Proof. This follows from Stampfli’s work and our Corollary 1. Theorems 4 and 5 suggest the possibility that every eigenspace of an isometry has an invariant complement. This holds for isometries of Hilbert space, since every isometry of a Hilbert space is the direct sum of a unitary operator and a unilateral shift (see [5]), but is apparently unknown for isometries of arbitrary Banach spaces.

References

uniform norm). This map has a unique extension to an isometry from the space \( \mathcal{F}(\mathcal{S}) \) (of continuous functions on \( \mathcal{S} \)) into the operators on \( \mathcal{L}^p \), since the polynomials are dense in \( \mathcal{F}(\mathcal{S}) \). Thus there is a unique definition of \( \varphi(M_\mathcal{S}) \) for every \( \varphi \in \mathcal{F}(\mathcal{S}) \). In fact, it is trivial to verify that \( \varphi(M_\mathcal{S}) \) is multiplication by \( \varphi \) for \( \varphi \in \mathcal{F}(\mathcal{S}) \). The existence of a “functional calculus” such as the above will be one of the elements of our characterization. (We should perhaps note that \( M_\mathcal{S} \) has a richer functional calculus than this: \( \varphi(M_\mathcal{S}) \) can be defined for \( \varphi \in \mathcal{L}^p \).)

Let \( f \) denote the function identically 1 on \( \mathcal{S} \). Then \( f \) is obviously a cyclic vector for \( M_\mathcal{S} \) (i.e., the smallest closed subspace of \( \mathcal{L}^p(\mathcal{S}, \mu) \) which contains \( (M_\mathcal{S}^k f)_{k=0}^\infty \) is \( \mathcal{L}^p(\mathcal{S}, \mu) \). Also

\[
\|g(M_\mathcal{S})f\| = \left( \int |g|^p d\mu \right)^{1/p} = \|g\| \cdot \|M_\mathcal{S}f\|
\]

for every \( g \in \mathcal{F}(\mathcal{S}) \).

We shall need one additional property of \( M_\mathcal{S} \). This property will be expressed in terms of a semi-inner-product. We recall that Lumer [3] has defined a semi-inner-product (s.i.p.) on a real or complex vector space \( X \) as a map from \( X \times X \) into the scalars that has all the properties of an inner product (including satisfying the Schwarz inequality) except for linearity in the second variable and a relation between \( \{f, g\} \) and \( \langle f, g \rangle \). We shall say that the s.i.p. \( \{\cdot, \cdot\} \) is compatible with the norm \( \|\cdot\| \) on \( X \) if \( \|f\| = |\langle f, f \rangle| \) for all \( f \in X \). Lumer shows that given a normed vector space there is at least one s.i.p. compatible with the norm.

We now seek an s.i.p. compatible with the norm on \( \mathcal{L}^p(\mathcal{S}, \mu) \). Giles [1] has exhibited a s.i.p. in the case of real \( \mathcal{L}^p, 1 < p < \infty \). It is easily verified that

\[
[f, g] = \frac{1}{|\langle f, f \rangle|^{1/p}} \int f |g|^{p-1} \text{sgn} g d\mu
\]

and

\[
[f, g] = \frac{1}{|\langle f, f \rangle|^{1/p}} \int f |g|^{p-1} \text{sgn} g d\mu
\]

define an s.i.p. on the real and complex spaces \( \mathcal{L}^p(\mathcal{S}, \mu) \) respectively, for \( 1 \leq p < \infty \).

Let \( \varphi \) be a non-negative continuous function on \( \mathcal{S} \) and \( f \) the function identically 1 on \( \mathcal{S} \). Then

\[
\|\varphi(M_\mathcal{S})f\|^p = \|\varphi(M_\mathcal{S})f, \varphi(M_\mathcal{S})f\| = \frac{1}{|\langle \varphi(f), f \rangle|^{p-1}} \int |\varphi(f)|^{p-1} \text{sgn} \varphi f d\mu = \frac{1}{|\langle f, f \rangle|^{p-1}} \int |\varphi(f)|^{p-1} \text{sgn} \varphi f d\mu.
\]

\[\text{Characterization of multiplication}\]

Now

\[
\|\varphi(M_\mathcal{S})f, f\| = \frac{1}{|\langle f, f \rangle|^{p-1}} \int |\varphi(f)|^{p-1} \text{sgn} \varphi f d\mu.
\]

Therefore

\[
\|\varphi(M_\mathcal{S})f\|^p = \frac{|\langle f, f \rangle|^{p-1}}{|\langle f, f \rangle|^{p-1}} \|\varphi(M_\mathcal{S})f, f\|.
\]

Equivalently,

\[
\|\varphi(M_\mathcal{S})f\|^p = \frac{|\langle f, f \rangle|^{p-2}}{|\langle f, f \rangle|^{p-2}} \|\varphi(M_\mathcal{S})f, f\|,
\]

or

\[
\|\varphi(M_\mathcal{S})f\|^p = |\langle f, f \rangle|^{p-2} |\varphi(M_\mathcal{S})f, f|.
\]

3. The characterization of \( M_\mathcal{S} \). The properties discussed above are sufficient to characterize \( M_\mathcal{S} \). The first two assumptions in the following theorem give, as indicated in the remarks above, a unique definition of \( \varphi(A) \) for \( \varphi \) continuous on \( \sigma(A) \).

**Theorem 1.** The bounded linear operator \( A \) on the complex (real) normed vector space \( \mathcal{X} \) is isometrically equivalent to \( M_\mathcal{S} \) on a complex (real) \( \mathcal{L}^p(\mathcal{S}, \mu) \), \( 1 \leq p < \infty \), if and only if \( A \) satisfies the following:

(a) \( \sigma(A) \) is real;

(b) \( \|\varphi(A)\| = \sup_{\varphi \in \sigma(A)} \|\varphi(a)\| \) for all complex (real) \( a \in \sigma(A) \);

(c) \( A \) has a cyclic vector \( f \) such that

(i) \( \|\varphi(A)f\| = \|\varphi(A)f\| \) for all complex (real) polynomials \( \varphi \);

(ii) there is an s.i.p. compatible with the norm such that \( \|\varphi(A)f\|^p = \|\varphi(A)\|^p \|\varphi(A)f, f\| \) for all continuous non-negative functions \( \varphi \) on \( \sigma(A) \).

**Proof.** We have seen that \( M_\mathcal{S} \) has the properties stated.

To prove the converse first define a linear functional on the complex (real) continuous functions on \( \sigma(A) \) by

\[ L(\varphi) = \|\varphi(A)f\|, \]

Then

\[ |L(\varphi)| \leq \|\varphi(A)f\| \cdot |f| \leq \|\varphi(A)f\| \cdot |f|^p = \sup_{\varphi \in \sigma(A)} \|\varphi(a)\|^p. \]

Also if \( \varphi \) is non-negative on \( \sigma(A) \), then

\[ L(\varphi) = \|\varphi(A)f\| = \|\varphi(A)\|^p \frac{|\langle f, f \rangle|^{p-2}}{|\langle f, f \rangle|^{p-2}} \geq 0. \]

Thus by the Riesz representation theorem there is a unique Borel measure \( \mu \) on \( \sigma(A) \) such that \( \|f\|^{-2} L(\varphi) = \int \varphi d\mu \) for all continuous functions \( \varphi \) on \( \sigma(A) \) (the reason for multiplying by \( \|f\|^{-2} \) will be apparent from the following).
Consider the linear transformation $U$ mapping the polynomials into $X$ by $Uq = q(A)f$. We claim that $U$ is an isometry on the polynomials as a subset of $\mathcal{L}^p(\mathcal{S}, \mu)$. For

$$
|q(A)f|^p = \|q(A)f\|^p = \|f\|^p - \|q^p(A)f, f\| = \|f\|^p - \int |q^p du = \|q\|^p.
$$

Therefore $|q(A)f| = |q\|^p$.

Since the polynomials are dense in $\mathcal{L}^p$ and $f$ is a cyclic vector for $A$, $U$ has a unique extension to an isometry $V$ taking $\mathcal{L}^p(\mathcal{S}, \mu)$ onto $X$. Then, if $q$ is any polynomial,

$$
V^{-1}AVq = V^{-1}(Aq(A)f) = Mrq.
$$

Thus $V^{-1}AV = Mr$.

4. The characterization of $Mr$. The fact that the polynomials in $\mathcal{S}$ are not usually dense in $\mathcal{L}^p(\mathcal{S}, \mu)$ for $\mathcal{S}$ a compact subset of the plane means that we cannot directly apply the above to characterize $Mr$. However, if we assume at the outset that an operator has a functional calculus with certain properties, then a similar result can be obtained. The following generalized Theorem 1:

**Theorem 2.** The bounded linear operator $A$ on the complex normed vector space $X$ is isometrically equivalent to $Mr$ on some $\mathcal{L}^p(\mathcal{S}, \mu)$, $1 < p < \infty$, if and only if:

(a) there is an isometric algebra isomorphism between $\mathcal{S}(\sigma(A))$ and a subalgebra of the operators on $X$ such that $\sigma$ corresponds to $A$;

(b) there is a vector $f$ such that

(i) the closure of $\{\varphi(A)f : \varphi \in \mathcal{S}(\sigma(A))\}$ is $X$ (where $\varphi(A)$ is the operator corresponding to $\varphi$ under the given isomorphism);

(ii) $\|\varphi(A)f\|^p = \|\varphi(A)f\|$ for all $\varphi \in \mathcal{S}(\sigma(A))$;

(iii) there is an admissible compatible norm such that $\|\varphi(A)f\|^p = |f|^p\|\varphi^p(A)f, f\|$ for all non-negative $\varphi \in \mathcal{S}(\sigma(A))$.

**Proof.** The proof given for Theorem 1 applies here with trivial modifications. Instead of defining $U$ on just the polynomials define $Uq = \varphi(A)f$ for $\varphi \in \mathcal{S}(\sigma(A))$. Then $U$ is isometric and its unique isometric extension $V$ satisfies $V^{-1}AV = Mr$.

5. Remarks. It might be interesting to attempt to characterize $Mr$ and $Mr$ when $\mu$ is finite; in this case $Mr$ is unbounded and the above techniques do not seem to apply.

The above could be used to give a characterization of multiplications by $\mathcal{S}$ functions. One could assume that there is a decomposition of $X$ into a “direct sum” of invariant subspaces for $A$ on each of which $A$ is isometrically equivalent to $Mr$. Then the measure spaces can be pieced together as suggested in [2].

It seems likely that other interesting operators could be characterized in terms of semi-inner-products.

**References**

