on the structure of Pontryagin spaces.

LEMMA 6.1 (Pontryagin [23], Naĭmark [21]). *For every representation of a connected solvable group in the space $\Pi_\varkappa$, there exists a $\varkappa$-dimensional subspace invariant with respect to all operators of the representation.*

This theorem together with Lie theorem and Theorem 3.6 lead to the following:

THEOREM 6.2. *Let $(U, \Pi_\varkappa)$ be an irreducible representation of a connected semisimple group G with finite center and the Iwasava decomposition $G = KAN$. Then there exists a character χ of the solvable subgroup AN and a nuclear operator intertwining for U^χ and U.*

Thus the Gelfand and Naĭmark list of irreducible representations of classical groups given in [3] contains all representations of this type.

APPENDIX

1. Proof of Lemma A.1.1° With the notation from section 3, let $\mathscr{D}_0$ denote the kernel of σ. The projective space $\pi(\tilde{\mathscr{D}}(G)) = \tilde{\mathscr{D}}(G)/\mathscr{D}_0$ is algebraically isomorphic to Φ, being so is the space of the algebraically irreducible representation of the algebra $\mathscr{D}(G)$. Let us lift B to the space $(\tilde{\mathscr{D}}(G)/\mathscr{D}_0)\times\mathscr{D}(G)$ and define for some $\varphi_0 \in X\mathscr{D}/\mathscr{D}_0$ the continuous functional od $\mathscr{D}(G)\times\mathscr{D}(G)$ by the formula

$$\mathscr{D}(G)\times\mathscr{D}(G)\ni(\varphi, \psi)\to B(\varphi_0, \varphi*\psi).$$

We can chose convex symmetric neighbourhoods $\mathscr{U}_1, \mathscr{U}_2$ in $\mathscr{D}(G)$ such that

$$\sup_{\varphi\in\mathscr{U}_1, \psi\in\mathscr{U}_2} |B(\varphi_0, \varphi*\psi)| < \infty$$

hence

$$\sup_{\varphi\in\mathscr{U}_1, \psi\in\mathscr{U}_2} |B(U(\varphi)\varphi_0, \psi)| < \infty.$$

It turns out that the convex set

$$\tilde{\mathscr{U}} = \{\tilde{\varphi}\in\tilde{\mathscr{D}}(G)/\mathscr{D}_0 \colon \tilde{\varphi} = U(\varphi)\varphi_0, \varphi\in\mathscr{U}_1\}$$

is the neighbourhood in $\pi(\tilde{\mathscr{D}}(G))$.

It suffice to prove that $\pi^{-1}\tilde{\mathscr{U}}$ absorbs all bounded sets defining the topology in $\tilde{\mathscr{D}}(G)$. Let $\tilde{B}_{\varphi_1,\mathscr{B}}/\mathscr{D}_0 = \{\tilde{\varphi}\in\tilde{\mathscr{D}}(G)/\mathscr{D}_0 \colon \tilde{\varphi} = U(\varphi)\varphi_1$ where $\varphi_1\in\pi(X\mathscr{D})$ and $\varphi\in\mathscr{B}\}$.

As remarked above $\varphi_1 = U(f)\varphi_0$ for some $f\in\mathscr{D}(G)$. Hence

$$\pi(\tilde{B}_{\varphi_1,\mathscr{B}}) = \tilde{B}_{\varphi_0,\mathscr{B}*f}/\mathscr{D}_0.$$

Since bounded set $\mathscr{B}*f \subset \mathscr{D}(G)$ is absorbed by $\mathscr{U}_1$, it follows that $\tilde{\mathscr{U}}$ absorbs $\tilde{B}_{\varphi_0,\mathscr{B}*f}$. In turn $\pi^{-1}(\tilde{\mathscr{U}})$ absorbs all $B_{\varphi,\mathscr{B}}$ being so is open in $\tilde{\mathscr{D}}(G)$.

We have $\sup_{\varphi\in\tilde{\mathscr{U}}, \psi\in\mathscr{U}_2} |B(\varphi, \psi)| < \infty$ what was to be proved.

2° The continuous hermitian symmetric form B on $\tilde{\mathscr{D}}(G)\times\mathscr{D}(G)$ induces a functional T_0 on the space $\mathscr{D}(E)$ with $E = \tilde{\mathscr{D}}(G)$ (cf. Bruhat [1], Proposition 1.1). The very definition of $\tilde{\mathscr{D}}(G)$ shows that the space $\mathscr{D}(E)$ is invariant under the continuous map θ defined by the formula

$$\theta\varphi(y) = L_y\overline{(\varphi(y))}.$$

The functional $T_0\cdot\theta$ is continuous and G — invariant, being so it is of the form $T\otimes dg$ where $T\in\bar{E}'$ (Bruhat [1], Proposition 3.3) what is clearly equivalent to the required result.

2. There are reasons to believe that condition 2° in definition 5.9 can be removed, i.e. results from 1°. Unfortunately, we shall not be able to admit the case of an arbitrary distribution $\tilde{T}$.

Exactly one of the orbits of AN in $\hat{S}$ has the dimension equal to $\dim\hat{S} = \dim AN$. This orbit is locally diffeomorphic to $A\times N$; in the sequel it is denoted by O^*. Denote by w^* the element of M' such that $O^* = AN\hat{\Pi}(w^*)$.

PROPOSITION A.2. *Let (U, J) be a completely irreducible representation of G and let $\omega_0\in\bar{\Phi}'$ be fixed under ${}^tU\,|\,MN$. If $\tilde{T}$ is a measure concentrated on O^*, then the representation is conical.*

Proposition A.2 states in particular that definition 5.9 generalizes the notion introduced by Helgason. Indeed, distribution $\tilde{T}$ associated to an eigenvector $\omega_0 \epsilon J$ is given by a continuous function on $\hat{S}$ and, by Proposition A.1, is conical. Thus a representation conical in the sense given by Helgason remains conical in our meaning.

*

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