Proof. This follows from Stampfli's work and our Corollary 1. Theorems 4 and 5 suggest the possibility that every eigenspace of an isometry has an invariant complement. This holds for isometries of Hilbert space, since every isometry of a Hilbert space is the direct sum of a unitary operator and a unilateral shift (see [5]), but is apparently unknown for isometries of arbitrary Banach spaces.

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UNIVERSITY OF MIAMI OXFORD, OHIO, U. S. A.
UNIVERSITY OF TORONTO, TORONTO, CANADA

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A characterization of multiplication by the independent variable on $\mathscr{L}^p$

by

PETER ROSENTHAL (Toronto)

1. Introduction. One way of viewing the spectral theorem for Hermitian operators on complex Hilbert spaces is: every Hermitian operator is unitarily equivalent to a multiplication operator. This formulation of the spectral theorem has been popularized by Halmos [2]. The essence of the spectral theorem is then the statement that an operator A is a Hermitian operator with a cyclic vector if and only if there is a compact subset $\mathscr{S}$ of $\boldsymbol{R}$ and a finite measure μ on $\mathscr{S}$ such that A is unitarily equivalent to multiplication by the independent variable on $\mathscr{L}^2(\mathscr{S}, \mu)$. The proof of this assertion in the case where A is an operator on a real Hilbert space can proceed exactly as the proof of the complex case in [2] once it is known that $\|q(A)\| = \sup_{t\in\sigma(A)}|q(t)|$ for all (real) polynomials q.

In this note we consider the problem of characterizing the operator M_x defined on $\mathscr{L}^p(\mathscr{S}, \mu)$ (where $\mathscr{S}$ is a compact subset of $\boldsymbol{R}$ and $1 \leqslant p < \infty$) by

$$(M_x f)(x) = xf(x) \quad \text{for } f \in \mathscr{L}^p.$$

That is, we find a necessary and sufficient condition that an operator A on a Banach space be isometrically equivalent to M_x on $\mathscr{L}^p(\mathscr{S}, \mu)$. Our proof will be very similar to the proof of the case $p = 2$ presented in [2].

We give a similar characterization of multiplication by z on $\mathscr{L}^p(\mathscr{S}, \mu)$ where $\mathscr{S}$ is a compact subset of the complex plane.

2. Properties of M_x. Let μ be a finite Borel measure on $\boldsymbol{R}$ with compact support $\mathscr{S}$ and fix $p, 1 \leqslant p < \infty$. We consider some properties of the operator M_x on $\mathscr{L}^p(\mathscr{S}, \mu)$. We consider the real and complex cases simultaneously unless otherwise specified.

Clearly $\sigma(M_x) = \mathscr{S}$, and $\|q(M_x)\| = \sup_{x\in\mathscr{S}}|q(x)|$ for all polynomials q. This means that the map $q \to q(M_x)$ is an isometry from the polynomials (with sup norm) into the algebra of bounded operators on $\mathscr{L}^p$ (with uniform norm). This map has a unique extension to an isometry from the space $\mathscr{C}(\mathscr{S})$ (of continuous functions on $\mathscr{S}$) into the operators on $\mathscr{L}^p$, since the polynomials are dense in $\mathscr{C}(\mathscr{S})$. Thus there is a unique definition of $\varphi(M_x)$ for every $\varphi \in \mathscr{C}(\mathscr{S})$. In fact, it is trivial to verify that $\varphi(M_x)$ is multiplication by φ for $\varphi \in \mathscr{C}(\mathscr{S})$. The existence of a "functional calculus" such as the above will be one of the elements of our characterization. (We should perhaps note that M_x has a richer functional calculus than this: $\varphi(M_x)$ can be defined for $\varphi \in \mathscr{L}^\infty$).

Let f denote the function identically 1 on $\mathscr{S}$. Then f is obviously a cyclic vector for M_x (i.e., the smallest closed subspace of $\mathscr{L}^p(\mathscr{S}, \mu)$ which contains $\{M_x^n f\}_{n=0}^\infty$ is $\mathscr{L}^p(\mathscr{S}, \mu)$). Also

$$\|q(M_x)f\| = \left(\int |q|^p d\mu\right)^{1/p} = \| |q|(M_x)f\|$$

for every $q \in \mathscr{C}(\mathscr{S})$.

We shall need one additional property of M_x. This property will be expressed in terms of a semi-inner-product. We recall that Lumer [3] has defined a semi-inner-product (s.i.p.) on a real or complex vector space X as a map from $X \times X$ into the scalars that has all the properties of an inner product (including satisfying the Schwarz inequality) except for linearity in the second variable and a relation between $[f, g]$ and $[g, f]$. We shall say that the s.i.p. $[\cdot, \cdot]$ is *compatible* with the norm $\|\cdot\|$ on X if $[f, f] = \|f\|^2$ for all $f \in X$. Lumer shows that given a normed vector space there is at least one s.i.p. compatible with the norm.

We now seek an s.i.p. compatible with the norm on $\mathscr{L}^p(\mathscr{S}, \mu)$. Giles [1] has exhibited a s.i.p. in the case of real $\mathscr{L}^p, 1 < p < \infty$. It is easily verified that

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int f|g|^{p-1} \operatorname{sgn} g \, d\mu$$

and

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int f|g|^{p-1} \operatorname{sgn} \bar{g} \, d\mu$$

define an s.i.p. on the real and complex spaces $\mathscr{L}^p(\mathscr{S}, \mu)$ respectively, for $1 \leqslant p < \infty$.

Let φ be a non-negative continuous function on $\mathscr{S}$ and f the function dentically 1 on $\mathscr{S}$. Then

$$\|\varphi(M_x)f\|^2 = [\varphi(M_x)f, \varphi(M_x)f] = \frac{1}{\|\varphi\|_p^{p-2}} \int \varphi|\varphi|^{p-1} \operatorname{sgn} \varphi \, d\mu = \frac{1}{\|\varphi\|_p^{p-2}} \int \varphi^p \, d\mu.$$

Now

$$[\varphi^p(M_x)f, f] = \frac{1}{\|f\|^{p-2}} \int \varphi^p \, d\mu.$$

Therefore

$$\|\varphi(M_x)f\|^2 = \frac{\|f\|^{p-2}}{\|\varphi\|_p^{p-2}} [\varphi^p(M_x)f, f].$$

Equivalently,

$$\|\varphi(M_x)f\|^2 = \frac{\|f\|^{p-2}}{\|\varphi(M_x)f\|^{p-2}} [\varphi^p(M_x)f, f],$$

or

$$\|\varphi(M_x)f\|^p = \|f\|^{p-2} [\varphi^p(M_x)f, f].$$

3. The characterization of M_x. The properties discussed above are sufficient to characterize M_x. The first two assumptions in the following theorem give, as indicated in the remarks above, a unique definition of $\varphi(A)$ for φ continuous on $\sigma(A)$.

THEOREM 1. *The bounded linear operator A on the complex (real) normed vector space X is isometrically equivalent to M_x on a complex (real) $\mathscr{L}^p(\mathscr{S}, \mu)$, $1 \leqslant p < \infty$, if and only if A satisfies the following:*

(a) *$\sigma(A)$ is real;*

(b) *$\|q(A)\| = \sup_{x \in \sigma(A)} |q(x)|$ for all complex (real) polynomials q;*

(c) *A has a cyclic vector f such that*

(i) *$\|q(A)f\| = \| |q|(A)f\|$ for all complex (real) polynomials q,*

(ii) *there is an s.i.p. compatible with the norm such that $\|\varphi(A)\|^p = \|f\|^{p-2}[\varphi^p(A)f, f]$ for all continuous non-negative functions φ on $\sigma(A)$.*

Proof. We have seen that M_x has the properties stated.

To prove the converse first define a linear functional on the complex (real) continuous functions on $\sigma(A)$ by $L(\varphi) = [\varphi(A)f, f]$.

Then

$$|L(\varphi)| \leqslant \|\varphi(A)f\| \cdot \|f\| \leqslant \|\varphi(A)\| \cdot \|f\|^2 = \sup_{x \in \sigma(A)} |\varphi(x)| \|f\|^2.$$

Also if φ is non-negative on $\sigma(A)$, then

$$L(\varphi) = [\varphi(A)f, f] = [(\varphi^{1/p})^p(A)f, f] = \frac{\|\varphi^{1/p}(A)f\|^p}{\|f\|^{p-2}} \geqslant 0.$$

Thus by the Riesz representation theorem there is a unique Borel measure μ on $\mathscr{S} = \sigma(A)$ such that $\|f\|^{p-2} L(\varphi) = \int \varphi \, d\mu$ for all continuous functions φ on $\sigma(A)$ (the reason for multiplying by $\|f\|^{p-2}$ will be apparent from the following).

Consider the linear transformation U mapping the polynomials into X by $Uq = q(A)f$. We claim that U is an isometry on the polynomials as a subset of $\mathscr{L}^p(\mathscr{S}, \mu)$. For

$$\|q(A)f\|^p = \big\| |q|(A) \big\|^p = \|f\|^{p-2}[|q|^p(A)f, f] = \|f\|^{p-2}L(|q|^p) = \int |q|^p d\mu = \|q\|_p^p.$$

Therefore $\|q(A)f\| = \|q\|_p$.

Since the polynomials are dense in $\mathscr{L}^p$ and f is a cyclic vector for A, U has a unique extension to an isometry V taking $\mathscr{L}^p(\mathscr{S}, \mu)$ onto X. Then, if q is any polynomial,

$$V^{-1}AVq = V^{-1}\big(Aq(A)f\big) = M_x q.$$

Thus $V^{-1}AV = M_x$.

4. The characterization of M_z. The fact that the polynomials in z are not usually dense in $\mathscr{L}^p(\mathscr{S}, \mu)$ for $\mathscr{S}$ a compact subset of the plane means that we cannot directly apply the above to characterize M_z. However, if we assume at the outset that an operator has a functional calculus with certain properties, then a similar result can be obtained. The following generalizes Theorem 1:

THEOREM 2. *The bounded linear operator A on the complex normed vector space X is isometrically equivalent to M_z on some $\mathscr{L}^p(\mathscr{S}, \mu)$, $1 \leqslant p < \infty$, if and only if:*

(a) *there is an isometric algebra isomorphism between $\mathscr{C}\big(\sigma(A)\big)$ and a subalgebra of the operators on X such that z corresponds to A;*

(b) *there is a vector f such that*

(i) *the closure of $\{\varphi(A)f\colon \varphi \in \mathscr{C}\big(\sigma(A)\big)\}$ is X (where $\varphi(A)$ is the operator corresponding to φ under the given isomorphism).*

(ii) $\|\varphi(A)f\| = \big\||\varphi|(A)f\big\|$ *for all* $\varphi \in \mathscr{C}\big(\sigma(A)\big)$.

(iii) *there is an s.i.p. compatible with the norm such that* $\|\varphi(A)\|^p = \|f\|^{p-2}[\varphi^p(A)f, f]$ *for all non-negative* $\varphi \in \mathscr{C}\big(\sigma(A)\big)$.

Proof. The proof given for Theorem 1 applies here with trivial modifications. Instead of defining U on just the polynomials define $U\varphi = \varphi(A)f$ for $\varphi \in \mathscr{C}\big(\sigma(A)\big)$. Then U is isometric and its unique isometric extension V satisfies $V^{-1}AV = M_z$.

5. Remarks. It might be interesting to attempt to characterize M_x and M_z when μ is not finite; in this case M_x is unbounded and the above techniques do not seem to apply.

The above could be used to give a characterization of multiplications by $\mathscr{L}^\infty$ functions. One could assume that there is a decomposition of X into a "direct sum" of invariant subspaces for A on each of which A is isometrically equivalent to M_z. Then the measure spaces can be pieced together as suggested in [2].

It seems likely that other interesting operators could be characterized in terms of semi-inner-products.

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UNIVERSITY OF TORONTO
TORONTO, CANADA

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