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Finite-dimensional perturbations of spectral problems and variational approximation methods for eigenvalue problems

Part I. Finite-dimensional perturbations

by

N. ARONSZAJN and R. D. BROWN (Lawrence, Kansas)

Table of contents

INTRODUCTION

The general idea of finite-dimensional perturbations of spectral problems was conceived in 1956. The first author gave an outline of this theory in several lectures in the following years; he also mentioned this idea and its applications in several papers [4], [5], [6]. The main application of this notion was to general spectral problems, which the first author had started investigating some time before.

In particular this general notion of finite-dimensional perturbation allows one to unify the theoretical background for almost all existing variational approximation methods for eigenvalue problems(1). There are many such methods; e.g., the Ritz method, the Weinstein method, and methods based on the second monotony principle (see [3]). In all these methods the connection between the eigenvalues of the auxiliary problem and those of the intermediate problems was given by a determinant — a meromorphic function of ζ — each of these determinants being obtained by different constructions. (In the case of Weinstein's method this determinant is called Weinstein's determinant.) One of the main aims of the general notion of finite-dimensional perturbations was to put each of these determinants corresponding to different methods into a general framework as the determinant of the corresponding perturbation. This aim is obtainable by considering only perturbations which do not change the domain of the operators (since the approximation methods can be so formulated that the operators which appear are all bounded). On the other hand, for application to general spectral problems the finite-dimensional perturbations have to admit a change of domain; e.g. in ordinary differential problems, when one changes the (not necessarily self-adjoint) boundary conditions.

Since 1956 quite a number of papers have been published pertaining to the theory of spectral problems on one hand and to the theory of perturbations on the other. Many of our results were found independently by several other authors — sometimes in weaker form, sometimes in much stronger form — usually in quite different setting and with quite different methods.

Already in 1957-58 two important papers by Gokhberg-Krein [14] and by Kato [19] completely superceded the partial results of the first author concerning quasi-resolvent sets, isolated eigenvalues and corresponding

(1) This notion obviously can only be applied to those methods where the consecutive intermediate problems are obtained from an auxiliary problem by finite dimensional perturbations. To our knowledge there exists only one variational approximation method for eigenvalue problems where this condition does not hold. It was proposed by N. Aronszajn in 1949 and developed in 1950 by A. K. Jennings in his Master's thesis [17].

elementary divisors for spectral systems. We take advantage of this fact and simply review (in Section II.3) the results of these papers needed here.

As concerns the theory of perturbations, most of the literature is concerned with the classical "ε-perturbations" or with more recently developed infinite-dimensional perturbations of various special classes — perturbations of trace class, compact perturbations, etc. In his recent excellent book [20] T. Kato gives a quite complete exposition of these kinds of perturbations, and we refer the reader to this book for the relevant literature. Even though finite-dimensional perturbations have also been treated in the framework of this general research, it seems that in all the available literature there is no treatment presented for the general kind of finite-dimensional perturbation we are considering, the main difference being that in the available literature all perturbations preserve the domain of the operator, while we consider also those which change the domain.

The notion of the matrix of a perturbation and its use for actual computation of orders of elementary divisors (see Theorem II.6.1) does not seem to have been noticed in the literature. However, the notion of the determinant of a perturbation has been used in many special cases. In his book, Kato introduces them in general (in a simplified setting, without change of domain) under the name of W-A determinants of first and second kind, and he proves (in this setting) our Theorem II.6.2 under the name of W-A formula(2).

Our main theorem of Chapter III (Theorem III.1.3), which evaluates the change of multiplicity of any measure when we change a self-adjoint operator A into a self-adjoint operator B by a general finite-dimensional perturbation, does not seem to have been noticed. However, its corollary (III.1.3′), that the absolutely continuous parts of A and B are unitarily equivalent, was obtained by several authors — Kuroda [21], [22], Birman [11], Kato [18] and others — by completely different methods, as a result of investigation of the existence of the complete generalized wave operators between A and B. Also, their result is much stronger than our corollary since they admit perturbations of trace classes which are not finite-dimensional(3).

(2) It should be noted that what Kato calls "W-A determinants of second kind" were introduced by Weinstein [24] in 1937 in special cases and investigated in general by Aronszajn [2] in 1948, who called them "Weinstein's determinants". Those of "first kind" were introduced by Aronszajn in 1950 [3] in connection with variational approximation methods based on the second monotony principle. Also the "W-A formula" was discovered by Aronszajn in 1943, published first — in case of Weinstein's determinant — in 1948 [2], and — for the other kind of determinant — in 1950 [3].

(3) These authors avoid dealing with perturbations changing the domain by replacing the perturbation between A and B by the one between the bounded operators $(A-\zeta_0)^{-1}$ and $(B-\zeta_0)^{-1}$ for a fixed non-real ζ_0.

The present paper presents Part I of our research on finite-dimensional perturbations of spectral problems and variational approximation methods for eigenvalue problems. This first part is concerned with the general theory of finite-dimensional perturbations and its direct applications. In Part II we will present different variational approximation methods for eigenvalue problems in a framework based on the concepts and results of this first part. We will also investigate there the important question of convergence for these approximation methods.

In Part III we will describe and investigate the different ways in which these abstract approximation methods can be used in concrete differential eigenvalue problems. We will show that for quite general elliptic eigenvalue problems it is possible to apply one of these methods, or some combination of these methods, to obtain as precise approximations as one wishes to the eigenvalues.

Summary of Part I. In Chapter I the concept of finite-dimensional perturbation is introduced, as well as some associated concepts and properties. A linear transformation B with domain $\mathfrak{D}(B)$ from a linear space V into a linear space W is a *finite-dimensional perturbation* of the linear transformation A with domain $\mathfrak{D}(A)$ from V into W if and only if there exist decompositions(4).

$$\mathfrak{D}(A) = \mathfrak{D}\dot{+}[a_1, \ldots, a_n], \quad \mathfrak{D}(B) = \mathfrak{D}\dot{+}[b_1, \ldots, b_n] \tag{0.1}$$

such that $A = B$ on $\mathfrak{D}$. The decomposition (0.1) is called a *representation* of the perturbation. The smallest integer n for which such a decomposition exists is called the *dimension* of the perturbation. It is shown in Section I.1 that the relation "$A \sim B$ if and only if A is a finite-dimensional perturbation of B" is an equivalence relation.

If a linear transformation T is a finite-dimensional perturbation of the identity on V, then, corresponding to each decomposition

$$V = \mathfrak{D}\dot{+}[a_1, \ldots, a_n] \tag{0.2}$$

such that $T = I$ on $\mathfrak{D}$, there exists an $n\times n$ matrix M, called a *matrix representation* of the perturbation $T \sim I$ corresponding to the decomposition (0.2). The determinant of M, called the *determinant of the perturbation*, is independent of the decomposition.

More generally, when (0.1) is a representation for a perturbation $A \sim B$ and when A^{-1} exists — which means A^{-1} is defined on all of W — we use the canonical linear isomorphism S_{BA} which equals the identity on $\mathfrak{D}$ and assigns b_k to a_k. We then define the matrix and determinant of the perturbation $B \sim A$ by using the operator $A^{-1}BS_{BA}$ which transforms $\mathfrak{D}(A)$ into $\mathfrak{D}(A)$ and is a perturbation of the identity. This determinant is essentially independent of the choice of the representation (0.1).

In section I.2 it is shown that the index of an operator is invariant under finite-dimensional perturbations while the nullity and deficiency change by at most the dimension of the perturbation.

In most of the present paper we consider the situation where V is a topological vector space and W is a Banach space, instead of dealing with algebraic vector spaces. In Section I.3 we consider finite-dimensional perturbations in such cases. The domains $\mathfrak{D}(A)$ of operators A are assumed to be Banach subspaces of V (i.e., the injection mapping $\mathfrak{D}(A) \to V$ is continuous) and the operators A are assumed to be bounded mappings of $\mathfrak{D}(A)$ into W. In addition, in representations (0.1) of a perturbation, $\mathfrak{D}$ is always assumed to be closed in $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$.

Chapter II is concerned with spectral problems for spectral systems. In Section II.1 the concept of a *spectral system* $[V, W, \mathscr{P}]$ is introduced, where V and W are complex vector spaces and $\mathscr{P}$ is a two-dimensional pencil of linear transformations of V into W. We may thus write the system as a four-tuple $[V, W, H, G]$, where G and H generate $\mathscr{P}$. The general spectral problem for such systems is discussed. Such general algebraic spectral problems were investigated by N. Aronszajn and U. Fixman in [8].

The special case of spectral problems for *finite* spectral systems (i.e., where $\dim V$ and $\dim W$ are finite) is considered in Section II.2, and some results (see [8]) are summarized.

In Section II.3 the concept of the *quasi-resolvent set* $\mathscr{R}$ for a *Banach system* $[V, W, H, G]$ is introduced. (Here V, W are Banach spaces and G, H are bounded.) This set consists of all those complex numbers λ such that the range of $A_\lambda = G-\lambda H$ is closed and either the nullity or deficiency (or both) of A_λ is finite. Gokhberg and Krein and, independently, Kato have shown that $\mathscr{R}$ is the disjoint union of countably many open components $\mathscr{R}_j$, in each of which the index of A_λ is constant and in each of which the nullity and deficiency of A_λ are constant except at countably many isolated points, called the *isolated eigenvalues* of the system. To each isolated eigenvalue λ there are associated a *generalized eigenspace* and finitely many *elementary divisors*, which give a corresponding spectral decomposition of the system at λ.

In Section II.4 the concept of a finite-dimensional perturbation of a Banach system $[V, W, H, G]$ is introduced, and it is shown that $\mathscr{R}$ remains invariant under such perturbations, though the isolated eigenvalues may change.

(4) $[a_1, \ldots, a_n]$ denotes the subspace spanned by the linearly independent vectors $a_1, \ldots, a_n$.

In Section II.5 the Invariant Factor Theorem for matrices is applied to matrices $M(\lambda)$ whose entries are analytic functions of λ. An example of such matrices is given by any matrix $M(\lambda)$ corresponding to $A_\lambda^{-1}B_\lambda S_{BA} \sim I$, where $A_\lambda = G - \lambda H$, $B_\lambda = G_1 - \lambda H_1$, the system (II): $[V_1, W, H_1, G_1]$ is a finite-dimensional perturbation of (I): $[V, W, H, G]$, S_{BA} is the canonical isomorphism corresponding to a representation of the perturbation, and λ is in the resolvent set of (I). In Sections II.6 and II.7 it is shown how, knowing all necessary information about system (I), $M(\lambda)$ and $\det M(\lambda)$ can be used to obtain information (concerning, e.g., the isolated eigenvalues, generalized eigenspaces, and elementary divisors) about system (II). Some simple examples are given at the end of Section II.7. More involved examples are given in an appendix at the end of the paper.

In Chapter III we consider finite-dimensional perturbations between two standard self-adjoint systems; i.e., between systems $[\mathfrak{D}(A), \mathscr{H}, I, A]$, where $\mathscr{H}$ is a Hilbert space, I is the identity operator, A is a self-adjoint operator in $\mathscr{H}$, and $\mathfrak{D}(A)$ is the domain of A (with its graph norm). In Section III.1 we first prove (Theorem III.1.1) that, if two self-adjoint operators A, B in $\mathscr{H}$ are equal on a subspace $\mathfrak{D}$ contained in $\mathfrak{D}(A) \cap \mathfrak{D}(B)$, and if $\mathfrak{D}$ has finite codimension in $\mathfrak{D}(A)$, then $\mathfrak{D}$ has finite-codimension in $\mathfrak{D}(B)$ and the two corresponding systems are in the same perturbation class. Next we prove (Theorem III.1.2) that, if the perturbation between two standard self-adjoint systems is of dimension m, then one can be obtained from the other by a succession of m perturbations of dimension one. These perturbations are of two types — type one being a perturbation which does not change the domain of the operator and type two being a perturbation which does.

We then state the main result of this chapter, Theorem III.1.3, to be proved in Section III.6. In this theorem we give evaluations for the change in multiplicities of measures relative to self-adjoint operators A and B differing by a perturbation of dimension m, and obtain as an easy corollary (Corollary III.1.3′) that the absolutely continuous parts of A and B are unitarily equivalent.

Since we deal here with possibly non-separable Hilbert spaces, we give, in Section III.2, a review of definitions and results concerning multiplicities of measures relative to self-adjoint operators, following (except for a few changes) Halmos [15]. In Section III.3 we review the well known results concerning multiplicity relative to operators with simple spectrum; i.e., where the Hilbert space $\mathscr{H}$ is generated relative to the operator A by a single vector. Also we give some results needed for functions in class P (functions analytic in the upper half plane with positive imaginary part).

In Sections III.4 and III.5 we apply the information given in the preceding sections to determine the changes in multiplicities of measures induced by one-dimensional perturbations of types one and two respectively. The proof of Theorem III.1.3 and its corollary, given in Section III.6, follows immediately from these investigations. We also give in Section III.6 a useful result concerning eigenvalues of two self-adjoint systems in the same perturbation class, the perturbation being of dimension m; namely, in any interval $\mathscr{I}$ of the real axis which contains no points of the essential spectrum, the number (counting multiplicities) of eigenvalues of one system differs from the number of eigenvalues of the perturbed system by at most m.

In the Appendix we apply the theorems and techniques of Chapter II to spectral problems arising from differential operators with boundary conditions. We consider the perturbation between two Hilbert systems (I): $[V, L^2(-1,1), I, d^4/dt^4]$ and (II): $[V_1, L^2(-1,1), I, d^4/dt^4]$, where V and V_1 are closed subspaces of the space of potentials of order four, $P^4(-1,1) \subset L^2(-1,1)$. V is characterized by the boundary conditions

$$x(-1) = x(1) = x''(-1) = x''(1) = 0,$$

and thus (I) is a standard self-adjoint system, whereas V_1 is defined by four arbitrary linearly independent homogeneous boundary conditions (of orders $\leqslant 3$).

The matrix and determinant corresponding to such perturbations is calculated explicitly in terms of the boundary conditions of V_1. These formulas, together with easily obtained information about the original operator, are used to investigate the changes in the character of the meromorphy domain, the isolated eigenvalues, and the elementary divisors which are induced by the perturbation.

Under such a perturbation the original character $(0, 0)$ of the meromorphy domain (the entire complex plane) changes to (n, n). We find that always $0 \leqslant n \leqslant 2$ and derive specific relations which the boundary conditions of V_1 must satisfy in order that $n = 0, 1$, or 2. Examples are given for each case. Also, among others, an example is given where, corresponding to the eigenvalue $\lambda_1 = \pi^4/16$ of the perturbed system, there are two elementary divisors — one of order one and one of order three — while, corresponding to the eigenvalue λ_1 of the original system, there is a single elementary divisor of order one.

I. FINITE-DIMENSIONAL PERTURBATIONS

1. Basic concepts. Let V, W be linear spaces over the complex number field and A, B be linear transformations from V into W. Then A is said to be a *finite-dimensional perturbation* of B, written $A \sim B$, if there exists a direct decomposition of the domains of A and B of the form

$$\mathfrak{D}(A) = \mathfrak{D} \dot{+} [a_1, \ldots, a_n], \quad \mathfrak{D}(B) = \mathfrak{D} \dot{+} [b_1, \ldots, b_n] \tag{I.1.1}$$

such that $A = B$ on $\mathfrak{D}$. ($\dotplus$ means direct sum, and $[a_1, \ldots, a_n]$ is the subspace generated by the linearly independent vectors $a_1, \ldots, a_n$; similarly for $[b_1, \ldots, b_n]$.) Equation (I.1.1) is called a *decomposition corresponding to the perturbation* $A \sim B$. The smallest integer n for which there exists such a decomposition is called the *dimension of the perturbation*.

It is easy to form a decomposition corresponding to $A \sim B$ realizing the dimension of the perturbation. To this effect put

$$\mathfrak{D}' = [u\colon\ u \in \mathfrak{D}(A) \cap \mathfrak{D}(B),\ Au = Bu].$$

We have $\mathfrak{D} \subset \mathfrak{D}' \subset \mathfrak{D}(A) \cap \mathfrak{D}(B)$ and since $\mathfrak{D}$ has the same co-dimension in $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ the same is true for $\mathfrak{D}'$ and hence we can find decompositions

$$\mathfrak{D}(A) = \mathfrak{D}' \dotplus [a_1', \ldots, a_{n'}'], \qquad \mathfrak{D}(B) = \mathfrak{D}' \dotplus [b_1', \ldots, b_{n'}'].$$

Here, obviously, the number $n' = \text{co-dim}\,\mathfrak{D}'$ is the smallest possible.

It is easy to see that the relation "$\sim$" is an equivalence relation. The symmetry and reflexivity are obvious; transitivity follows from the fact that the intersection of two spaces of finite co-dimension is again of finite co-dimension. Equivalence classes corresponding to the relation "$\sim$" are called *perturbation classes*. We denote by $\langle A\rangle$ the perturbation class consisting of all B such that $B \sim A$.

In the special case that the transformations in question are defined on all of V, $A \sim B$ and $C \sim D$ imply that $\lambda A + \mu C \sim \lambda B + \mu D$ for any scalars λ, μ. If, in addition, $V = W$, then $A \sim B$ and $C \sim D$ also imply $AC \sim BD$.

If $A \sim B$, then to any decomposition of type (I.1.1) corresponds a canonical linear isomorphism S_{BA} of $\mathfrak{D}(A)$ onto $\mathfrak{D}(B)$, defined by $S_{BA} = I$ on $\mathfrak{D}$ and $S_{BA}a_i = b_i$, $i = 1, \ldots, n$. On the other hand, let $A \sim B$ and let S_{BA} be any isomorphism of $\mathfrak{D}(A)$ onto $\mathfrak{D}(B)$ which equals I on a subspace $\mathfrak{D}'$ of $\mathfrak{D}(A)$ of finite co-dimension. Then, if (I.1.1) is a decomposition corresponding to $A \sim B$, $\mathfrak{D}'' = \mathfrak{D}' \cap \mathfrak{D}$ has finite co-dimension in $\mathfrak{D}(A)$ and $S_{BA} = I$ on $\mathfrak{D}''$. Thus, there exist linearly independent vectors $c_1, \ldots, c_m$ such that

$$\mathfrak{D}(A) = \mathfrak{D}'' \dotplus [c_1, \ldots, c_m], \qquad \mathfrak{D}(B) = \mathfrak{D}'' \dotplus [S_{BA}c_1, \ldots, S_{BA}c_m]$$

is a decomposition corresponding to $A \sim B$, and S_{BA} is the canonical isomorphism it determines.

Let $\langle A\rangle$ be a perturbation class of transformations from V into W. For any two transformations $B, C \in \langle A\rangle$, choose a decomposition corresponding to $B \sim C$, and a corresponding isomorphism S_{CB}. The set $\{S_{CB}\colon B, C \in \langle A\rangle\}$ is called *coherent* if for any B, C, D in $\langle A\rangle$,

$$S_{CD}S_{DB} = S_{CB}.$$

In particular, $S_{BB}S_{BC} = S_{BC}$, hence S_{BB} is the identity on B.

A coherent set of isomorphisms for $\langle A\rangle$ can always be chosen. For example, a decomposition corresponding to $B \sim A$ and the associated isomorphism S_{BA} can be chosen for each $B \in \langle A\rangle$. Then a coherent set $\{S_{CB}\colon B, C \in \langle A\rangle\}$ of isomorphisms is defined by $S_{CB} = S_{CA}S_{BA}^{-1}$. In what follows, when dealing with a perturbation class, we shall always assume a coherent set of isomorphisms has been chosen.

Those transformations T of V into itself such that $T \sim I$ turn out to be of special importance. If $T \sim I$, then there is a decomposition

$$V = \mathfrak{D} \dotplus [a_1, \ldots, a_n] \tag{I.1.2}$$

such that $T = I$ on $\mathfrak{D}$. Let P be the corresponding projection of V onto $[a_1, \ldots, a_n]$. The subspace $[a_1, \ldots, a_n]$ is an invariant subspace of PT.

THEOREM I.1.1. *Let* (I.1.2) *be a decomposition corresponding to* $T \sim I$ *and* P *be the corresponding projection of* V *onto* $[a_1, \ldots, a_n]$. *Then*

$$Su = u - Tu$$

defines an isomorphism between $N[PT] \cap [a_1, \ldots, a_n]$ *and* $N[T]$ (5).

Proof. If $u \in [a_1, \ldots, a_n]$ and $PTu = 0$, then $Tu \in \mathfrak{D}$ and $Su = u - Tu \in N[T]$. Let $Q = I - P$ and $v \in N[T]$. Then

$$Tv = T(Pv + Qv) = TPv + Qv = 0$$

so that $TPv = -Qv$. Thus, if R is the restriction of P to $N[T]$, then $R\colon N[T] \to [a_1, \ldots, a_n]$ and

$$PT(Rv) = PTPv = -PQv = 0$$

for $v \in N[T]$. Hence Rv is in the null space of PT. To complete the proof we need only note that for $v \in N[T]$,

$$SRv = Pv - TPv = Pv + Qv = v$$

so that $S = R^{-1}$.

In Theorem I.1.1, let M be the matrix representation of the restriction of PT to $[a_1, \ldots, a_n]$ with respect to the basis $a_1, \ldots, a_n$. Then $\dim N[T]$ is equal to the rank of M. M is called the *matrix representation of* T *with respect to the decomposition* (I.1.2).

If a decomposition for $T \sim I$ different from (I.1.2) is used, a different matrix representation is obtained. If for instance a different basis $b_1, \ldots, b_n$ is chosen for $[a_1, \ldots, a_n]$, then

$$V = \mathfrak{D} \dotplus [b_1, \ldots, b_n] \tag{I.1.3}$$

is also a decomposition for $T \sim I$. For $k = 1, \ldots, n$,

$$Ta_k = \sum_{i=1}^{n} a_i \alpha_{ik} + d_k, \qquad Tb_k = \sum_{i=1}^{n} b_i \beta_{ik} + d_k',$$

(5) $N[T] = \{u\colon\ u \in \mathfrak{D}(T)$ and $Tu = 0\}$

where $d_k, d'_k \in \mathfrak{D}$. The matrix representation of $T \sim I$ with respect to (I.1.2) is then $M = (\alpha_{ik})$, while the matrix with respect to (I.1.3) is $N = (\beta_{ik})$. Since $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are bases for the same finite dimensional space, however,

$$b_k = \sum_{i=1}^{n} a_i \gamma_{ik}, \quad k = 1, \ldots, n,$$

where the $n \times n$ matrix $S = (\gamma_{ik})$ is non-singular. Writing $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $d = (d_1, \ldots, d_n)$ and $d' = (d'_1, \ldots, d'_n)$, we have then that $b = aS$, so

$$Tb = T(aS) = (Ta)S = (aM+d)S = aMS + dS.$$

But also,

$$Tb = bN + d' = aSN + d'$$

so $d' = dS$ and $MS = SN$. Thus N is related to M by the formula

$$N = S^{-1} MS.$$

In particular, $\det N = \det M$.

Another way in which the decomposition (I.1.2) for $T \sim I$ can be changed is by choosing linearly independent elements $d_1, \ldots, d_m \in \mathfrak{D}$ to obtain a representation

(I.1.4) $$V = \mathfrak{D}' \dot{+} [a_1, \ldots, a_n, d_1, \ldots, d_m], \quad \mathfrak{D}' \subset \mathfrak{D}.$$

For $k = 1, \ldots, n$,

$$Ta_k = \sum_{i=1}^{n} a_i \alpha_{ik} + \sum_{i=1}^{m} d_i \delta_{ik} + d'_k$$

while for $k = 1, \ldots, m$, $Td_k = d_k$. Hence the matrix representation for T with respect to (I.1.4) is the $(n+m)\times(n+m)$ matrix

$$M' = \begin{pmatrix} M & 0 \\ D & I \end{pmatrix},$$

where $M = (\alpha_{ik})$ is the $n \times n$ matrix representation for T with respect to (I.1.2), D is the $m \times n$ matrix (δ_{ik}) and I is the $m \times m$ identity matrix. Again $\det M' = \det M$.

Suppose next that M is the matrix representation for $T \sim I$ with respect to (I.1.2) and N is the matrix representation of $T \sim I$ with respect to the decomposition

(I.1.5) $$V = \mathfrak{D}' \dot{+} [b_1, \ldots, b_m].$$

Put $\mathscr{A} = [a_1, \ldots, a_n]$ and $\mathscr{B} = [b_1, \ldots, b_m]$. We choose decompositions

$$\mathfrak{D} = \mathfrak{D} \cap \mathfrak{D}' \dot{+} \mathfrak{D}_1, \quad \mathfrak{D}' = \mathfrak{D} \cap \mathfrak{D}' \dot{+} \mathfrak{D}'_1.$$

Defining $\mathscr{A}' = \mathscr{A} \dot{+} \mathfrak{D}_1$, $\mathscr{B}' = \mathscr{B} \dot{+} \mathfrak{D}'_1$, we get

$$V = \mathfrak{D} \cap \mathfrak{D}' \dot{+} \mathscr{A}' = \mathfrak{D} \cap \mathfrak{D}' \dot{+} \mathscr{B}'.$$

We decompose further

$$\mathfrak{D} \cap \mathfrak{D}' = \mathfrak{D}'' \dot{+} [(\mathscr{A}' + \mathscr{B}') \cap \mathfrak{D} \cap \mathfrak{D}'].$$

We arrive thus at the decomposition

$$V = \mathfrak{D}'' \dot{+} (\mathscr{A}' + \mathscr{B}'),$$

which leads to two decompositions for the perturbation $T \sim I$:

$$V = \mathfrak{D}'' \dot{+} [a_1, \ldots, a_n, d_1, \ldots, d_q] \quad \text{with } d_i \in \mathfrak{D},\ i = 1, 2, \ldots, q,$$
$$V = \mathfrak{D}'' \dot{+} [b_1, \ldots, b_m, d'_1, \ldots, d'_p] \quad \text{with } d'_i \in \mathfrak{D}',\ i = 1, 2, \ldots, p.$$

In view of the preceeding considerations we obtain the following theorem:

THEOREM I.1.2. *If (I.1.2) and (I.1.5) are two decompositions for the perturbation $T \sim I$ and M, N the corresponding matrices, then there exist two integers $p \geqslant 0$, $q \geqslant 0$ with $m+p = n+q$ and a $(n+q)\times(n+q)$ non-singular matrix S, p, q and S depending only on the decompositions, so that*

$$\begin{pmatrix} N & 0 \\ N' & I \end{pmatrix} = S^{-1} \begin{pmatrix} M & 0 \\ M' & I \end{pmatrix} S.$$

Here the matrices M' and N' are $q \times n$ and $p \times m$ respectively; they depend in general on the operator T and not only on M, N and the decompositions.

As an immediate corollary we obtain

COROLLARY I.1.2'. $\det M = \det N$.

THEOREM I.1.3. *If $T_1 \sim I$ and $T_2 \sim I$, then there exists a decomposition for $T_1T_2 \sim I$ which is at the same time a decomposition for $T_1 \sim I$ and $T_2 \sim I$. Moreover, if M_1, M_2, M_3 are the matrix representations of T_1, T_2, and T_1T_2 respectively with respect to any such common decomposition, then $M_3 = M_1M_2$.*

Proof. If $V = \mathfrak{D} \dot{+} [a_1, \ldots, a_n]$, $V = \mathfrak{D}_1 \dot{+} [b_1, \ldots, b_m]$ are decompositions for $T_1 \sim I$, $T_2 \sim I$ respectively, then $\mathfrak{D}' = \mathfrak{D} \cap \mathfrak{D}_1$ is of finite codimension and there exist elements $c_1, \ldots, c_p$ in V such that

(I.1.6) $$V = \mathfrak{D}' \dot{+} [c_1, \ldots, c_p],$$

and $T_1T_2 = T_1 = T_2 = I$ on $\mathfrak{D}'$. Hence (I.1.6) is the common decomposition of the theorem. Putting $c = (c_1, \ldots, c_p)$, we have

$$T_1 c = cM_1 + d, \quad T_2 c = cM_2 + d', \quad T_1T_2 c = cM_3 + d'',$$

where d, d', and d'' are p vectors with components in $\mathfrak{D}'$. But also

$$T_1T_2c = T_1(cM_2+d') = (T_1c)M_2+d' = c(M_1M_2)+(dM_2+d'),$$

so $d'' = dM_2+d'$ and $M_3 = M_1M_2$.

The determinant of $T \sim I$, written $\det T$, is defined as $\det M$, where M is any matrix representation of T. From the above discussion it follows that $\det T$ is independent of the particular representation of $T \sim I$ used to define it. From theorem I.1.3 follows

COROLLARY I.1.3'. *If $T_1 \sim I$ and $T_2 \sim I$, then $T_1T_2 \sim I$ and $\det(T_1T_2) = (\det T_1)(\det T_2)$.*

Let (I.1.2) be any decomposition corresponding to $T \sim I$, and P be the projection of V onto $[a_1, \dots, a_n]$ determined by (I.1.2). Then $Ta_i - PTa_i \in \mathfrak{D}$, $i = 1, \dots, n$, and if $\mathscr{C}$ is the finite-dimensional space spanned by $a_1, \dots, a_n$, $Ta_1-PTa_1, \dots, Ta_n-PTa_n$, then $V = \mathfrak{D}' \dotplus \mathscr{C}$, where $T = I$ on $\mathfrak{D}' \subset \mathfrak{D}$, and C is an invariant subspace of T. Clearly, $\det T$ is the determinant of the restriction of T to the finite-dimensional space C. The following theorem is then obvious:

THEOREM I.1.4. *Let $T \sim I$. Then T has an inverse if and only if $\det T \neq 0$, and $\det T^{-1} = (\det T)^{-1}$.*

Given a perturbation $T \sim I$ and a corresponding decomposition (I.1.2), the corresponding matrix representation M can be obtained using linear functionals defined on V. For if

$$Ta_k = \sum_{i=1}^{n} a_i a_{ik} + d_k, \qquad k = 1, \dots, n,$$

and if $F_1, \dots, F_n$ are linearly independent linear functionals on V which vanish on $\mathfrak{D}$, then for $k, l = 1, \dots, n$,

$$F_l(Ta_k) = \sum_{i=1}^{n} F_l(a_i)a_{ik}, \tag{I.1.7}$$

and the $n \times n$ matrix $[F_l(a_i)]$ is non-singular. Hence (I.1.7) can be solved for a_{ik} to obtain

$$M = \{F_l(a_i)\}^{-1}\{F_l(Ta_k)\}, \tag{I.1.8}$$

$$\det T = \det M = \frac{\det[F_l(Ta_k)]}{\det[F_l(a_i)]}. \tag{I.1.9}$$

These formulas become especially simple when the functionals F_i form a dual system (biorthogonal) to the basis $\{a_k\}$, i.e. when $F_i(a_k) = \delta_{i,k}$. Then

$$M = \{F_i(Ta_k)\}, \qquad \det T = \det\{F_i(Ta_k)\}.$$

Let A be a linear transformation from V into W and $\langle A\rangle$ be its perturbation class. For any $B, C \in \langle A\rangle$ such that B^{-1} exists, we define the transformation $B\backslash C: \mathfrak{D}(A) \to \mathfrak{D}(A)$ by

$$B\backslash C = S_{AB}B^{-1}CS_{CA}, \tag{I.1.10}$$

where $\{S_{BC}: B, C \in \langle A\rangle\}$ is a coherent set of isomorphisms for $\langle A\rangle$. Clearly $B\backslash C$ is a finite-dimensional perturbation of the identity on $\mathfrak{D}(A)$. Obviously $B\backslash C$ depends on the fixed operator A in the perturbation class $\langle A\rangle = \langle B\rangle = \langle C\rangle$.

THEOREM I.1.5. *Let A be a fixed linear transformation from V into W and let $B \sim C \sim D \sim A$ such that B^{-1} and C^{-1} exist. Then*

$$B\backslash C\, C\backslash D = B\backslash D, \qquad \det B\backslash C \det C\backslash D = \det B\backslash D.$$

Proof. $B\backslash C C\backslash D = S_{AB}B^{-1}CS_{CA}S_{AC}C^{-1}DS_{DA} = S_{AB}B^{-1}DS_{DA}$. The second equality in the theorem follows from Corollary I.1.3'.

THEOREM I.1.6. *Let $A \sim A' \sim B \sim C$ and let B^{-1} exist. Denote by $(B\backslash C)'$ the operator $B\backslash C$ with A replaced by A'. Then $(B\backslash C)'$ is isomorphic to $B\backslash C$ by the isomorphism $S_{A'A}$, i.e. $(B\backslash C)' = S_{A'A}B\backslash CS_{AA'}$. Moreover, $\det B\backslash C = \det(B\backslash C)'$.*

Proof. By the coherence of the isomorphisms S_{BC} we have

$$S_{A'B}B^{-1}CS_{CA'} = S_{A'A}S_{AB}B^{-1}CS_{CA}S_{AA'}.$$

The isomorphism of the two operators implies the equality of their determinants.

Consider now a common coherent decomposition for $A \sim B \sim C$ where B^{-1} exists:

$$\left.\begin{aligned} \mathfrak{D}(A) &= \mathfrak{D}\dotplus[a_1, \dots, a_n] \\ \mathfrak{D}(B) &= \mathfrak{D}\dotplus[S_{BA}a_1, \dots, S_{BA}a_n] \\ \mathfrak{D}(C) &= \mathfrak{D}\dotplus[S_{CA}a_1, \dots, S_{CA}a_n] \end{aligned}\right\} \quad Au = Bu = Cu \text{ for } u \in \mathfrak{D}.$$

Obviously $\mathfrak{D}(A) = \mathfrak{D}\dotplus[a_1, \dots, a_n]$ is a decomposition corresponding to $B\backslash C \sim I$ on $\mathfrak{D}(A)$. Writing $\operatorname{mat} B\backslash C$ for the corresponding matrix representation of $B\backslash C$ and introducing n independent linear functionals F_l on $\mathfrak{D}(A)$ which vanish on $\mathfrak{D}$ we obtain from (I.1.8) and (I.1.9)

$$\operatorname{mat}(B\backslash C) = \{F_l(a_i)\}^{-1}\{F_l(S_{AB}B^{-1}CS_{CA}a_k)\}, \tag{I.1.11}$$

$$\det(B\backslash C) = \frac{\det\{F_l(S_{AB}B^{-1}CS_{CA}a_k)\}}{\det\{F_l(a_i)\}}. \tag{I.1.12}$$

2. Nullity and deficiency. Let V, W be linear spaces and let A be a linear transformation from V into W. The *nullity* $\alpha(A)$ is defined to be the dimension of $N[A]$, and the *deficiency* $\beta(A)$ is the codimension of the range $R[A]$ in W. The *index* $\gamma(A)$ is the difference $\gamma(A) = \beta(A) - \alpha(A)$, and is defined only when either $\alpha(A)$ or $\beta(A)$ (or both) is finite. Thus $\gamma(A)$ may be finite, $+\infty$ or $-\infty$.

THEOREM I.2.1. *Let $B \overset{\cdot}{\sim} A$, and n be the dimension of the perturbation. Then $\alpha(A)$ is finite if and only if $\alpha(B)$ if finite and $\beta(A)$ is finite if and only if $\beta(B)$ is finite. If $\alpha(A)$ is finite, $|\alpha(A) - \alpha(B)| \leqslant n$ and if $\beta(A)$ is finite, $|\beta(A) - \beta(B)| \leqslant n$. Moreover, $\gamma(A)$ is defined if and only if $\gamma(B)$ is defined, in which case $\gamma(A) = \gamma(B)$.*

Proof. Let $\mathfrak{D} = \{u\colon u \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$ and $Au = Bu\}$. We may then write $N[A], N[B]$ as direct sums

$$N[A] = N \dot{+} N_A, \qquad N[B] = N \dot{+} N_B,$$

where $N = N[A] \cap N[B] \subset \mathfrak{D}$, and choose a decomposition corresponding to $A \sim B$ of the form

$$\mathfrak{D}(A) = \mathfrak{D} \dot{+} N_A \dot{+} M_A, \qquad \mathfrak{D}(B) = \mathfrak{D} \dot{+} N_B \dot{+} M_B.$$

Since n is the dimension of the perturbation we have

$$\dim N_A + \dim M_A = \dim N_B + \dim M_B = n. \tag{I.2.1}$$

In particular, $\dim N_A$ and $\dim N_B$ are finite. Thus since $\alpha(A) = \dim N + \dim N_A$ and $\alpha(B) = \dim N + \dim N_B$, $\alpha(B)$ is finite if and only if $\alpha(A)$ is, in which case (I.2.1) shows that

$$|\alpha(A) - \alpha(B)| = |\dim N_A - \dim N_B| \leqslant n.$$

On the other hand, $A(M_A) \cap A(\mathfrak{D}) = \{0\}$ and $B(M_B) \cap B(\mathfrak{D}) = \{0\}$, so

$$W = A(\mathfrak{D}) \dot{+} A(M_A) \dot{+} W_A,$$
$$W = B(\mathfrak{D}) \dot{+} B(M_B) \dot{+} W_B,$$

where $\dim W_A = \beta(A)$, $\dim W_B = \beta(B)$. Note that $A(\mathfrak{D}) = B(\mathfrak{D})$, that $\dim A(M_A) = \dim M_A < \infty$, and that $\dim B(M_B) = \dim M_B < \infty$. Hence

$$\dim M_A + \beta(A) = \dim M_B + \beta(B),$$

so $\beta(B)$ is finite if and only if $\beta(A)$ is finite, in which case (I.2.1) shows

$$|\beta(B) - \beta(A)| = |\dim M_B - \dim M_A| \leqslant n.$$

We next consider the quantities $\gamma(A)$ and $\gamma(B)$. Since $\alpha(A)$ is finite if and only if $\alpha(B)$ is finite, and similarly for $\beta(A)$ and $\beta(B)$, it is clear that if $\gamma(A)$ is $+\infty$ or $-\infty$, then $\gamma(B)$ must also be $+\infty$ or $-\infty$ respectively. If $\gamma(A)$ is finite, on the other hand, then both $\alpha(A)$ and $\beta(A)$ are finite so that $\alpha(B), \beta(B)$ must also be finite. Hence, using (I.2.1),

$$\alpha(A) - \alpha(B) = \dim N_A - \dim N_B = \dim M_B - \dim M_A = \beta(A) - \beta(B),$$

so that $\gamma(A) = \gamma(B)$.

3. Continuous operators on Banach spaces. In most of our considerations we shall deal with operators A having for domain $\mathfrak{D}(A)$ a Banach subspace of a fixed Hausdorff topological vector space V and transforming $\mathfrak{D}(A)$ continuously (in its topology) into a fixed Banach space W.

We remind the reader of a few elementary facts about Banach subspaces.

A Banach space $\mathfrak{D}$ which is a subspace of a Hausdorff topological vector space V is called a *Banach subspace* of V if the injection mapping (identity mapping), $\mathfrak{D} \to V$, is continuous. If $\mathfrak{D}$ and $\mathfrak{D}_1$ are Banach subspaces, then $\mathfrak{D} \cap \mathfrak{D}_1$ and $\mathfrak{D} + \mathfrak{D}_1$ are also Banach subspaces of V with the following norms:

$$\|u\|_{\mathfrak{D} \cap \mathfrak{D}_1} = \max[\|u\|_{\mathfrak{D}}, \|u\|_{\mathfrak{D}_1}], \tag{I.3.1}$$

$$\|u\|_{\mathfrak{D} + \mathfrak{D}_1} = \inf_{\substack{v + v_1 = u \\ v \in \mathfrak{D}, v_1 \in \mathfrak{D}_1}} [\|v\|_{\mathfrak{D}} + \|v_1\|_{\mathfrak{D}_1}]. \tag{I.3.2}$$

In case $\mathfrak{D}$ and $\mathfrak{D}_1$ are Hilbert subspaces we can replace the above norms by equivalent ones making the corresponding spaces into Hilbert spaces. These norms are:

$$\|u\|^2_{\mathfrak{D} \cap \mathfrak{D}_1} = \|u\|^2_{\mathfrak{D}} + \|u\|^2_{\mathfrak{D}_1}, \tag{I.3.1'}$$

$$\|u\|^2_{\mathfrak{D} + \mathfrak{D}_1} = \inf_{\substack{v + v_1 = u \\ v \in \mathfrak{D}, v_1 \in \mathfrak{D}_1}} [\|v\|^2_{\mathfrak{D}} + \|v_1\|^2_{\mathfrak{D}_1}]. \tag{I.3.2'}$$

By using the closed graph theorem one immediately obtains the following two propositions:

PROPOSITION I.3.1. *If $\mathfrak{D}_1 \subset \mathfrak{D}$ and $\mathfrak{D}$ and $\mathfrak{D}_1$ are Banach subspaces of V, then $\mathfrak{D}_1$ is a Banach subspace of $\mathfrak{D}$.*

PROPOSITION I.3.2. *If $\mathfrak{D}$ is a subspace of the Hausdorff topological vector space V, then there exists at most one Banach topology on $\mathfrak{D}$ making it into a Banach subspace of V.*

PROPOSITION I.3.3. *If A is a continuous mapping of the Banach space $\mathfrak{D}$ into the Banach space W, then the range, $A(\mathfrak{D})$, is a Banach subspace of W.*

Proof. Consider the null-space $N[A]$ which is a closed subspace of $\mathfrak{D}$. The mapping A induces an algebraic isomorphism of the quotient space $\mathfrak{D}/N[A]$ onto $A(\mathfrak{D})$. Using this isomorphism, and transferring the norm of $\mathfrak{D}/N[A]$ to $A(\mathfrak{D})$ we obtain the required Banach topology.

PROPOSITION I.3.4. *If $\mathfrak{D}_1$ is a Banach subspace of $\mathfrak{D}$, and $\mathfrak{D}_1$ is of finite codimension in $\mathfrak{D}$, then $\mathfrak{D}_1$ is a closed subspace of $\mathfrak{D}$.*

Proof. We have $\mathfrak{D} = \mathfrak{D}_1 \dot{+} S$, where S is finite-dimensional. Any norm chosen on S makes S into a Banach subspace of $\mathfrak{D}$. The norm I.3.2 on $\mathfrak{D}_1 \dot{+} S$ is here the direct sum norm in which $\mathfrak{D}_1$ is a closed subspace of $\mathfrak{D}$. But this norm makes $\mathfrak{D}$ a Banach subspace of $\mathfrak{D}$; hence by Proposition I.3.2 it is equivalent to the original norm of $\mathfrak{D}$.

Proposition I.3.4 together with I.3.3 gives

PROPOSITION I.3.5. *If A is a continuous mapping of the Banach space $\mathfrak{D}$ into the Banach space W and if the deficiency $\beta(A)$ is finite, then $A(\mathfrak{D})$ is a closed subspace of W.*

When dealing with finite-dimensional perturbations of operators transforming continuously their domains — Banach subspaces of V — into the Banach space W we will assume that in the corresponding decomposition (I.1.1)

(I.3.3) *The subspace $\mathfrak{D}$ is a closed subspace of $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$.*

This assumption is not an essential restriction since in the minimal decomposition, $\mathfrak{D} = [u\colon u \in \mathfrak{D}(A) \cap \mathfrak{D}(B), Au - Bu = 0]$ is a closed subspace of both $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$. To see this we remark that with the norm (I.3.1) on $\mathfrak{D}(A) \cap \mathfrak{D}(B)$ this intersection becomes a Banach subspace of V which is transformed continuously into W by A and B. Hence $\mathfrak{D}$ is a closed subspace of $\mathfrak{D}(A) \cap \mathfrak{D}(B)$ and therefore a Banach subspace of $\mathfrak{D}(A)$ as well as $\mathfrak{D}(B)$. Proposition I.3.4 then gives our statement.

In consequence of the assumption (I.3.3) the linear functionals F_l used in Section 1 to establish the matrix representing the perturbation $T \sim I$ are necessarily *continuous*. Also the isomorphisms S_{BC} between the domains of transformations in the same perturbation class are necessarily topological.

II. SPECTRAL PROBLEMS AND THE QUASI-RESOLVENT SET

1. The spectral problem. We will call an *algebraic spectral system* or, briefly, a *system*, a triple $[V, W, \mathscr{P}]$ composed of two vector spaces V and W and a 2-dimensional pencil $\mathscr{P}$ of linear transformations of V into W. Such a pencil is formed by a linear mapping $\mathscr{L}$ of the 2-dimensional complex space C^2 into the space $L(V, W)$ of all linear transformations of V into W. However, the same pencil is obtained if we replace $\mathscr{L}$ by $\mathscr{L}S$, where S is any linear automorphism of C^2 onto C^2. It follows that the pencil is determined if we give the images $H = \mathscr{L}(\omega_1)$ and $G = \mathscr{L}(\omega_2)$ of any basis ω_1, ω_2 in C^2 without specification of the basis. H and G will be called *generators* of the pencil $\mathscr{P}$.

If a transformation Z is the image of an element in C^2 by a mapping determining the pencil we will say, for brevity, that *Z belongs to $\mathscr{P}$, $Z \in \mathscr{P}$.*

If H, G are generators for $\mathscr{P}$, the quadruple $[V, W, H, G]$ will be called a *representation* of the system; sometimes we will just speak about the system $[V, W, H, G]$.

Let H, G be fixed generators for $\mathscr{P}$. Then

(II.1.1) $$Z \in \mathscr{P} \Leftrightarrow Z = \alpha G + \beta H, \quad (\alpha, \beta) \in C^2.$$

Any other couple of generators H', G' of $\mathscr{P}$ is given by a non-singular complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

(II.1.2) $$G' = aG + bH, \quad H' = cG + dH.$$

Consider the linear mapping $C^2 \to \mathscr{P}$ given by (II.1.1). We will not be interested in the image of $(0, 0)$ and in our investigations two transformations Z_1 and Z_2 of $\mathscr{P}$ which differ only by a non-zero scalar multiple will have the same properties and will be considered as equivalent. Accordingly, we can consider $C^2 - (0, 0)$ as divided into equivalence classes of vectors differing by a non-zero scalar multiple (they form the 1-dimensional complex projective space) and these equivalence classes will be transformed (by (II.1.1)) into equivalence classes in $\mathscr{P}$. This set of equivalence classes of $\mathscr{P}$ will be denoted by $\hat{\mathscr{P}}$ — the reduced pencil $\mathscr{P}$. The equivalence classes in $C^2 - (0, 0)$ are in classical 1-1 correspondence with the complex numbers on the Riemann sphere; namely, we put for $(\alpha, \beta) \in C^2 - (0, 0)$, $\zeta = -\beta/\alpha$ if $\alpha \neq 0$ or $\zeta = \infty$ for $\alpha = 0$. We thus obtain a mapping of the Riemann sphere on the reduced pencil $\zeta \to \tau(G - \zeta H)$ if $\zeta \neq \infty$, $\infty \to \tau H$ $(\tau \neq 0)$. In the equivalence class of $\hat{\mathscr{P}}$ corresponding to ζ we pick a representative transformation A_ζ as follows:

(II.1.3) $$A_\zeta = G - \zeta H \quad \text{for } \zeta \neq \infty, \; A_\infty = H.$$

The mapping of the Riemann sphere into $\hat{\mathscr{P}}$ and the transformations A_ζ obviously depend on the choice of the generators H, G in $\mathscr{P}$. If we choose other generators H', G' given by (II.1.2) and denote the corresponding transformation assigned to ζ as A'_ζ, then we have the formulas

(II.1.4) *The equivalence class of $A'_{\zeta'}$ is the same as the one of A_ζ, where*

$$\zeta' = -\frac{b - d\zeta}{a - c\zeta} \quad \text{or} \quad \zeta = \frac{a\zeta' + b}{c\zeta' + d}.$$

Remark II.1.1. There is an unavoidable lack of continuity in the determination of A_ζ for $\zeta \to \infty$. In most of our considerations we will treat only the case $\zeta \neq \infty$ but the results will be valid also for $\zeta = \infty$, due to the fact that we can choose the generators so that to a given equivalence class in $\mathscr{P}$ a finite value of ζ is assigned. In the few cases where a special couple of generators has to be used we will consider explicitly the case $\zeta = \infty$.

If $[V, W, \mathscr{P}]$ is an algebraic spectral system, V is called the *domain* and W the *range space* of the system. For any $V_1 \subset V$, let $R[\mathscr{P}, V_1]$

be the subspace of W generated by all elements Zv with $v \in V_1$ and $Z \in \mathscr{P}$; $R[\mathscr{P}, V]$ is the *range* of the system. The system is exact if $R[\mathscr{P}, V] = W$; it is *proper* if, for every pair G, H of generators of $\mathscr{P}$, $N[G] \cap N[H]$ contains only the zero vector. The system is *singular* if no transformation in $\mathscr{P}$ has an inverse defined on all of W; otherwise the system is *non-singular*.

If V_1, W_1 are subspaces of V, W respectively, and if $R[\mathscr{P}, V_1] \subset W_1$, then by restricting the transformations in $\mathscr{P}$ to V_1 we obtain a pencil, again called $\mathscr{P}$, of transformations of V_1 into W_1. Then $[V_1, W_1, \mathscr{P}]$ is also an algebraic spectral system, and is called an *algebraic subsystem* of $[V, W, \mathscr{P}]$. It is called an *algebraic spectral subsystem* if there exists another subsystem $[V_2, W_2, \mathscr{P}]$ such that $V = V_1 \dotplus V_2$ and $W = W_1 \dotplus W_2$. The two subspaces then determine a spectral decomposition of $[V, W, \mathscr{P}]$, written

$$[V, W, \mathscr{P}] = [V_1, W_1, \mathscr{P}] \dotplus [V_2, W_2, \mathscr{P}].$$

The general spectral problem is the determination and investigation of all spectral decompositions of a given spectral system. Two systems $[V, W, \mathscr{P}]$ and $[V_0, W_0, \mathscr{P}_0]$ are *isomorphic* if there exist isomorphisms S, T of V onto V_0 and W onto W_0 respectively such that $\mathscr{P}_0 = T\mathscr{P}S^{-1}$. Since isomorphic spectral systems have isomorphic subsystems, the spectral problems for isomorphic systems are equivalent.

Remark II.1.2. It might appear to the reader that our notion of a 2-dimensional pencil is nothing more than a 2-dimensional subspace of $L(V, W)$. This would be so if we did not have the degenerate pencils with linearly dependent generators, in which case the transformations of the pencil form a 1-dimensional, or even 0-dimensional subspace of $L(V, W)$. We cannot systematically avoid the consideration of degenerate pencils since even if the generators of a pencil are not linearly dependent, they may very well become so in a subsystem. In a degenerate pencil we can always choose a couple of generators with $H = 0$. In this case $A_\zeta = G$ for all finite ζ and $A_\infty = 0$.

In case V and W are locally convex linear topological spaces we shall consider only *topological spectral systems*; i.e., those algebraic systems $[V, W, \mathscr{P}]$ where $\mathscr{P}$ consists entirely of continuous transformations. Accordingly, $[V_1, W_1, \mathscr{P}]$ is a *topological* subsystem if, in addition to being an algebraic subsystem, V_1 and W_1 are closed subspaces of V and W respectively. We shall then consider only *topological* spectral decompositions

$$[V, W, \mathscr{P}] = [V_1, W_1, \mathscr{P}] \dotplus [V_2, W_2, \mathscr{P}];$$

namely, those where $[V_i, W_i, \mathscr{P}]$, $i = 1, 2$, are topological subsystems and where the projections of V onto V_i and W onto W_i are all continuous.

Two topological spectral systems $[V, W, \mathscr{P}]$ and $[V_0, W_0, \mathscr{P}_0]$ are *isomorphic* if in addition to being isomorphic as algebraic systems the isomorphisms $S\colon V \to V_0$ and $T\colon W \to W_0$ are topological isomorphisms.

Some special topological systems are of importance. A topological spectral system $[V, W, \mathscr{P}]$ is called a *Hilbert system, Banach reflexive system, Banach system*, or *normed system* respectively if V and W are both Hilbert spaces, Banach reflexive spaces, Banach spaces, or normed spaces respectively. It is called a *weak topological system* if both V and W are weak topological spaces; i.e., if V, W have the X-topology, Y-topology respectively, where X, Y are some total subspaces of the algebraic antiduals V', W' respectively of V, W.

Both algebraic systems and topological systems can always be made into weak topological systems. To see how this change can be made in the case of algebraic systems, let $[V, W, \mathscr{P}]$ be an algebraic spectral system and X, Y be any total subspaces of V', W' respectively. If V, W are provided with the X, Y topologies respectively, then $[V, W, \mathscr{P}]$ becomes a weak topological system if and only if each transformation $Z \in \mathscr{P}$ is continuous with respect to these topologies. But a transformation Z of V into W is continuous with respect to the X, Y topologies on V, W respectively if and only if $Z^*(Y) \subset X$, where Z^* is the algebraic adjoint of Z. Thus $[V, W, \mathscr{P}]$ becomes a weak topological system if and only if X, Y are so chosen that $Z^*(Y) \subset X$ for every $Z \in \mathscr{P}$. In the case of an *algebraic* system $[V, W, \mathscr{P}]$, we shall always make it into a weak topological system by choosing $X = V'$, $Y = W'$.

In case $[V, W, \mathscr{P}]$ is a topological system, however, we shall assign to V, W the V^*, W^* topologies respectively, where V^*, W^* are the anti-conjugate spaces with respect to the original topologies (i.e., the spaces of continuous *anti-linear* functionals). Since every linear mapping of V into W which is continuous relative to the original topologies remains continuous relative to the newly assigned weak topologies (see the preceding paragraph), the assignment of the weak topologies makes $[V, W, \mathscr{P}]$ into a (weak) topological system.

Let

$$\text{(II.1.5)} \qquad [V, W, \mathscr{P}] = [V_1, W_1, \mathscr{P}] \dotplus [V_2, W_2, \mathscr{P}]$$

be a spectral decomposition in the original topologies. Subspaces, being convex, are closed in the original topologies if and only if they are closed in the weak topologies, and projections of V or W onto closed subspaces which are continuous in the original topologies remain continuous in the weak topologies. Hence (II.1.5) remains a spectral decomposition for the weak system. However, spectral decompositions for the weak system are not in general also spectral decompositions for the original system, since projections which are continuous in the weak topology need not be continuous in the original topology. If, however, the original system is a Banach system, then its spectral decompositions and those of the corresponding weak system are precisely the same. This conclusion follows from the fact that a transformation of one Banach space into another is continuous if and only if it is continuous relative to the associated weak topologies.

Another case in which it may be convenient to replace one spectral system by another is in the case of incomplete normed systems. If $[V, W, \mathscr{P}]$ is a normed system and if $\tilde{V}$, $\tilde{W}$ are the completions of V, W respectively, then every transformation in $\mathscr{P}$ can be extended by continuity to a transformation of $\tilde{V}$ into $\tilde{W}$. Thus we obtain a pencil of transformations of $\tilde{V}$ into $\tilde{W}$ (again denoted by $\mathscr{P}$), and a Banach system $[\tilde{V}, \tilde{W}, \mathscr{P}]$, the *completion* of $[V, W, \mathscr{P}]$. Topological spectral decompositions

$$[V, W, \mathscr{P}] = [V_1, W_1, \mathscr{P}] \dotplus [V_2, W_2, \mathscr{P}]$$

determine corresponding spectral decompositions

$$[\tilde{V}, \tilde{W}, \mathscr{P}] = [\tilde{V}_1, \tilde{W}_1, \mathscr{P}] \dotplus [\tilde{V}_2, \tilde{W}_2, \mathscr{P}],$$

where $\tilde{V}_1$ is the closure of V_1 in $\tilde{V}$ and $\tilde{V}_2$, $\tilde{W}_1$, $\tilde{W}_2$ are defined in a similar way. All spectral decompositions of $[\tilde{V}, \tilde{W}, \mathscr{P}]$ are not necessarily obtainable in this way, however. Thus, by completing the original normed system the class of spectral decompositions may be enriched.

Hence every system can be replaced by a weak topological system and normed systems can be replaced by their completed systems, though in each case the class of spectral decompositions may be enriched. Our main interest will therefore be in weak topological systems and in Banach systems. For a system $[V, W, \mathscr{P}]$ of either of these types we can also consider the associated dual system $[W^*, V^*, \mathscr{P}^*]$, where V^*, W^* are the anti-conjugate spaces to V, W respectively and

$$\mathscr{P}^* = \{Z^*\colon Z \in \mathscr{P}\}.$$

If $[V, W, \mathscr{P}]$ is a Banach system, V^* and W^* are given the canonical topologies determined by the norms of V, W respectively. If it is a weak topological system, V^*, W^* are provided with the V, W topologies respectively. For weak topological systems we have the following (see Aronszajn [5]):

THEOREM (The duality theorem). *Let $[V, W, \mathscr{P}]$ be a weak topological system and $[W^*, V^*, \mathscr{P}^*]$ its dual system. Then*

1° *The dual of the system $[W^*, V^*, \mathscr{P}^*]$ is the system $[V, W, \mathscr{P}]$.*

2° *If $[V_1, W_1, \mathscr{P}]$ is a topological subsystem of $[V, W, \mathscr{P}]$, then $[W_1^\perp, V_2^\perp, \mathscr{P}^*]$ is a topological subsystem of $[W^*, V^*, \mathscr{P}^*]$ ($V_1^\perp = \{v^* \in V^*\colon v^*[V_1] = 0\}$ and similarly for $W_1^\perp$).*

3° *Corresponding to each spectral decomposition*

$$[V, W, \mathscr{P}] = [V_1, W_1, \mathscr{P}] \dotplus [V_2, W_2, \mathscr{P}]$$

there is a dual spectral decomposition

$$[W^*, V^*, \mathscr{P}^*] = [W_2^\perp, V_2^\perp, \mathscr{P}^*] \dotplus [W_1^\perp, V_1^\perp, \mathscr{P}^*].$$

Further, there is a canonical isomorphism of the spectral subsystems $[W_2^\perp, V_2^\perp, \mathscr{P}^*]$, $[W_1^\perp, V_1^\perp, \mathscr{P}^*]$ onto $[W_1^*, V_1^*, \mathscr{P}^*]$, $[W_2^*, V_2^*, \mathscr{P}^*]$ respectively.

For a Banach system we have the full duality theorem only in case the system is reflexive. For in this case the weak system corresponding to $[W^*, V^*, \mathscr{P}]$ coincides with the dual of the weak system corresponding to $[V, W, \mathscr{P}]$. In the non-reflexive case, the dual of the weak system corresponding to $[V, W, \mathscr{P}]$ has coarser topologies than the weak system corresponding to $[W^*, V^*, \mathscr{P}^*]$. Thus, when the duality theorem is required we shall replace non-reflexive Banach systems by the corresponding weak systems.

The close connection between spectral problems for spectral systems and more classical spectral problems is illustrated by the ordinary spectral systems. An *ordinary spectral system* is a Banach system of the form $[V, V, \mathscr{P}]$ where $\mathscr{P}$ contains the identity I on V. For an ordinary system, if

$$[V, V, \mathscr{P}] = [V_1, W_1, \mathscr{P}] \dotplus [V_2, W_2, \mathscr{P}],$$

then, for each $Z \in \mathscr{P}$, $Z(V_1) \subset W_1$ and $Z(V_2) \subset W_2$. Since $I \in \mathscr{P}$, therefore, $V_1 \subset W_1$ and $V_2 \subset W_2$. But also $V = V_1 \dotplus V_2 = W_1 \dotplus W_2$, so $V_1 = W_1$ and $V_2 = W_2$. Thus the spectral problem for ordinary systems is the determination of all decompositions of V into the direct sum of closed subspaces invariant under each operator in $\mathscr{P}$. If $G \in \mathscr{P}$ is so chosen that G, I generate $\mathscr{P}$, then the system can be written $[V, V, I, G]$, and the representatives for the equivalence classes in $\mathscr{P}$ are $A_\lambda = G - \lambda I$. The spectral problem for the system is then equivalent to the classical spectral problem for G.

Every non-singular Banach system $[V, W, \mathscr{P}]$ is isomorphic to an ordinary system, for at least one transformation $Z \in \mathscr{P}$ has an inverse (necessarily bounded, since V, W are Banach spaces). Thus Z is a topological isomorphism of V onto W and $[V, V, Z^{-1}\mathscr{P}]$ is a Banach system isomorphic to $[V, W, \mathscr{P}]$. Since $I \in G^{-1}\mathscr{P}$, the system is ordinary. For singular systems, however, such a simplification is not possible.

2. Finite spectral systems. $[V, W, \mathscr{P}]$ is a *finite spectral system* if both V and W are finite-dimensional. In this section we shall give a summary of the results of Aronszajn and Fixman [8] concerning finite systems. In the case of finite systems, the distinctions between algebraic and topological spectral systems disappear.

$[V, W, \mathscr{P}]$ is a *minimal system* if it has no non-trivial spectral decompositions. Every minimal system $[V, W, H, G]$ is of one of the following types:

I^n: $\dim V = n$, $\dim W = n-1$ and $\dim N[A_\lambda] = 1$ for all λ (including ∞).

II^n_θ: $\dim V = n$, $\dim W = n$, and $\dim N[A_\lambda] = 0$ for $\lambda \neq \theta$, and $=1$ for $\lambda = \theta$.

III^n: $\dim V = n-1$, $\dim W = n$, and $\dim N[A_\lambda] = 0$ for all λ.

The index n in all three types passes through all positive integers $n = 1, 2, 3, \ldots$ Moreover, a system of one of the above types is minimal. The classification of a minimal spectral system $[V, W, \mathscr{P}]$ is therefore made with respect to a particular pair of generators G, H of $\mathscr{P}$. However, a minimal system of type I^n or III^n with respect to one pair of generators is of the same type with respect to any pair of generators, while a system of type II^n_ζ for one pair of generators is of type $\mathrm{II}^n_{\zeta'}$, if the new generators are given by (II.1.2) and ζ, ζ' are related by (II.1.4). Note also that systems of type I^n or III^n are singular while those of type II^n are non-singular.

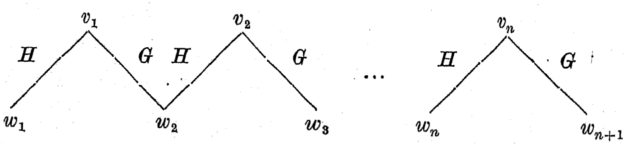
If G, H are linearly dependent; e.g., if $\alpha H = \beta G$, $(\alpha, \beta) \neq (0, 0)$, then a minimal system $[V, W, H, G]$ must be of either type I^1, $\mathrm{II}^1_{\alpha/\beta}$, or III^1.

Every finite system can be decomposed into a direct sum of minimal systems

$$[V, W, H, G] = [V_1, W_1, H, G] + \ldots + [V_r, W_r, H, G].$$

The decomposition is not unique but each such decomposition contains not only the same number r of minimal subspaces but also the same number of each type of minimal subspace.

A second way of classifying minimal systems is by means of chains. A *chain* Γ of length n with respect to the system $[V, W, H, G]$ is a subset $\{v_1, \ldots, v_n\}$ of V such that $Gv_i = Hv_{i+1}$, $i = 1, \ldots, n-1$. If we set $w_i = Hv_i$, $i = 1, \ldots, n$, and $w_{n+1} = Gv_n$, then Γ can be represented schematically as follows:



The space spanned by $v_1, \ldots, v_n$ is called the *domain* $D(\Gamma)$ of Γ; the space spanned by $w_1, \ldots, w_{n+1}$ is the *range* $R(\Gamma)$ of Γ. A chain is *proper* if the elements $w_1, \ldots, w_{n+1}$ are linearly independent except that w_1 or w_{n+1} or both may be zero. It follows that the elements $v_1, \ldots, v_n$ are also linearly independent.

Γ is said to *determine* the system $[V, W, H, G]$ if it is a chain relative to $[V, W, H, G]$ and if $V = D(\Gamma)$, $W = R(\Gamma)$. Then a minimal system $[V, W, H, G]$ is of type

I^n: if and only if there exists a proper chain Γ of length n which determines $[V, W, H, G]$ such that $w_1 = w_{n+1} = 0$.

II^n_θ: if and only if there exists a proper chain of length n which determines $[V, W, H, A_\theta]$, if $\theta \neq \infty$ (or $[V, W, G, H]$, if $\theta = \infty$), such that $w_1 \neq 0$, $w_{n+1} = 0$.

III^n: if and only if there exists a proper chain Γ of length $n-1$ which determines $[V, W, H, G]$ such that $w_1 \neq 0$, $w_n \neq 0$. (For type III^1 we consider Γ to have $D(\Gamma) = \{0\}$ and $R(\Gamma) = [w_1]$.)

The above statements are true whether or not G and H are linearly independent.

A third way of characterizing minimal systems is by means of matrices. (For a classical discussion see Turnbull and Aitken [23].) For if $[V, W, H, G]$ is a minimal system of type I^n, and $\Gamma = \{v_1, \ldots, v_n\}$ is a proper chain which determines it (with $w_1 = w_{n+1} = 0$), then G and H can be represented as matrices with respect to the basis $v_1, \ldots, v_n$ for V and the basis $w_2, \ldots, w_n$ for W. Then, for any λ, the $(n-1)\times n$ matrix representing A_λ with respect to this basis is

$$\begin{pmatrix} 1 & -\lambda & 0 & & & 0 \\ 0 & 1 & -\lambda & & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -\lambda & 0 \\ 0 & & & 0 & 1 & -\lambda \end{pmatrix}.$$

If $[V, W, H, G]$ is minimal of type II^n_θ, $\theta \neq \infty$, and Γ is a proper chain of length n which determines $[V, W, H, A_\theta]$ (with $w_1 \neq 0$, $w_{n+1} = 0$), then, using $v_1, \ldots, v_n$ as a basis for V and noting that $Gv_i = w_{i+1} + \theta w_i$, $i = 1, \ldots, n$, we see that A_λ can be represented by the $n \times n$ matrix

$$\begin{pmatrix} \theta-\lambda & 0 & & & 0 \\ 1 & \theta-\lambda & & & \\ & \ddots & \ddots & \ddots & \\ & & & \theta-\lambda & 0 \\ 0 & & & 1 & \theta-\lambda \end{pmatrix}.$$

If $\theta = \infty$, A_λ is represented by the $n \times n$ matrix

$$\begin{pmatrix} 1 & 0 & & & 0 \\ -\lambda & 1 & & & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & 0 \\ 0 & & & -\lambda & 1 \end{pmatrix}.$$

In similar fashion if $[V, W, H, G]$ is minimal of type III^n, A_λ can be represented by the $n\times(n-1)$ matrix

$$\begin{pmatrix} -\lambda & 0 & & 0 \\ 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \\ 0 & & 0 & 1 \end{pmatrix}.$$

Thus for each minimal system $[V, W, H, G]$, $A_\lambda = G-\lambda H$ can be represented, with respect to a proper choice of basis for V, by a matrix of one and only one of the above canonical forms, thus exhibiting the type of the system.

The close connection of the concepts given here with those of classical spectral theory is again illustrated by ordinary systems. If $[V, V, I, G]$ is a minimal system of type II^n_θ, then there is a proper chain Γ of length n such that

$$w_j = Iv_j = (G-\theta I)v_{j-1} = (G-\theta I)^{j-1}v_1, \quad j = 1, \ldots, n,$$

and $w_{n+1} = (G-\theta I)^n v_1 = 0$. Thus

$$V = D(\Gamma) = [v_1, (G-\theta I)v_1, \ldots, G(-\theta I)^{n-1}v_1],$$

where $(G-\theta I)^n v_1 = 0$. Moreover, the matrix representation for $A_0 = G$ is simply the Jordan canonical matrix representation for G.

3. The quasi-resolvent set $\mathscr{R}$. In this section we discuss the notions of quasi-resolvent set and of isolated eigenvalues for a Banach system $[V, W, H, G]$. Most of the results stated are due to Gokhberg-Krein [14] and to Kato [19].

Recall that the *nullity* $\alpha(A_\lambda)$ is defined to be the dimension of $N[A_\lambda]$, $A_\lambda = G-\lambda H$, and the *deficiency* $\beta(A_\lambda)$ is the codimension of the range $A_\lambda(V)$ in W. When there is no danger of confusion we shall often write $\alpha(\lambda), \beta(\lambda)$ for $\alpha(A_\lambda), \beta(A_\lambda)$ respectively. The *quasi-resolvent set* $\mathscr{R}$ of the Banach system $[V, W, H, G]$ is defined to be the set of all λ on the Riemann sphere such that $R[A_\lambda]$ is closed and either $\alpha(\lambda)$ or $\beta(\lambda)$ (or both) is finite. The complement of $\mathscr{R}$ is called the *essential spectrum* of the system.

$\mathscr{R}$ has the following properties:

1° $\mathscr{R}$ is the disjoint union of a countable number of open connected components $\mathscr{R}_i$.

2° In each component $\mathscr{R}_i$ the index $\gamma(\lambda) = \beta(\lambda)-\alpha(\lambda)$ is constant.

3° In each component $\mathscr{R}_i$ both $\alpha(\lambda)$ and $\beta(\lambda)$ are constant except at a countable number of isolated points $\lambda_{i,k}$, called the *isolated eigenvalues* of the system.

4° About each isolated eigenvalue $\lambda_{i,k}$ there is a neighborhood $N_{i,k}$ with the property that, for all λ in $N_{i,k}$, $\lambda \neq \lambda_{i,k}$,

$$\alpha(\lambda_{i,k}) = \alpha(\lambda)+r_{i,k}, \quad \beta(\lambda_{i,k}) = \beta(\lambda)+r_{i,k},$$

where $r_{i,k}$ is an integer, $0 < r_{i,k} < \infty$.

The pair $(\alpha(\lambda), \beta(\lambda))$ is called the *character of the point* λ. In each component $\mathscr{R}_i$ of $\mathscr{R}$ all points except the isolated eigenvalues have the same character; i.e., $\alpha(\lambda) = \alpha_i$, $\beta(\lambda) = \beta_i$ for all $\lambda \in \mathscr{R}_i$ which are not isolated eigenvalues. The pair (α_i, β_i) is called the *character of the component* $\mathscr{R}_i$. $\mathscr{R}_i$ is said to be of *finite character* if both α_i and β_i are finite; if either α_i or β_i is infinite, $\mathscr{R}_i$ is of *infinite character*. The union of all components of $\mathscr{R}$ of finite character is the *finite part* $\mathscr{R}^f$ of $\mathscr{R}$, while the union of those components of $\mathscr{R}$ of infinite character is the *infinite part* $\mathscr{R}^\infty$ of $\mathscr{R}$. Clearly, $\mathscr{R} = \mathscr{R}^f \cup \mathscr{R}^\infty$.

The union of all components of $\mathscr{R}$ of character $(0, 0)$ is called the *meromorphy domain* of the system, while the set of all λ with character $(0, 0)$ is its *resolvent set*. For an ordinary system $[V, V, I, G]$ these definitions coincide with the classical definitions of meromorphy domain and resolvent set for G. Note that the resolvent set of a Banach system $[V, W, H, G]$ is non-empty if and only if the system is non-singular. For a non-singular system the generators G and H can always be chosen so that H^{-1} exists. Then, for λ in the resolvent set $\mathscr{R}_0$ of $[V, W, H, G]$, $A_\lambda^{-1} = (G-\lambda H)^{-1} = (H^{-1}G-\lambda I)^{-1}H^{-1}$, and it follows from the classical theory that $\mathscr{R}_0$ is open and A_λ^{-1} is analytic in λ for $\lambda \in \mathscr{R}_0$.

The following theorem is due to Gokhberg-Krein [14] and Kato [19]:

THEOREM G.-K.-K. *Let Δ be a component of the quasi-resolvent set $[V, W, H, G]$ with character (α_0, β_0) and let λ be an isolated eigenvalue in Δ. Then $[V, W, H, G]$ admits a (topological) spectral decomposition*

$$[V, W, H, G] = [V_0, W_0, H, G]\dotplus[V_1, W_1, H, G]\dotplus\ldots\dotplus[V_r, W_r, H, G],$$

where:

1° *r is determined by the equations*

$$\alpha_0+r = \alpha(\lambda), \quad \beta_0+r = \beta(\lambda).$$

2° *For each $i = 1, \ldots, r$, $[V_i, W_i, H, G]$ is a finite spectral system of type $\mathrm{II}^{n_i}_\lambda$, where $n_i = \dim V_i$.*

3° *The systems $[V_0, W_0, H, G]$ and $[V, W, H, G]$ have the same quasi-resolvent set and its components have the same character with respect to both systems. Moreover, both systems have the same isolated eigenvalues, except that λ is not an isolated eigenvalue of $[V_0, W_0, H, G]$.*

The finite minimal systems $[V_i, W_i, H, G]$, $i = 1, \ldots, r$, of type $\mathrm{II}^{n_i}_\lambda$ are called *elementary divisors of $[V, W, H, G]$ corresponding to λ*. The finite-dimensional space $V' = V_1\dotplus\ldots\dotplus V_r$ is called a *generalized* *eigenspace for* $[V, W, H, G]$ *corresponding to* λ. In case $\lambda \in \mathscr{R}$ is not an isolated eigenvalue for the system we take $V' = \{0\}$ to be the generalized eigenspace corresponding to λ. Hence, for all $\lambda \in \mathscr{R}$, there exists a spectral decomposition

$$[V, W, H, G] = [V_0, W_0, H, G] \dotplus [V', W', H, G],$$

where V' is a generalized eigenspace corresponding to λ. Moreover, λ is not an isolated eigenvalue for the system $[V_0, W_0, H, G]$, while the system $[V', W', H, G]$ has no isolated eigenvalue except, possibly, λ. If λ is in the meromorphy domain of $[V, W, H, G]$, both V' and W' are uniquely determined. For arbitrary $\lambda \in \mathscr{R}$, $\dim V'$ and $\dim W'$ are unique, though V' and W' may not be.

4. Behavior of $\mathscr{R}$ under a finite-dimensional perturbation. Let $[V, W, H, G]$ and $[V_1, W, H_1, G_1]$ be Banach systems. Then $[V_1, W, H_1, G_1]$ is said to be a *finite dimensional perturbation* of $[V, W, H, G]$, written $[V_1, W, H_1, G_1] \sim [V, W, H, G]$, if V and V_1 are closed subspaces of a common Banach space and if $G_1 \sim G$ and $H_1 \sim H$. (The domain of G and H is V while that of G_1 and H_1 is V_1.) In this case $B_\lambda \sim A_\lambda$, where $B_\lambda = G_1 - \lambda H_1$ and $A_\lambda = G - \lambda H$.

Our major interest in this section is an investigation of the behavior of the quasi-resolvent set when a Banach system $[V, W, H, G]$ is subjected to a finite-dimensional perturbation.

THEOREM II.4.1. *Let* $[V, W, H, G], [V_1, W, H_1, G_1]$ *be Banach systems with quasi-resolvent set* $R = R^f \cup R^\infty$, $R_1 = R_1^f \cup R_1^\infty$. *If* $[V, W, H, G] \sim [V_1, W, H_1, G_1]$, *then* $R_1^f = R^f$ *and* $R_1^\infty = R^\infty$.

The proof of Theorem II.4.1 immediately follows from Theorem I.2.1, Proposition I.3.4, and the assumption (I.3.3).

Even though the quasi-resolvent set R for a Banach system is preserved under finite-dimensional perturbations of the system, a component which has character (α_0, β_0) as a component of $\mathscr{R}$ for $[V, W, H, G]$ will not necessarily have character (α_0, β_0) as a component of $\mathscr{R}$ for the perturbed system $[V_1, W, H_1, G_1]$. In particular, the meromorphy domain of $[V, W, H, G]$ need not be the same as that of $[V_1, W, H_1, G_1]$. The type of behavior which can occur is illustrated by the following example:

Let $V = L^2(0, 2\pi)$ and consider the system $[V, V, I, G]$, where G is defined by $Gx(\varphi) = e^{i\varphi}x(\varphi)$ for $x \in V$. We immediately see that $\|Gx\| = \|x\|$ for all $x \in V$, so that G is a unitary operator in V.

For each complex λ,

$$A_\lambda x(\varphi) = (G - \lambda I)x(\varphi) = (e^{i\varphi} - \lambda)x(\varphi),$$

and A_λ has a bounded inverse if and only if $|\lambda| \neq 1$. The quasi-resolvent set for $[V, V, I, G]$ thus consists of two components:

$$C_0 = \{\lambda\colon\ |\lambda| > 1\} \quad \text{and} \quad C_1 = \{\lambda\colon\ |\lambda| < 1\},$$

and coincides with the meromorphy domain. The essential spectrum is the unit circle $\{\lambda\colon |\lambda| = 1\}$; there are no isolated eigenvalues.

Let a, b, c be any non-zero elements of V and form the orthogonal decomposition

$$V = \mathfrak{D} \oplus [a].$$

Define bounded operators I_1, G_1 by $I_1 = I$, $G_1 = G$ on $\mathfrak{D}$ and $I_1 a = b$, $G_1 a = ce^{i\varphi}$. The system $[V, V, I_1, G_1]$ is a 1-dimensional perturbation of $[V, V, I, G]$, and its quasi-resolvent set is therefore also $C_0 \cup C_1$. We shall see, however, that by proper choice of a, b, c we can insure that C_0, C_1, or $C_1 \cup C_0$ is not in the meromorphy domain of the system $[V, V, I_1, G_1]$.

If $|\lambda| \neq 1$, then A_λ^{-1} exists and so B_λ^{-1} exists if and only if $\det(A_\lambda^{-1}B_\lambda) \neq 0$. From (I.1.9) we see easily (take $F_1(x) = (x, a)$) that

$$\det(A_\lambda^{-1}B_\lambda) = (A_\lambda^{-1}B_\lambda a, a)/(a, a).$$

Straightforward computation then shows that

$$\det(A_\lambda^{-1}B_\lambda) = \frac{1}{(a, a)}\left\{\int_0^{2\pi} \frac{[c(\varphi) - b(\varphi)]\overline{a(\varphi)}}{e^{i\varphi} - \lambda} e^{i\varphi}d\varphi + \int_0^{2\pi} b(\varphi)\overline{a(\varphi)}\,d\varphi\right\}.$$

We make the following choices for a, b, and c:

1° If $a = 1$, $b = 1 + e^{i\varphi}$, and $c = e^{i\varphi}$, then

$$\det(A_\lambda^{-1}B_\lambda) = \frac{1}{2\pi}\left\{\int_0^{2\pi} \frac{-1}{e^{i\varphi} - \lambda} e^{i\varphi}d\varphi + \int_0^{2\pi}(1 + e^{i\varphi})\,d\varphi\right\} = \begin{cases} 0 & \text{for } |\lambda| < 1, \\ 1 & \text{for } |\lambda| > 1. \end{cases}$$

Thus, in this case, C_0 is in the meromorphy domain of the perturbed system, while C_1 is not.

2° If $a = 1$, $b = e^{i\varphi}$, and $c = 1 + e^{i\varphi}$, then

$$\det(A_\lambda^{-1}B_\lambda) = \begin{cases} 1 & \text{for } |\lambda| < 1, \\ 0 & \text{for } |\lambda| > 1, \end{cases}$$

so that C_1 is in the meromorphy domain for $[V, V, I_1, G_1]$ while C_0 is not.

3° If $a = 1$ and $b = c = e^{i\varphi}$, then $\det(A_\lambda^{-1}B_\lambda) = 0$ for all $|\lambda| \neq 1$, so, in this case, the system $[V, V, I_1, G_1]$ has no meromorphy domain.

Thus, if a Banach system is subjected to even a 1-dimensional perturbation, the meromorphy domain may not only be changed but may disappear altogether.

5. Degrees and type of a matrix. In the next section certain facts are needed concerning analytic $m \times m$ matrices $M(\lambda)$; i.e., matrices whose components are analytic functions of λ in a neighborhood of some point λ_0 of the complex plane. These facts follow from known theorems concerning matrices whose components are elements of an integral domain which is at the same time a principal ideal domain.

Let R be the ring of germs of functions analytic at λ_0 (i.e., the ring of functions analytic at λ_0, where two functions are identified whenever they are equal on some neighborhood of λ_0). Then R is clearly a (commutative) ring with identity, and the units of R are precisely those elements f of R such that $f(\lambda_0) \neq 0$. Thus every element f in R can be written in the form

$$f(z) = z^n g(z),$$

where $n \geqslant 0$ is an integer and g is a unit. It easily follows that the product fg of two elements in R is zero if and only if at least one of the factors f, g is zero; thus R is an integral domain. It also follows that every ideal in R is generated by the element $z^n \in R$ for some integer $n = 0, 1, \ldots$; thus R is a principal ideal domain.

The following theorem [12] thus applies to R:

THEOREM (Invariant Factor Theorem for Matrices). *Let R be a ring with identity which is at the same time an integral domain and a principal ideal domain, and R_m be the ring of $m \times m$ matrices with components in R. If $M \in R_m$, then there exist non-zero elements $\delta_1, \ldots, \delta_r \in R$, unique up to unit factors, such that, for $i = 1, \ldots, r-1 < m$, $\delta_i | \delta_{i+1}$, and unimodular matrices $X_1, X_2 \in R_m$ such that $A = X_1 D X_2$, where $D \in R_m$ is the diagonal matrix $\{\delta_1, \ldots, \delta_r, 0, \ldots, 0\}$.*

The elements $\delta_1, \ldots, \delta_r$ in the above theorem are called the *invariant factors* of the matrix M. The proof of the theorem shows they may be obtained in either of two ways:

1. The elementary matrix operations of interchanging two rows (or columns) of a matrix, of adding to a row (or column) another row (or column) multiplied by an element of R, and of multiplying or dividing a row (or column) by a unit can be effected by multiplication on the right or left by unimodular matrices. By successive application of such elementary matrix operations, M can be transformed into the diagonal matrix D, which gives $\delta_1, \ldots, \delta_r$ directly.

2. The *i-th determinental divisor* $d_i(M)$ *of* M, defined to be the greatest common divisor of all the $i \times i$ minors of the matrix M, remains invariant (up to unit factors) if M is multiplied on the right or left by a unimodular matrix in R_m, $i = 1, \ldots, m$. Thus the determinental divisors of M are (up to unit factors) the same as those of D. Since clearly

$$\delta_1 = d_1(D), \quad \delta_i = d_i(D) d_{i-1}(D)^{-1}, \quad i = 2, \ldots, r,$$

the invariant factors can be computed directly from the determinental divisors of M:

(II.5.1) $\delta_1 = \varepsilon_1 d_1(M), \quad \delta_i = \varepsilon_i d_i(M) d_{i-1}(M)^{-1}, \quad i = 2, \ldots, r,$

where $\varepsilon_1, \ldots, \varepsilon_r$ are units. Note also that $d_i(M) = 0$ for $i = r+1, \ldots, m$.

If R is the ring of germs of analytic functions at the point λ_0 in the complex plane, then a matrix $X \in R_m$ is unimodular if and only if $\det X(\lambda_0) \neq 0$. Also, if $M \in R_m$ and $\delta_1, \ldots, \delta_r$ are its invariant factors, then for each $i = 1, \ldots, r$ there exists a *unique* non-negative integer ν_i such that $\delta_i(\lambda) = \varepsilon_i(\lambda)\lambda^{\nu_i}$, where ε_i is a unit. Since $\delta_i | \delta_{i+1}$ for $i = 2, \ldots, r$, clearly

$$0 \leqslant \nu_1 \leqslant \ldots \leqslant \nu_r.$$

For $i = r+1, \ldots, m$, let $\nu_i = \infty$, and let ν_{k+1} be the first element of the sequence $\{\nu_1, \ldots, \nu_m\}$ which is > 0. Then the sequence $\{\nu_{k+1}, \ldots, \nu_m\}$ will be called the *type* of the $m \times m$ matrix $M(\lambda)$ at λ_0, and the numbers $\nu_{k+1}, \ldots, \nu_m$ will be called the *type exponents* of $M(\lambda)$ at λ_0. There are at most m type exponents in an $m \times m$ matrix. If $\nu_i = 0$ for all $i = 1, \ldots, m$, then $M(\lambda)$ is said to be *without type exponents*.

It follows from the Invariant Factor Theorem that the type of $M(\lambda)$ at λ_0 is not changed if $M(\lambda)$ is multiplied by a matrix $X \in R_m$ with $\det X(\lambda_0) \neq 0$, and that there exist $X_1, X_2 \in R_m$ with $\det X_1(\lambda_0) \neq 0$, $\det X_2(\lambda_0) \neq 0$ such that $D = X_1 M X_2$, where $D(\lambda)$ is the diagonal matrix $\{\lambda^{\nu_1}, \ldots, \lambda^{\nu_r}, 0, \ldots, 0\}$. In fact $M(\lambda)$ can be transformed into $D(\lambda)$ by successive applications of the elementary matrix transformations defined above, thus giving the type directly.

Let $d_i(M)$, $i = 1, \ldots, m$, be the determinental divisors of M. For $i = 1, \ldots, r$, $d_i(M) \neq 0$, so there exists a unique non-negative integer $d_i(\lambda_0)$ such that $d_i(M)(\lambda) = \varepsilon_i'(\lambda)\lambda^{d_i(\lambda_0)}$, with ε_i' a unit. If $d_i(M)(\lambda) \equiv 0$ (as is the case for $i = r+1, \ldots, m$), then by convention $d_i(\lambda_0) = \infty$. $d_i(\lambda_0)$ is called the *i-th degree of the matrix* $M(\lambda)$ *at* λ_0. Clearly

$$0 \leqslant d_1(\lambda_0) \leqslant \ldots \leqslant d_m(\lambda_0) \leqslant \infty.$$

The type exponents of $M(\lambda)$ can be computed directly using the degrees, for by (II.5.1)

$$\nu_1 = d_1(\lambda_0), \quad \nu_i = d_i(\lambda_0) - d_{i-1}(\lambda_0), \quad i = 1, \ldots, r,$$

while for $i = r+1, \ldots, m$, $\nu_i = d_i(\lambda_0) = \infty$. Conversely, the type exponents $\nu_{k+1}, \ldots, \nu_m$ completely determine the degrees $d_i(\lambda_0)$. For obviously they completely determine all the numbers ν_i (the ν_i with the first k indices are all zero), and

$$d_i(\lambda_0) = \begin{cases} \sum_{j=1}^{i} \nu_j & \text{for } i < r, \\ \infty & \text{for } i \geqslant r. \end{cases} \tag{II.5.2}$$

Note that $M(\lambda_0)$ is invertible if and only if $d_i(\lambda_0) = 0$, $i = 1, \ldots, m$; i.e., if and only if $M(\lambda)$ is without type exponents at λ_0.

PROPOSITION II.5.1. *Let $T(\lambda)$ be an operator-valued analytic function for λ in a neighborhood of λ_0, $T(\lambda)$ being bounded linear operators of V into V. Suppose that $T(\lambda)\sim I$ with a corresponding decomposition $V=\mathfrak{D}\dotplus[a_1,\ldots,a_m]$ independent of λ, and with the $m\times m$ matrix representation $M(\lambda)$. Let $V=\mathfrak{D}'\dotplus[a_1',\ldots,a_{m'}']$ be another decomposition for $T(\lambda)\sim I$, also independent of λ and with a corresponding $m'\times m'$ matrix $M'(\lambda)$. Then the types at λ_0 of $M(\lambda)$ and $M'(\lambda)$ are the same.*

Proof. Formula (I.1.8) assures us that $M(\lambda)$ and $M'(\lambda)$ are analytic matrices (since the functionals F_l can be taken independent of λ). Theorem I.1.2 and the last stages of its proof show that we can construct a decomposition $V=\mathfrak{D}''\dotplus C$ for $T(\lambda)\sim I$ depending only on the two given decompositions — hence independent of λ — such that $\mathfrak{D}''\subset\mathfrak{D}\cap\mathfrak{D}'$ and

$$C=[a_1,\ldots,a_m,d_1,\ldots,d_p]=[a_1',\ldots,a_{m'}',d_1',\ldots,d_{p'}'].$$

with $d_i\in\mathfrak{D}$ and $d_i'\in\mathfrak{D}'$. The corresponding matrix $M''(\lambda)$, formed for the first basis of C is then

$$M''(\lambda)=\begin{pmatrix}M(\lambda) & 0\\ M_1(\lambda) & I\end{pmatrix}=S^{-1}\begin{pmatrix}M'(\lambda) & 0\\ M_1'(\lambda) & I\end{pmatrix}S,$$

where the matrices $M_1(\lambda)$ and $M_1'(\lambda)$ are $p\times m$ and $p'\times m'$ respectively and S is an invertible $(m+p)\times(m+p)$ matrix with constant coefficients. Since a matrix of the form $\begin{pmatrix}R(\lambda) & 0\\ R_1(\lambda) & I\end{pmatrix}$ can obviously be transformed into $\begin{pmatrix}I & 0\\ 0 & R(\lambda)\end{pmatrix}$ by multiplication with analytic $(m+p)\times(m+p)$ matrices invertible at λ_0, the type of such a matrix is the same as that of $R(\lambda)$. Hence the types of the three matrices $M''(\lambda)$, $M(\lambda)$ and $M'(\lambda)$ are the same.

Remark II.5.1. The hypothesis in the preceeding proposition that the decomposition corresponding to $T(\lambda)\sim I$ is independent of λ will be satisfied in all cases to which we apply this proposition in the next section. However, this hypothesis can be considerably weakened. It is enough to assume that the decomposition depends analytically on λ near λ_0: i.e., that $V=\mathfrak{D}(\lambda)\dotplus[a_1(\lambda),\ldots,a_m(\lambda)]$, where $a_k(\lambda)$ are vector-valued analytic functions and $\mathfrak{D}(\lambda)=\{u\in V\colon F_l(u;\lambda)=0,\ l=1,\ldots,m\}$, where F_l are linearly independent bounded linear functionals depending analytically on λ. The existence of such decompositions for $T(\lambda)\sim I$ can be easily proved, and the proposition can be extended to prove that for all such decompositions the corresponding matrices have the same type.

6. Determination of the isolated elementary divisors and of the character of a perturbed system. Special cases. Let (I): $[V,W,H,G]$ and (II): $[V_1,W,H_1,G_1]$ be Banach systems and suppose (I) $\sim$ (II).

As usual, set $A_\lambda=G-\lambda H$, $B_\lambda=G_1-\lambda H_1$. The quasi-resolvent sets for (I) and (II) are the same, but in each component Δ of the quasi-resolvent set the characters of (I) and (II) may differ, and also the isolated eigenvalues and the corresponding elementary divisors might be completely different. Our aim in this and the next sections will be to establish methods for getting all desired information about (II) (such as the character in a component Δ, the isolated eigenvalues, the number and orders of the corresponding elementary divisors) assuming that we know all that is needed about system (I).

In this section we will give two theorems concerning special cases and in the next will apply them for the consideration of the general case.

Note first that for such an investigation no generality is lost by assuming $V_1=V$. For in any case $V_1=SV$, where S is a topological isomorphism, so that the system $[V,W,H_1S,G_1S]$ is a Banach system isomorphic to (II). We therefore assume from now until the end of the next section that $V_1=V$.

Remark II.6.1. It should be stressed, however, that for other investigations this kind of simplification may not be feasible, since it may change the nature of the operators and many desirable features of the original setting may be lost. See, for instance, the developments of Chapter III.

The next proposition will lead to another simplifying assumption.

PROPOSITION II.6.1. *Suppose that the system $[V,W,H,G]$ is such that there is no essential spectrum. Then either the character of the system on the whole Riemann sphere is infinite or the system is finite dimensional.*

Proof. Suppose that the character of the system on the whole Riemann sphere is finite. In particular for $\lambda=\infty$, when $A_\lambda=H$, we will have $\alpha(H)=p<\infty$ and $\beta(H)=q<\infty$. Consider the null space N of H and the direct decomposition into closed subspaces $V=V_1\dotplus N$, $W=H(V)\dotplus W_1$, $\dim N=p$ and $\dim W_1=q$. Further, let V_2 and W_2 be two spaces with $\dim V_2=q$ and $\dim W_2=p$.

We define now a new system $[\tilde V,\tilde W,\tilde H,\tilde G]$ as follows:

$$\tilde V=V\dotplus V_2=V_1\dotplus N\dotplus V_2,\qquad \tilde W=W\dotplus W_2=H(V)\dotplus W_1\dotplus W_2,$$

$$\tilde H=\begin{cases}H & \text{on } V_1,\\ S & \text{on } N,\\ T & \text{on } V_2,\end{cases}$$

where S and T are isomorphisms of N and V_2 onto W_2 and W_1 respectively;

$$\tilde G=\begin{cases}G & \text{on } V,\\ 0 & \text{on } V_2.\end{cases}$$

It is clear that the new system is obtained from the old by a finite-dimensional extension followed by a finite-dimensional perturbation. Therefore the quasi-resolvent set of the new system is the same as of the old; i.e., the whole Riemann sphere. However, now $\alpha(\tilde{H}) = \beta(\tilde{H}) = 0$. Hence the system $[\tilde{V}, \tilde{W}, \tilde{H}, \tilde{G}]$ is isomorphic to the ordinary system $[\tilde{V}, \tilde{V}, I, \tilde{H}^{-1}\tilde{G}]$. Since, by a classical theorem, an ordinary system in an infinite-dimensional Banach space must have an essential spectrum, it follows that $\tilde{V}$ and $\tilde{W}$ are finite-dimensional, hence also V and W, which proves our proposition.

Remark II.6.2. It is quite clear that for a finite-dimensional system the quasi-resolvent set is always the whole Riemann sphere. The quasi-resolvent may actually be the whole Riemann sphere for an infinite-dimensional system, but then the character must be infinite — either (p, ∞) or (∞, q). The first case can be illustrated by taking V and W as sequential Hilbert spaces and putting $H\{\xi_k\} = \{\eta_k\}$, where $\eta_{2k-1} = \xi_k$ and $\eta_{2k} = 0$ for $k = 1, 2, \ldots$; $G\{\xi_k\} = \{\zeta_k\}$, where $\zeta_{2k-1} = 0$ and $\zeta_{2k} = \xi_k$ for $k = 1, 2, \ldots$ Here the character is $(0, \infty)$. To illustrate the second case take $H\{\xi_k\} = \{\xi_{2k-1}\}$ and $G\{\xi_k\} = \{\xi_{2k}\}$. Here the character is $(\infty, 0)$.

Since our main interest will be in the finite part of the quasi-resolvent set for infinite-dimensional systems, we may assume, following Proposition II.6.1, that the essential spectrum is not empty. In consequence, we can always choose a basis H, G for the pencil of our system so that $\lambda = \infty$ be in the essential spectrum, i.e., so that ∞ does not belong to the quasi-resolvent set.

From now on, until the last remark of this section, we will assume that $\lambda = \infty$ is not in the quasi-resolvent set of $[V, W, H, G]$. This assumption is obviously preserved by finite-dimensional perturbations.

THEOREM II.6.1. *Let* (I): $[V, W, H, G] \sim$ (II): $[V, W, H_1, G_1]$ *and let*

$$V = \mathfrak{D} + [a_1, \ldots, a_m] \tag{II.6.1}$$

be a decomposition (independent of λ) for $A_\lambda \sim B_\lambda$, where $A_\lambda = G - \lambda H$ and $B_\lambda = G_1 - \lambda H_1$. Let Δ be a component of the meromorphy domain for (I) *and $\lambda_0 \in \Delta$ be an element of the resolvent set for* (I). *Let, further,*

$$[V, W, H_1, G_1] = [V_0, W_0, H_1, G_1] + \sum_{k=1}^{r} \circ [V_k, W_k, H_1, G_1] \tag{II.6.2}$$

be a spectral decomposition corresponding to λ_0, existing by Theorem G.-K.-K., § 3, where the elementary divisors $[V_k, W_k, H_1, G_1]$ are arranged so that their orders n_k form a non-decreasing sequence. Then 1° *the character of Δ for system* (II) *is (n_0, n_0) for some $n_0 \geqslant 0$ satisfying the condition $n_0 + r \leqslant m$.* 2° *The sequence*

$$n_1 \leqslant n_2 \leqslant \ldots \leqslant n_r < \underbrace{\infty = \infty \ldots = \infty}_{n_0 \text{ times}} \tag{II.6.3}$$

gives the type of the matrix $M(\lambda)$ representing $A_\lambda^{-1} B_\lambda \sim I$ with respect to the decomposition (II.6.1).

Proof. Since λ_0 is an element of the resolvent set of (I), then for λ near λ_0, $A_\lambda^{-1} B_\lambda$ is a well defined perturbation of the identity, and $M(\lambda)$ is an $m \times m$ matrix analytic in λ. Thus its type is well defined. Also Δ is a component of the meromorphy domain of (I), so by Theorem II.4.1 it is a component of the finite part of the quasi-resolvent set of (II), and by Theorem I.2.1

$$\alpha(B_\lambda) - \beta(B_\lambda) = 0 \quad \text{for } \lambda \in \Delta.$$

Hence the character of Δ for system (II) can be denoted by (n_0, n_0), $n_0 \geqslant 0$.

Since $\alpha(B_{\lambda_0}) = n_0 + r$ and $\alpha(A_{\lambda_0}) = 0$ and the perturbation $A_{\lambda_0} \sim B_{\lambda_0}$ is, by (II.6.1), at most of dimension m, we get by Theorem I.2.1 that $n_0 + r \leqslant m$.

We may assume without loss of generality that $\lambda_0 = 0$; hence $A_{\lambda_0} = G$ and $B_{\lambda_0} = G_1$.

By Theorem G.-K.-K. the nullity and deficiency of B_λ for $\lambda = 0$ in the subsystem $[V_0, W_0, H_1, G_1]$ of the spectral decomposition (II.6.2) must be equal to n_0. Therefore, denoting the null space of G_1 in V_0 by N_0, we can find direct decompositions into closed subspaces $V_0 = V_0' + N_0$, $W_0 = G_1(V_0') + W_0'$, where $\dim N_0 = \dim W_0' = n_0$. Hence there exists a 1-1 linear mapping G_1' of N_0 onto W_0' and we may define a new linear transformation $\hat{G}_1$ of V into W by putting

$$\hat{G}_1 = \begin{cases} G_1 & \text{on } V_0', \\ G_1' & \text{on } N_0, \\ G_1 - \varepsilon H_1 & \text{on } V' = \sum_1^r \circ V_k, \end{cases}$$

where $\varepsilon \neq 0$. The system $(\hat{\text{II}})$: $[V, W, H_1, \hat{G}_1]$ is then a finite-dimensional perturbation of system (II) and hence also of (I). Furthermore, it is clear that $\lambda = 0$ is in the resolvent set of $(\hat{\text{II}})$. Put $\hat{B}_\lambda = \hat{G}_1 - \lambda H_1$. It follows that there is a common decomposition corresponding to the perturbations (I) $\sim$ (II) $\sim$ $(\hat{\text{II}})$ and hence a common decomposition for the perturbations $A_\lambda^{-1} B_\lambda \sim A_\lambda^{-1} \hat{B}_\lambda \sim \hat{B}_\lambda^{-1} B_\lambda \sim I$. By denoting the corresponding matrix representations $M'(\lambda)$, $P'(\lambda)$ and $N'(\lambda)$ we have $M'(\lambda) = P'(\lambda) N'(\lambda)$.

Since $P'(\lambda)$ is analytic and invertible at $\lambda = 0$, the type of $M'(\lambda)$ is the same as the type of $N'(\lambda)$ (the Invariant Factor Theorem). By Proposition II.5.1 the type of $N'(\lambda)$ is the same as the type of $M(\lambda)$ at $\lambda = 0$. It remains therefore to determine the type of $N'(\lambda)$ to prove our theorem. We do this by choosing a suitable decomposition for the perturbation, $(\hat{\mathrm{II}}) \sim (\mathrm{II})$ — hence also for the perturbation $\hat{B}_\lambda^{-1}B_\lambda \sim I$ — such that the type of the corresponding matrix $N(\lambda)$ at $\lambda = 0$ can be readily determined; again the type of $N(\lambda)$ is the same as that of $N'(\lambda)$.

Let $v_{0,1}, \ldots, v_{0,n_0}$ be a basis for N_0 and $\{v_{k,i}\}_{i=1,\ldots,n_k}$ be the chain determining $[V_k, W_k, H_1, G_1]$ as an elementary divisor of type II^{n_k}. Then

$$V = V_0' \dotplus [v_{0,1}, \ldots, v_{0,n_0} v_{1,1}, \ldots, v_{1,n_1}, \ldots, v_{r,1}, \ldots, v_{r,n_r}]$$

is the desired decomposition.

It is immediately seen that the resulting matrix representation $N(\lambda)$ for the perturbation $\hat{B}_\lambda^{-1}B_\lambda \sim I$ decomposes in the form

$$N(\lambda) = \begin{pmatrix} N_0(\lambda) & & & 0 \\ & N_1(\lambda) & & \\ & & \ddots & \\ 0 & & & N_r(\lambda) \end{pmatrix},$$

where $N_k(\lambda)$ is an $n_k \times n_k$ matrix representing the perturbation $\hat{B}_\lambda^{-1}B_\lambda \sim I$ on the subspaces $\eta_0, V_1, \ldots, V_r$ respectively.

For $k > 0$, $\hat{B}_\lambda^{-1}B_\lambda$ on V_k is the product of two operators B_λ: $V_k \to W_k$ and $\hat{B}_\lambda^{-1}$: $W_k \to V_k$. By choosing on W_k the basis $w_{ki} = H_1 v_{ki}$ we get the corresponding matrix $N_k(\lambda) = \hat{L}_k^{-1}(\lambda)L_k(\lambda)$, where $\hat{L}_k(\lambda)$ is invertible at $\lambda = 0$. Hence the type of $N_k(\lambda)$ at 0 is the same as that of $L_k(\lambda)$. But $L_k(\lambda)$ is clearly the matrix

$$\begin{pmatrix} -\lambda & 0 & & & 0 \\ 1 & -\lambda & & & \\ & \ddots & \ddots & \ddots & \\ & & & -\lambda & 0 \\ 0 & & & 1 & -\lambda \end{pmatrix}.$$

For this matrix an elementary computation gives the type as composed of a single type-exponent n_k. For $k = 0$, however, the matrix $N_0(\lambda)$ is the matrix representation of the operator $\hat{B}_\lambda^{-1}B_\lambda \sim I$, restricted to the subspace V_0, for the decomposition $V_0 = V_0' \dotplus [v_{0,1} \ldots v_{0,n_0}]$. Since B_λ on V_0 has exactly an n_0-dimensional null space for λ near zero, the same is true of the $n_0 \times n_0$ matrix $N_0(\lambda)$ (see Theorem I.1.1). This means that the type of the matrix $N(\lambda)$ is exactly the sequence II.6.3, which finishes the proof of the theorem.

Before passing to the next theorem we will define the notion of index for meromorphic functions. If $f(\lambda)$ is meromorphic in a neighborhood of λ_0 there exists a well-determined integer N so that $f(\lambda) = (\lambda-\lambda_0)^N g(\lambda)$, where $g(\lambda)$ is regular and non-vanishing in a neighborhood of λ_0. We call N the *index* of f at λ_0. If $f(\lambda)$ is regular at λ_0, then the index is non-negative. If λ_0 is a pole of $f(\lambda)$, then the index is negative, and its absolute value is the order of the pole. Obviously, the index of a product of two meromorphic functions is the sum of their indices.

THEOREM II.6.2. *Assume all the hypotheses of Theorem* II.6.1 *except that Δ is now also supposed to be in the meromorphy domain of* (II), *but λ_0 is now an arbitrary element of Δ. Consider the determinant of $M(\lambda)$, where $M(\lambda)$ is the matrix representation of $A_\lambda^{-1}B_\lambda \sim I$ with respect to the decomposition* (II.6.1). *Then* $\det M(\lambda)$ *is a meromorphic function in Δ and*

(II.6.4) *index of* $\det M(\lambda)$ *at* $\lambda_0 =$ *difference of the dimensions of the λ_0-eigenspaces of* (II) *and of* (I).

Proof. Since B_λ is an operator valued function holomorphic in Δ and A_λ^{-1} is meromorphic in the same domain, their product $A_\lambda^{-1}B_\lambda$ is a meromorphic, operator valued function; hence $M(\lambda)$ is a meromorphic, matrix-valued function and $\det M(\lambda)$ is a meromorphic function.

Now let

$$[V, W, H, G] = [V_0'', W_0'', H, G] \dotplus \sum_{k=1}^{r'} \cdot [V_k'', W_k'', H, G]$$

be the spectral decomposition corresponding to the system (I) at λ_0, where $[V_k'', W_k'', H, G]$ is an elementary divisor with eigenvalue λ_0 and order n_k', $k = 1, \ldots, r'$, and the system $[V_0'', W_0'', H, G]$ has no eigenvalue at λ_0. The eigenspace of (I) at λ_0 is $V'' = \sum_{k=1}^{r'} \cdot V_k''$ and its dimension is

$$\dim V'' = \sum_{k=1}^{r'} n_k'.$$

Similarly, following (II.6.2), the eigenspace of system (II) at λ_0 is $V' = \sum_{k=1}^{r} \cdot V_k$ and its dimension is

$$\dim V' = \sum_{k=1}^{r} n_k.$$

We now define the linear transformation $\hat{G}$ as follows:

$$\hat{G} = \begin{cases} G & \text{on } V_0'', \\ G - \varepsilon H & \text{on } V'', \end{cases}$$

where $\varepsilon \neq 0$. It follows that the system $(\hat{\mathrm{I}})$: $[V, W, H, \hat{G}]$ is a finite-dimensional perturbation of (I) for which λ_0 is in the resolvent set. Put

$\hat{A}_\lambda = \hat{G} - \lambda H$. We can choose a common decomposition for the perturbation $(\hat{\mathrm{I}}) \sim (\mathrm{I}) \sim (\mathrm{II})$ and in this common decomposition calculate the matrix representations $\hat{M}(\lambda)$, $\hat{N}(\lambda)$ and $\hat{P}(\lambda)$ corresponding to the perturbations $A_\lambda^{-1}B_\lambda \sim I$, $\hat{A}_\lambda^{-1}A_\lambda \sim I$ and $\hat{A}_\lambda^{-1}B_\lambda \sim I$ respectively. The same argument used for $M(\lambda)$ gives that $\hat{M}(\lambda)$, $\hat{N}(\lambda)$ and $\hat{P}(\lambda)$ are matrix-valued meromorphic functions in Δ. Furthermore, since $A_\lambda^{-1}\hat{A}_\lambda = (\hat{A}_\lambda^{-1}A_\lambda)^{-1}$ and $A_\lambda^{-1}B_\lambda = (\hat{A}_\lambda^{-1}A_\lambda)^{-1}(\hat{A}_\lambda^{-1}B_\lambda)$, we have by Theorem I.1.3 that $\hat{N}(\lambda)^{-1}$ is the matrix representation corresponding to $A_\lambda^{-1}\hat{A}_\lambda \sim I$ and that

$$\hat{M}(\lambda) = \hat{N}(\lambda)^{-1}\hat{P}(\lambda).$$

It follows that

$$\operatorname{index}_{\lambda_0}\det\hat{M}(\lambda) = \operatorname{index}_{\lambda_0}\det\hat{P}(\lambda) - \operatorname{index}_{\lambda_0}\det\hat{N}(\lambda).$$

Since $\hat{P}(\lambda)$ and $\hat{N}(\lambda)$ are regular at λ_0 and without infinite type-exponents, formula (II.5.2) gives immediately that their indices are the sums of their type-exponents. By the preceding theorem, applied first to the systems $(\hat{\mathrm{I}})$ and (II) and second to $(\hat{\mathrm{I}})$ and (I), we get

$$\operatorname{index}_{\lambda_0}\det\hat{P}(\lambda) = \sum_{k=1}^{r} n_k = \dim V', \quad \operatorname{index}_{\lambda_0}\det\hat{N}(\lambda) = \sum_{k=1}^{r'} n'_k = \dim V''.$$

Since, by Corollary I.1.2′, $\det\hat{M}(\lambda) = \det M(\lambda)$, our theorem is proved.

Remark II.6.3. The two preceding theorems allow us to determine the isolated eigenvalues of system (II) in the following cases: 1° When Δ is a component of the meromorphy domain of (I), Theorem II.6.1 allows us to determine all the isolated eigenvalues of (II) which are not also eigenvalues of (I), and the number and orders of the corresponding elementary divisors for such eigenvalues; this is done by establishing the type of the matrix $M(\lambda)$ for all such λ's.

2° If Δ is at the same time in the meromorphy domain of (II), Theorem II.6.2 allows us to determine all the isolated eigenvalues of (II) in Δ (not excepting those which coincide with eigenvalues of (I)) and the total dimension of the corresponding eigenspace of (II) by using the index of $\det M(\lambda)$. We should mention that in this last case the sole consideration of $\det M(\lambda)$ cannot provide any further information and if we want to know in detail the number and the orders of elementary divisors we have to revert to the consideration of the matrix $M(\lambda)$. Only when we are assured beforehand that all the elementary divisors are simple (i.e. of order 1) will Theorem II.6.2 give us completely the structure of the elementary divisors for a given eigenvalue (this is the case, for instance, with self-adjoint systems).

In order to perform the required calculations explicitly in the above mentioned cases, it is clear that we must be able to determine the matrix $M(\lambda)$ explicitly, which in turn requires a knowledge of the elements $F_l(A_\lambda^{-1}B_\lambda a_k)$ of the matrix $M(\lambda)$ as functions of $\lambda \in \Delta$.

Remark II.6.4. The simplifying restriction introduced at the beginning of this section, that ∞ is in the essential spectrum, can now be removed. If a component Δ of the quasi-resolvent set contains ∞, we change the basis H, G into H_1, G_1, which transforms Δ into Δ_1 by the corresponding Moebius transformation, so that Δ_1 lies in the finite plane. Applying our theorems to systems with the new basis and using the inverse Moebius transformation, we get the theorems in the same form in the original basis. Obviously we understand here in the usual sense the notion of meromorphic function (or matrix-valued function) in a domain containing ∞.

7. Determination of the isolated elementary divisors and of the character of a perturbed system. General case. We consider again two systems (I): $[V, W, H, G]$ and (II): $[V, W, H_1, G_1]$ such that (I) $\sim$ (II). Let

(II.7.1) $$V = \mathfrak{D} \dotplus [a_1, \ldots, a_m]$$

be a decomposition corresponding to the perturbation (I) $\sim$ (II). We consider a component Δ of the quasi-resolvent set and aim at obtaining all the information about system (II) in Δ concerning isolated eigenvalues and elementary divisors.

The simplest case is the one treated in the preceding section by Theorems II.6.1 and II.6.2; namely,

Case A. Δ is at the same time in the meromorphy domain for (I) and (II).

We compute the determinant $\det M(\lambda)$, establish the sequence of its zeros in Δ, adjoin to it the sequence of all eigenvalues of (I), and for the so-established sequence $\{\zeta_k\} \subset \Delta$ we calculate the indices $\operatorname{index}_{\zeta_k}\det M(\lambda)$. The eigenvalues of (II) are among the elements of $\{\zeta_k\}$. To know which ones are actually eigenvalues for (II) we consider the eigenspaces V'_{ζ_k} of (I) at ζ_k (when ζ_k is not an eigenvalue of (I) we have $V'_{\zeta_k} = (0)$). Then by (II.6.4) the dimension of the eigenspace at ζ_k for (II) is $\operatorname{index}_{\zeta_k}\det M(\lambda) + \dim V'_{\zeta_k}$. This sum is always non-negative, and ζ_k is an eigenvalue for (II) if and only if the sum is positive.

This procedure, however, does not allow one to determine the number and orders of the elementary divisors. For this purpose we have to use Theorem II.6.1. This theorem allows us to compute the numbers and orders of elementary divisors for eigenvalues ζ_k of (II) *which are not eigenvalues for* (I). For such an eigenvalue the number of elementary divisors in the present case is exactly the dimension of the null-space of $M(\zeta_k)$. To calculate the orders of the elementary divisors one must calculate the type-exponents of the matrix $M(\lambda)$ at ζ_k. We have still to establish a method for calculating the number and orders of elementary divisors of (II) for an eigenvalue ζ_k which is at the same time an eigenvalue of (I). The procedure proposed here will work in the more general case when Δ is not in meromorphy domain of (II) and will be presented in the next case.

Case B. Δ is in the meromorphy domain of (I) (but not necessarily of (II)).

Theorem II.6.1 here allows us to find all the eigenvalues of (II) in Δ which are not eigenvalues of (I), as well as the number and the orders of the corresponding elementary divisors and the type of (II) in Δ.

Now let $\zeta\in\Delta$ be an eigenvalue of (I). Consider the spectral decomposition of (I) corresponding to ζ:

$$\text{(II.7.2)}\qquad [V, W, H, G] = [V_0', W_0', H, G] + \sum_{k=1}^{r'} \cdot [V_k', W_k', H, G],$$

so that $V_\zeta' = \sum_{k=1}^{r'} \cdot V_k'$ is the eigenspace of (I) at ζ.

We define a new system $(\hat{\mathrm{I}})$: $[V, W, H, \hat{G}]$ by putting

$$G = \begin{cases} G & \text{on } V_0' \\ G-\varepsilon H & \text{on } V_\zeta', \text{ with some } \varepsilon\neq 0. \end{cases}$$

It is clear that if we can determine A_λ^{-1} explicitly we can do the same for $\hat{A}_\lambda^{-1} = (\hat{G}-\lambda H)^{-1}$. Further, $(\hat{\mathrm{I}})$ is obtained from (I) by a finite-dimensional perturbation. Hence $(\hat{\mathrm{I}}) \sim$ (II). $(\hat{\mathrm{I}})$ has the same character and the same spectral decompositions as (I) except that the eigenvalue ζ of (I) is now replaced by the eigenvalue $\zeta-\varepsilon$ for $(\hat{\mathrm{I}})$. We can therefore apply Theorem II.6.1 to $(\hat{\mathrm{I}}) \sim$ (II). A decomposition for this perturbation is readily obtained in the form

$$\text{(II.7.3)}\qquad V = \hat{\mathfrak{D}} + [\hat{a}_1, \dots, \hat{a}_{m'}, v_{1,1}, \dots, v_{1,n_1'}, \dots, v_{r',1}, \dots, v_{r',n_{r'}'}],$$

where $\hat{\mathfrak{D}} \subset \mathfrak{D}$, $[\hat{a}_1, \dots, v_{r',n_{r'}'}] = [a_1, \dots, a_m] + V_\zeta'$, and $v_{k,1}, \dots, v_{k,n_k'}$ is the basis of the chain determining the elementary divisor $[V_k', W_k', H, G]$. The type of the matrix $\hat{M}(\lambda)$ corresponding to $\hat{A}_\lambda^{-1}B_\lambda$ then gives, by Theorem II.6.1, the number r and the orders n_k of the elementary divisors of (II) at ζ and also the character (n_0, n_0) of (II) in Δ.

Remark II.7.1. If we want to know only the type of (II) in Δ the simplest way is to choose a $\zeta\in\Delta$ *which is not an eigenvalue of* (I) and apply

Theorem II.6.1 directly to (I) $\sim$ (II), since the matrix $M(\lambda)$ which is then considered is in general of much smaller order than $\hat{M}(\lambda)$.

Case C. Δ has character (p, p), $0 < p < \infty$, for system (I).

In this case (I) is a finite-dimensional perturbation of a system $(\hat{\mathrm{I}})$ for which Δ has the character $(0, 0)$. Our method will consist of choosing such a system $(\hat{\mathrm{I}})$ for which we will have all the needed information; i.e., its eigenvalues, elementary divisors, spectral decompositions in Δ, and for which we will be able to compute the inverse $\hat{A}_\lambda^{-1}$ as a function of λ. We may say that in the present case the nceessary information about (I) which we have to require includes the knowledge of such a system $(\hat{\mathrm{I}})$. Once such a system is known, everything is reduced to the cases A and B. In order to show how such a system can be effectively constructed we choose a $\zeta\in\Delta$ which is not an eigenvalue of (I). Then $\alpha(A_\zeta) = \beta(A_\zeta) = p$. We choose direct decompositions into closed subspaces, $V = V_1 + N$, $W = A_\zeta(V) + W_1$, where N is the null-space of A_ζ and $\dim N = \dim W_1 = p$. We consider the system $(\hat{\mathrm{I}})$: $[V, W, H, \hat{G}]$, where

$$\hat{G} = \begin{cases} G & \text{on } V_1, \\ S+\zeta H & \text{on } N, \end{cases}$$

and S is a linear isomorphism of N onto W_1.

It is clear that $\alpha(\hat{A}_\zeta) = \beta(\hat{A}_\zeta) = 0$ and therefore the character of $(\hat{\mathrm{I}})$ in Δ is $(0, 0)$.

The system $(\hat{\mathrm{I}})$ depends on the choice of decompositions of V and W and also on the choice of the transformation S. Therefore our assumption is that the knowledge of (I) allows us to find such a system $(\hat{\mathrm{I}})$ for which all the information necessary for application of Theorem II.6.1 is available. It is to be noted that the passage from (I) to $(\hat{\mathrm{I}})$ will not only change the character from (p, p) to $(0, 0)$ but may also change considerably the sequence of eigenvalues and the corresponding elementary divisors and spectral decompositions. However, if λ is an eigenvalue for (I) it will remain one for $(\hat{\mathrm{I}})$, since $|\alpha(A_\lambda) - \alpha(\hat{A}_\lambda)| \leqslant p$, $\alpha(A_\lambda) > p$ implies $\alpha(\hat{A}_\lambda) > 0$.

Case D. Δ has the character (p, q), $p\neq q$, p and q finite.

In the present case we cannot obtain from (I), by finite dimensional perturbation, a system with character $(0, 0)$ in Δ (since the index $p-q\neq 0$ would be maintained). However, by using a finite-dimensional *extension* and finite-dimensional perturbation we can arrive at a system $(\tilde{\mathrm{I}})$ with character $(0, 0)$ in Δ which will allow us to reduce our problem to the case treated by Theorem II.6.1. Again, the possibility of finding such an adequate extension and perturbation of (I) will have to be assumed as part of the required knowledge of (I). To describe how such an extension and perturbation of (I) can be achieved, consider again any $\zeta \in \Delta$ which is not an eigenvalue of (I). We then write the decompositions $V = V_1 + N$, $W = A_\zeta(V) + W_1$, where N is the null-space of A_ζ, $\dim N = p$, and $\dim W_1 = q$. We then put

$$\tilde{V} = V_1 + N + V_2 = V + V_2,$$
$$\tilde{W} = A_\zeta(V) + W_1 + W_2 = W + W_2,$$

with $\dim V_2 = q$, $\dim W_2 = p$. We form the system $(\tilde{\mathrm{I}})$: $[\tilde{V}, \tilde{W}, \tilde{H}, \tilde{G}]$ with

$$\tilde{H} = \begin{cases} H & \text{on } V, \\ 0 & \text{on } V_2, \end{cases} \qquad \tilde{G} = \begin{cases} G & \text{on } V_1, \\ S + \zeta H & \text{on } N, \\ T & \text{on } V_2, \end{cases}$$

where S and T are linear isomorphisms of N onto W_2 and V_2 onto W_1 respectively. If, in the definition of $\tilde{G}$, we took $\tilde{G} = G$ on N instead of $S + \zeta H$, the resulting system would be a pure extension of (I) to the spaces $\tilde{V}$, $\tilde{W}$. As defined, $(\tilde{\mathrm{I}})$ is a finite-dimensional extension followed by a finite-dimensional perturbation of the original system (I).

One checks immediately that for the operator $\tilde{A}_\zeta = \tilde{G} - \zeta \tilde{H}$ we have $\alpha(\tilde{A}_\zeta) = 0 = \beta(\tilde{A}_\zeta)$. Clearly, since the quasi-resolvent set is not changed by finite dimensional extension, the quasi-resolvent of $(\tilde{\mathrm{I}})$ is the same as for (I), and $(\tilde{\mathrm{I}})$ has the character $(0, 0)$ in Δ.

We next extend (II) to a system $(\tilde{\mathrm{II}})$: $[\tilde{V}, \tilde{W}, \tilde{H}_1, \tilde{G}_1]$ by putting

$$\tilde{H}_1 = \begin{cases} H_1 & \text{on } V, \\ 0 & \text{on } V_2, \end{cases} \qquad \tilde{G}_1 = \begin{cases} G_1 & \text{on } V, \\ 0 & \text{on } V_2. \end{cases}$$

Since we have the spectral decomposition $[\tilde{V}, \tilde{W}, \tilde{H}_1, \tilde{G}_1] = [V, W, H_1, G_1] + [V_2, W_2, \tilde{H}_1, \tilde{G}_1]$, it is clear that all the eigenvalues, as well as the numbers of corresponding elementary divisors and their orders, are the same for (II) and $(\tilde{\mathrm{II}})$. The essential difference between (II) and $(\tilde{\mathrm{II}})$ is in their character in Δ. If (p', q') is the character of (II) in Δ $(p' - q' = p - q)$, then clearly the character of $(\tilde{\mathrm{II}})$ is $(p' + q, q' + p)$. It is also clear that $(\tilde{\mathrm{II}})$ is a finite-dimensional perturbation of $(\tilde{\mathrm{I}})$ with a decomposition of the form $\tilde{V} = \tilde{\mathfrak{D}} + [\tilde{a}_1, \ldots, \tilde{a}_n]$, where $\tilde{\mathfrak{D}} \subset V_1 \cap \mathfrak{D}$ and $[\tilde{a}_1, \ldots, \tilde{a}_n] \supset [a_1, \ldots, a_m] + (N + V_2)$. The systems $(\tilde{\mathrm{I}}) \sim (\tilde{\mathrm{II}})$ are now in case B.

Remark II.7.2. In some cases (especially when dealing with ordinary differential eigenvalue problems) one can compute directly the null-space

$N(B_\lambda)$ for each λ. This is of great help in the investigation of system (II), since, in each component Δ, $\dim N(B_\lambda)$ is constant except for positive jumps at the eigenvalues of (II). The constant is the nullity in the character of Δ, whereas the jumps give the number of elementary divisors for the corresponding eigenvalues. This information shortens considerably our procedures in many of the cases considered above. However, if there are several elementary divisors for a given eigenvalue, we will need Theorem II.6.1 to determine their orders, since the knowledge of $N(B_\lambda)$ does not allow us to determine either these orders or the total dimension of the eigenspace.

Example. In the appendix we illustrate the cases A and B by rather involved examples from the theory of boundary value problems for ordinary differential equations of fourth order. Here we shall give a simple example to illustrate the cases A and D.

Let $\mathscr{H}$ be the Hilbert space of all functions $f(z)$ analytic and in L^2 in $|z| < 1$. As norm and scalar product we take

$$\|f\|^2 = \frac{1}{\pi} \int\limits_{|z|<1} |f(z)|^2 \, dx\, dy, \qquad (f, g) = \frac{1}{\pi} \int\limits_{|z|<1} f(z) \overline{g(z)} \, dx\, dy.$$

Consider the ordinary system (I): $[\mathscr{H}, \mathscr{H}, I, G]$, where $Gf = zf(z)$, and the ordinary system (II): $[\mathscr{H}, \mathscr{H}, I, G_1]$, where G_1 is obtained from G by a 1-dimensional perturbation corresponding to the decomposition

$$\mathscr{H} = \mathfrak{D} + [1], \qquad \mathfrak{D} = \{f : f(0) = 0\},$$

with $G_1 1 = h(z)$ for some $h(z) \in \mathscr{H}$.

It is well known and easily verified that the quasi-resolvent set for (I) decomposes into two components Δ_1: $|\lambda| < 1$ and Δ_2: $|\lambda| > 1$. Since both systems are ordinary, ∞ is in the resolvent set for both, and we can restrict ourselves to finite λ and not bother to change the bases of the systems so as to have ∞ in the essential spectrum.

The character of (I) in Δ_1 and Δ_2 is $(0, 1)$ and $(0, 0)$ respectively, and (I) has no isolated eigenvalues.

The situation in the domain Δ_2 pertains to case A. The matrix $M(\lambda)$ is of order 1. We have here

$$B_\lambda 1 = h(z) - \lambda, \qquad A_\lambda^{-1} f = \frac{f(z)}{z - \lambda}.$$

Using the linear functional $F_1(f) = (f, 1) = f(0)$, we get

$$M(\lambda) = \left(\frac{h(z) - \lambda}{z - \lambda}, 1 \right) = \frac{\lambda - h(0)}{\lambda}.$$

It is now clear that the character of (II) in Δ_2 is $(0,0)$. There is at most one eigenvalue of (II) in Δ_2: this eigenvalue exists if and only if $|h(0)| > 1$ and then it is $h(0)$. There is then only one elementary divisor of order 1.

The situation in Δ_1 belongs to case D. We take $\zeta = 0$. A_ζ has no null-space and $\mathscr{H} = A_\zeta(\mathscr{H}) \dotplus [1]$, where $A_\zeta(\mathscr{H}) = \mathfrak{D} = \{f\colon f(0) = 0\}$. By the procedure of case D we take an abstract element e generating a 1-dimensional space $[e]$. We make $\tilde{\mathscr{H}} = \mathscr{H} \dotplus [e]$ into a Hilbert space by putting $\|f+\theta e\|^2 = \|f\|^2 + |\theta|^2$. We then extend the system (I) to $(\tilde{\mathrm{I}})$: $[\tilde{\mathscr{H}}, \mathscr{H}, \tilde{I}, \tilde{G}]$ by defining $\tilde{I} = I$ on $\mathscr{H}$ and $= 0$ on $[e]$, $\tilde{G} = G$ on $\mathscr{H}$ and $\tilde{G}e = 1$. The system $(\tilde{\mathrm{I}})$ has then the character $(0,0)$ in Δ_1 and has no eigenvalues in Δ_1. For $\lambda \in \Delta_1$ and $f \in \mathscr{H}$ we have

$$\tilde{A}_\lambda^{-1} f = \frac{f(z) - f(\lambda)}{z-\lambda} + f(\lambda)e.$$

We now extend the system (II) to $(\tilde{\mathrm{II}})$: $[\tilde{\mathscr{H}}, \mathscr{H}, \tilde{I}, \tilde{G}_1]$ by defining $\tilde{I}$ as before and setting $\tilde{G}_1 = G_1$ on $\mathscr{H}$, $= 0$ on $[e]$. The decomposition corresponding to $(\tilde{\mathrm{I}}) \sim (\tilde{\mathrm{II}})$ is

$$\tilde{\mathscr{H}} = \mathfrak{D} \dotplus [1, e].$$

Using the scalar products in $\tilde{\mathscr{H}}$ with 1 and e respectively as linear functionals we obtain:

$$\tilde{B}_\lambda 1 = h(z) - \lambda, \qquad \tilde{B}_\lambda e = 0,$$

$$\tilde{A}_\lambda^{-1}\tilde{B}_\lambda 1 = \frac{h(z) - h(\lambda)}{z-\lambda} + \big(h(\lambda) - \lambda\big)e, \qquad \tilde{A}_\lambda^{-1}\tilde{B}_\lambda e = 0,$$

and the matrix $\tilde{M}(\lambda)$ becomes

$$\tilde{M}(\lambda) = \begin{Vmatrix} \dfrac{h(\lambda) - h(0)}{\lambda} & 0 \\ h(\lambda) - \lambda & 0 \end{Vmatrix}.$$

An easy, elementary investigation gives the following results:

1° There is a unique infinite type exponent for $\tilde{M}(\lambda)$ at each $\lambda \in \Delta_1$. Hence $(\tilde{\mathrm{II}})$ is of character $(1,1)$ in Δ_1.

2° There are no other type-exponents of $\tilde{M}(\lambda)$ at $\lambda_0 \in \Delta_1$ except for at most one λ_0. This exceptional λ_0 exists and is equal to $h(0)$ if and only if one of the following sets of conditions holds:

(a) $h(0) = h\big(h(0)\big)$, where $0 < |h(0)| < 1$;

(b) $h(0) = h'(0) = 0$.

When either of these conditions hold, the additional finite type-exponent at $\lambda_0 = h(0)$ is 1. Hence if the above conditions do not hold, $(\tilde{\mathrm{II}})$ has no eigenvalue in Δ_1. If the conditions are satisfied, then $(\tilde{\mathrm{II}})$ has one eigenvalue $\lambda_0 = h(0)$ in Δ_1 with one elementary divisor of order 1.

In consequence, the same results hold for (II) as concerns the eigenvalues and elementary divisors in Δ_1. The character of (II) is obtained from that of $(\tilde{\mathrm{II}})$ by subtracting $q = 1$ from the nullity and $p = 0$ from the deficiency. Thus the character of (II) in Δ_1 is $(0,1)$.

It should be mentioned that in this simple example most of the results for system (II) could be obtained by a direct analysis without much difficulty. However, checking that the elementary divisor is of order 1 would require investigation of the null-space of B_λ^2.

III. GENERAL SELF-ADJOINT SYSTEMS

1. General setting. The quadruple (I): $[\mathfrak{D}(A), \mathscr{H}, I, A]$ will be called a *standard self-adjoint system* provided A is a self-adjoint linear operator with (dense) domain $\mathfrak{D}(A)$ in the Hilbert space $\mathscr{H}$. Such systems are Hilbert systems in the sense of section II.1 if $\mathfrak{D}(A)$ is provided with the graph norm of A:

$$\|u\|_A^2 = \|u\|^2 + \|Au\|^2 \quad \text{for } u \in \mathfrak{D}(A).$$

Note, however, that standard self-adjoint systems are not necessarily self dual in the sense of section II.1. (A weak topological system or a Banach system $[V, W, \mathscr{P}]$ is self-dual if and only if $[V, W, \mathscr{P}] = [W^*, V^*, \mathscr{P}^*]$; i.e., if and only if $W = V^*$ and, for some pair of generators G, H of $\mathscr{P}$, $G = G^*$ and $H = H^*$. In the case of weak topological systems $W = V^*$ is given the weak-* topology; in the case of Banach systems, it is given the canonical norm topology induced by the norm of V and V is required to be reflexive.) We will not assume in this chapter that $\mathscr{H}$ is separable unless otherwise stated.

The standard self-adjoint system (II): $[\mathfrak{D}(B), \mathscr{H}, I, B]$ is a *finite-dimensional perturbation* of (I), written (I) $\sim$ (II), provided $A \sim B$. This definition agrees with that given in section II.4, for, using the fact that A, B are closed operators and that

$$\mathfrak{D} = \{u\colon u \in \mathfrak{D}(A) \cap \mathfrak{D}(B) \text{ and } Au = Bu\}$$

is a closed subspace of $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$, we see easily that $\mathfrak{D}(A), \mathfrak{D}(B)$, with their respective graph norms, form a *compatible couple*; i.e., the identification map of $\mathfrak{D} \subset \mathfrak{D}(A)$ onto $\mathfrak{D} \subset \mathfrak{D}(B)$ is closed. Thus (see Aronszajn and Gagliardo [9]) $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ can be continuously embedded in the Banach space

$$\mathfrak{D}(A) \dotplus \mathfrak{D}(B) = \mathfrak{D}(A) \dotplus \mathfrak{D}(B)/Z,$$

where $Z = \{(u, -u): u \in \mathfrak{D}\}$, with norm

$$\|u\|_{\mathfrak{D}(A)+\mathfrak{D}(B)} = \inf_{u=v+w} (\|v\|_A + \|w\|_B).$$

On $\mathfrak{D}(A), \mathfrak{D}(B)$ the norm $\| \|_{\mathfrak{D}(A)+\mathfrak{D}(B)}$ is equivalent to $\| \|_A, \| \|_B$ respectively, and $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim [\mathfrak{D}(B), \mathscr{H}, I, B]$ in the sense of section II.4.

Thus the results of the preceding sections, in particular Theorems II.6.1 and II.6.2, apply. Note also that, for a standard self-adjoint system, every λ with $\operatorname{Im}\lambda \neq 0$ is in the resolvent set. Thus the quasi-resolvent set coincides with the meromorphy domain and the essential spectrum, as well as every isolated eigenvalue, lies on the real axis.

THEOREM III.1.1. *Let* (I): $[\mathfrak{D}(A), \mathscr{H}, I, A]$, (II): $[\mathfrak{D}(B), \mathscr{H}, I, B]$ *be standard self-adjoint systems and* $\mathfrak{D} = \{u: u \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$ *and* $Au = Bu\}$. *If* $\mathfrak{D}$ *has finite codimension in* $\mathfrak{D}(A)$ (*or in* $\mathfrak{D}(B)$), *then* (I) $\sim$ (II).

Proof. Let $V = \overline{\mathfrak{D}}$. Since by hypothesis $\mathfrak{D}$ has finite codimension in $\mathfrak{D}(A)$, $V = \overline{\mathfrak{D}}$ has finite codimension in $\overline{\mathfrak{D}(A)} = \mathscr{H}$. Hence $\mathscr{H} = V \dot{+} X$, where $\dim X = m < \infty$. Let $\mathfrak{D}(A') = \mathfrak{D}(A) \cap V$, $\mathfrak{D}(B') = \mathfrak{D}(B) \cap V$. Then

$$\mathfrak{D}(A) = \mathfrak{D}(A') \dot{+} F_A, \quad \mathfrak{D}(B) = \mathfrak{D}(B') \dot{+} F_B,$$

where $\dim F_A, \dim F_B$ are finite.

Let $X = [x_1, \ldots, x_m]$. Then the linearly independent functionals $(u, x_1), \ldots, (u, x_m)$ are continuous on $\mathscr{H}$ and, a fortiori, continuous on $\mathfrak{D}(A)$ in the graph norm of A. Moreover,

$$\mathfrak{D}(A') = \{u: u \in \mathfrak{D}(A) \text{ and } (u, x_i) = 0, i = 1, \ldots, m\}.$$

Thus $\mathfrak{D}(A')$ has codimension m in $\mathfrak{D}(A)$, so $\dim F_A = m$. Similarly, $\dim F_B = m$.

Note that, since $\mathfrak{D}$ is contained in $\mathfrak{D}(A')$ and $\mathfrak{D}(B')$, then $\overline{\mathfrak{D}(A')} = \overline{\mathfrak{D}(B')} = V$. Thus $\mathscr{H} = V \dot{+} F_A = V \dot{+} F_B$. If $V \cap F_A \neq \{0\}$, then V would have codimension $< m$ in $\mathscr{H}$, and similarly for $V \cap F_B$. Thus

$$\mathscr{H} = V \dot{+} F_A = V \dot{+} F_B. \tag{III.1.1}$$

Let P be the projection of $\mathscr{H}$ onto V and A', B' be the restrictions of PA, PB to $\mathfrak{D}(A'), \mathfrak{D}(B')$ respectively. Then, clearly, A' and B' are symmetric operators with dense domains in V. Suppose $v \in V$ is such that $(A'd, v)$ is continuous for $d \in \mathfrak{D}(A')$. If $u \in \mathfrak{D}(A)$, then $u = d + f$ with $d \in \mathfrak{D}(A')$ and f in the finite-dimensional space F_A. In view of equation (III.1.1) therefore

$$(Au, v) = (A'd, v) + (Af, v)$$

is continuous in u. Hence $v \in \mathfrak{D}(A) \cap V = \mathfrak{D}(A')$ and A' is self-adjoint. Similarly, B' is self-adjoint.

Let T be the restriction of A' or B' to $\mathfrak{D}(T) = \mathfrak{D}$. Then T is clearly a closed symmetric operator, and A' and B' are self-adjoint extensions of T. Thus

$$\mathfrak{D}(A') = \mathfrak{D} \dot{+} F'_A, \quad \mathfrak{D}(B') = \mathfrak{D} \dot{+} F'_B,$$

where $\dim F'_A = \dim F'_B = m' < \infty$, and

$$\mathfrak{D}(A) = \mathfrak{D} \dot{+} (F'_A \dot{+} F_A), \quad \mathfrak{D}(B) = \mathfrak{D} \dot{+} (F'_B \dot{+} F_B),$$

where $\dim(F'_A \dot{+} F_A) = \dim(F'_B \dot{+} F_B) = m' + m$. Since $A = B$ on $\mathfrak{D}$, the theorem is proved.

Definition III.1.1. Let (I): $[\mathfrak{D}(A), \mathscr{H}, I, A]$, (II): $[\mathfrak{D}(B), \mathscr{H}, I, B]$ be standard self-adjoint systems in the same perturbation class. Then the perturbation (I) $\sim$ (II) is:

1° a *perturbation of type one* if and only if B is a 1-dimensional perturbation of A with $\mathfrak{D}(A) = \mathfrak{D}(B)$, representable in the form

$$\mathfrak{D}(A) = \mathfrak{D}(B) = \mathfrak{D} \dot{+} [f], \quad Bf = Af + \lambda Pf,$$

where P is the orthogonal projection of $\mathscr{H}$ onto a 1-dimensional subspace $[x]$, $\|x\| = 1$, $Pf = x$, $P(\mathfrak{D}) = 0$ and $\lambda \neq 0$ real;

2° a *perturbation of type two* if and only if A and B are self-adjoint extensions of a closed symmetric operator with dense domain in $\mathscr{H}$ and deficiency indices $(1, 1)$.

The importance of the above special perturbations is demonstrated by the following theorem:

THEOREM III.1.2. *Let* (I): $[\mathfrak{D}(A), \mathscr{H}, I, A]$, (II): $[\mathfrak{D}(B), \mathscr{H}, I, B]$ *be standard self-adjoint systems and let* (I) $\sim$ (II) *be a perturbation of dimension* m. *Then there exist integers* m_1, m_2, $m_1 + m_2 = m$, *such that the perturbation* (I) $\sim$ (II) *can be achieved by* m_1 *consecutive perturbations of type two followed by* m_2 *consecutive perturbations of type one.*

Proof. By hypothesis

$$\mathfrak{D}(A) = \mathfrak{D} \dot{+} F_A, \quad \mathfrak{D}(B) = \mathfrak{D} \dot{+} F_B,$$

where $\dim F_A = \dim F_B = m$ and $A = B$ on $\mathfrak{D}$. If $\mathfrak{D}(A) \neq \mathfrak{D}(B)$, then there exists an element $b_1 \in F_B$ such that $b_1 \notin \mathfrak{D}(A)$. We then define a new linear operator $\hat{A}^*$ with domain $\mathfrak{D}(\hat{A}^*) = \mathfrak{D}(A) \dot{+} [b_1]$ by setting $\hat{A}^* = A$ on $\mathfrak{D}(A)$ and $\hat{A}^* b_1 = Bb_1$. Clearly, $\hat{A}^*$ is a closed operator with dense domain in $\mathscr{H}$, and A is a restriction of $\hat{A}^*$. Thus $\hat{A}^*$ has a well defined adjoint $\hat{A}$ in $\mathscr{H}$ and, since A is self-adjoint, $\hat{A}$ is a restriction of A. It follows that $\hat{A}$ is closed and symmetric, with dense domain $\mathfrak{D}(\hat{A}) \subset \mathfrak{D}(A)$ in $\mathscr{H}$.

By definition of $\hat{A}$, $v \in \mathfrak{D}(\hat{A})$ if and only if there exists a $w \in \mathscr{H}$ such that $(\hat{A}^* u, v) = (u, w)$ for every $u \in \mathfrak{D}(A) \dotplus [b_1]$. In particular, taking $u \in \mathfrak{D}(A)$ we obtain $(Au, v) = (u, w)$; hence $v \in \mathfrak{D}(A)$ and $w = Av$, facts which we already know. Taking then $u = \xi b_1$, we get $(Bb_1, v) = (b_1, Av)$. Conversely, if $v \in \mathfrak{D}(A)$ satisfies

$$(Bb_1, v) - (b_1, Av) = 0, \tag{III.1.2}$$

then $v \in \mathfrak{D}(\hat{A})$. Thus

$$\mathfrak{D}(\hat{A}) = \{v \in \mathfrak{D}(A) \colon v \text{ satisfies (III.1.2)}\}.$$

The functional $(Bb_1, v) - (b_1, Av)$ is clearly continuous on $\mathfrak{D}(A)$ with respect to the graph norm $\| \ \|_A$ of A. Moreover, it is not identically zero on $\mathfrak{D}(A)$, else $(Bb_1, v) = (b_1, Av)$ for every $v \in \mathfrak{D}(A)$, implying $b_1 \in \mathfrak{D}(A)$. Hence $\mathfrak{D}(\hat{A})$ is a closed (with respect to $\| \ \|_A$) subspace of $\mathfrak{D}(A)$ of codimension one:

$$\mathfrak{D}(A) = \mathfrak{D}(\hat{A}) \dotplus [f]. \tag{III.1.3}$$

Clearly, $\mathfrak{D} \subset \mathfrak{D}(\hat{A})$, and since $\mathfrak{D}$ has codimension m in $\mathfrak{D}(A)$ we can further decompose $\mathfrak{D}(\hat{A})$ into $\mathfrak{D}(\hat{A}) = \mathfrak{D} \dotplus F'_A$, where $\dim F'_A = m-1$.

Equation (III.1.3) shows that the closed symmetric operator $\hat{A}$ has deficiency indices $(1, 1)$. Define a new operator A_1 with domain

$$\mathfrak{D}(A_1) = \mathfrak{D}(\hat{A}) \dotplus [b_1]$$

such that $A_1 = \hat{A} = A$ on $\mathfrak{D}(\hat{A})$ and $A_1 b_1 = Bb_1$. Then for $u = v + \xi b_1 \in \mathfrak{D}(A_1)$,

$$\begin{aligned}(A_1 u, u) &= \big(A_1(v+\xi b_1), v+\xi b_1\big)\\ &= (Av + \xi Bb_1, v + \xi b_1)\\ &= (Av, v) + |\xi|^2 (Bb_1, b_1) + \xi (Bb_1, v) + \bar{\xi}(Av, b_1)\\ &= (Av, v) + |\xi|^2 (Bb_1, b_1) + 2\,\mathrm{Re}(\xi Bb_1, v)\end{aligned}$$

is real (note $(Av, b_1) = \overline{(Bb_1, v)}$ since $v \in \mathfrak{D}(\hat{A})$). Hence A_1 is a proper symmetric extension of $\hat{A}$ and is therefore self-adjoint.

Thus $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim [\mathfrak{D}(A_1), \mathscr{H}, I, A_1]$ is a perturbation of type two. Moreover,

$$\mathfrak{D}(A_1) = \mathfrak{D} \dotplus [b_1] \dotplus F'_A, \qquad \mathfrak{D}(B) = \mathfrak{D} \dotplus [b_1] \dotplus F'_B,$$

where $A_1 = B$ on $\mathfrak{D} \dotplus [b_1]$ and $\dim F'_A = \dim F'_B = m-1$. Thus the perturbation $A_1 \sim B$ is of dimension $m-1$.

If $\mathfrak{D}(A_1) \neq \mathfrak{D}(B)$, then there exists an element $b_2 \in F'_B$ such that $b_2 \notin \mathfrak{D}(A_1)$ and we can proceed, the same way as above, to find a perturbation $[\mathfrak{D}(A_1), \mathscr{H}, I, A_1] \sim [\mathfrak{D}(A_2), \mathscr{H}, I, A_2]$ of type two such that $A_2 \sim B$ is a perturbation of dimension $m-2$ with decomposition

$$\mathfrak{D}(A_2) = \mathfrak{D} \dotplus [b_1] \dotplus [b_2] \dotplus F''_A, \qquad \mathfrak{D}(B) = \mathfrak{D} \dotplus [b_1] \dotplus [b_2] \dotplus F''_B,$$

where $A_2 = B$ on $\mathfrak{D} \dotplus [b_1] \dotplus [b_2]$ and $\dim F''_A = \dim F''_B = m-2$. Continuing in this way, we arrive after m_1 perturbations of type two, $0 \leqslant m_1 \leqslant m$, at a self-adjoint operator $A_{m_1} = A'$ such that $A' \sim B$ is a perturbation of dimension $m_2 = m - m_1$ and $\mathfrak{D}(A') = \mathfrak{D}(B)$.

Thus

$$\mathfrak{D}(A') = \mathfrak{D}(B) = \mathfrak{D}' \dotplus F,$$

where $\mathfrak{D}' = \{u \in \mathfrak{D}(B) \colon A'u = Bu\}$ and $\dim F = m_2$.

It follows that the operator $B - A'$, defined on $\mathfrak{D}(B)$, has $\mathfrak{D}'$ as its nullspace, that its range $(B-A')\big(\mathfrak{D}(B)\big) = (B-A')(F)$ is exactly m_2-dimensional, and that $B - A'$ is 1-1 on F.

Since $B - A'$ is obviously symmetric it is closable, and its closure $\overline{B-A'}$ has the same m_2-dimensional range as $B - A'$. It follows that $\overline{B-A'}$ is bounded, defined on the whole of $\mathscr{H}$ and self-adjoint. But $\mathscr{H} = \overline{\mathfrak{D}(B)} = \overline{\mathfrak{D}'} + F$ and $\overline{B-A'}(\overline{\mathfrak{D}'}) = 0$. Hence $\overline{\mathfrak{D}'}$ is the nullspace of $\overline{B-A'}$, $X = \mathscr{H} \ominus \overline{\mathfrak{D}'}$ is the range of $\overline{B-A'}$, and X has an orthonormal basis of m_2 eigenvectors $x_1, x_2, \ldots, x_{m_2}$ of $\overline{B-A'}$ with corresponding real non-zero eigenvalues λ_k, $k = 1, \ldots, m_2$.

Let f_k be the unique element in F with $\overline{B-A'} f_k = \lambda_k x_k$, and denote by P_k the orthogonal projection of $\mathscr{H}$ onto $[x_k]$. Then

$$P = \sum_1^{m_2} P_k \text{ is the projection on } X,$$

$$(f_k, x_i) = \frac{1}{\lambda_i}(f_k, \overline{B-A'} x_i) = \frac{1}{\lambda_i}(\overline{B-A'} f_k, x_i) = \delta_{ki},$$

$$\overline{B-A'} = \sum_{i=1}^{m_2} \lambda_i P_i, \quad \text{and} \quad B = A' + \sum_{i=1}^{m_2} \lambda_i P_i \text{ on } \mathfrak{D}(B).$$

We introduce now the operators A'_k, $k = 0, \ldots, m_2$, defined on $\mathfrak{D}(B)$ by

$$A'_0 = A', \qquad A'_k = A' + \sum_{i=1}^{k} \lambda_i P_i \text{ for } k = 1, 2, \ldots, m_2,$$

$$A'_{m_2} = B.$$

It is clear that these operators are self-adjoint and that $A'_{k-1} \sim A'_k$ for $k = 1, \ldots, m_2$.

The perturbation is of dimension 1 and is represented by

$$\mathfrak{D}(A'_{k-1}) = \mathfrak{D}(A'_k) = \{\mathfrak{D}' + [f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{m_2}]\} + [f_k],$$

where $A'_k f_k = A'_{k-1} f_k + \lambda_k P_k f_k$. Thus the corresponding perturbation is of type one, and the theorem is proved.

Remark III.1.1. It follows from Theorem III.1.2 and its proof that every one-dimensional perturbation between standard self-adjoint systems is either of type one or type two.

We next state the main theorem of this chapter, which will be proved in the following sections:

THEOREM III.1.3. *Consider two standard self-adjoint systems in the same perturbation class — $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim [\mathfrak{D}(B), \mathscr{H}, I, B]$ — the perturbation being of dimension m. Then there exists a singular Borel measure μ_0 on R^1 with the following properties:*

1° *Every measure $\mu \neq 0$ orthogonal to μ_0 is stationary rel. A if and only if it is stationary rel. B, in which case the multiplicity of μ rel. A equals the multiplicity of μ rel. B: $m^{(A)}(\mu) = m^{(B)}(\mu)$.*

2° *For every measure $\mu \neq 0$ which is stationary rel. A and B and which is absolutely continuous with respect to μ_0,*

$$|m^{(A)}(\mu) - m^{(B)}(\mu)| \leqslant m \text{ (6)}.$$

Theorem III.1.3. has the following corollary:

COROLLARY III.1.3′. *If $[\mathfrak{D}(B), \mathscr{H}, I, B] \sim [\mathfrak{D}(A), \mathscr{H}, I, A]$, then the absolutely continuous parts of the self-adjoint operators A, B are unitarily equivalent.*

Remark III.1.2. It will also follow from the proof of Theorem III.1.3 that for two standard self-adjoint systems $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim [\mathfrak{D}(B), \mathscr{H}, I, B]$ there exist subspaces $\mathscr{H}_1, \mathscr{H}_2$ which reduce both A and B and such that $A = B$ on $\mathfrak{D}(A) \cap \mathscr{H}_2 = \mathfrak{D}(B) \cap \mathscr{H}_2$ and $\mathscr{H}_1$ is separable.

In Sections III.2 and III.3 are reviewed certain (essentially known) facts about multiplicity of measures for a self-adjoint operator which are needed for the proof of Theorem III.1.3. In view of Theorem III.1.2 it will suffice to prove Theorem III.1.3 for perturbations of types one and two. These proofs are given in Sections III.4, III.5 respectively. In Section III.6 Theorem III.1.3 and Corollary III.1.3′ are proved. In addition, a theorem is proved which states that for self-adjoint operators $A \sim B$ and for any interval $\mathscr{I}$ of R^1 in the meromorphy domain of A and B the numbers of eigenvalues of A and B in $\mathscr{I}$ differ by at most the dimension of the perturbation.

(6) Here, since $m^{(A)}(\mu)$, $m^{(B)}(\mu)$ are cardinal numbers, the inequality is interpreted to mean that $m^{(A)}(\mu)$ is infinite if and only if $m^{(B)}(\mu)$ is infinite, in which case $m^{(A)}(\mu) = m^{(B)}(\mu)$; otherwise they are both finite and $|m^{(A)}(\mu) - m^{(B)}(\mu)| \leqslant m$.

2. Review of some notions about measure theory and the theory of multiplicity. In this section we give statements without proofs. We refer the reader to [16] for the proofs relevant to the first part of the section and to [15] for those in the second part.

Let $\mathscr{M}$ be the class of all non-negative Borel measures on the real line R^1. We will sometimes use the correspondence between such measures μ and the non-decreasing functions $\mu(\lambda)$ on the real line, normalized by:

$$\mu(0) = 0, \quad \mu(\lambda) = \tfrac{1}{2}[\mu(\lambda-) + \mu(\lambda+)].$$

In connection with this representation we will sometimes denote the measure μ by the symbol $d\mu(\lambda)$, a convention which will be especially useful when we want to define a (not necessarily positive nor real) measure ν by the formula

$$d\nu(\lambda) = \varphi(\lambda)\, d\mu(\lambda),$$

where $\varphi(\lambda)$ is locally integrable relative to μ.

Lebesgue measure will be denoted by Λ and, by an understandable deviation from the general rule, the corresponding function will be denoted by λ, and instead of $d\Lambda$ we will write $d\lambda$.

The class $\mathscr{M}$ is a boundedly complete lattice under the order relation $\mu \leqslant \nu$ and lattice operations $\sup(\mu, \nu)$, $\inf(\mu, \nu)$. A *support* S for a measure μ is any Borel set with $\mu(R^1 - S) = 0$. Two measures μ_1, μ_2 are *orthogonal*, in symbols $\mu_1 \perp \mu_2$, if they possess respective supports S_1, S_2 such that $S_1 \cap S_2 = 0$.

Any bounded family $\{\mu_i\}$ of mutually orthogonal measures is at most enumerable and, for such a family, $\sup\{\mu_i\} = \Sigma^{\perp}\mu_i$. (We will use the notation $\Sigma^{\perp}\mu_i$ to indicate that the terms in the sum are mutually orthogonal measures(7).) If $\mu = \Sigma^{\perp}\mu_i$, then for any support S of μ we can find supports S_i for μ_i such that the S_i's are mutually disjoint and contained in S. Each measure μ_i is then completely determined as the part of the measure μ in S_i.

We introduce in $\mathscr{M}$ the new partial order relation $\mu \ll \nu$, meaning μ is *absolutely continuous rel.* ν; i.e., every support of ν is a support of μ. If $\mu \ll \Lambda$, we will say simply that μ is *absolutely continuous*. If $\mu \ll \nu$ and $\nu \ll \mu$, we write $\mu \sim \nu$ (in words, μ is *equivalent* to ν), which means that any support of one of the two measures is also a support of the other. The set of all equivalence classes of measures in $\mathscr{M}$ is denoted by $\mathscr{M}^e$. The equivalence class of a measure $\mu \in \mathscr{M}$ is denoted by μ^e, so that $\nu \sim \mu$ means the same thing as $\nu \in \mu^e$. If $\mu \perp \nu$, then, for every $\mu_1 \in \mu^e$ and every $\nu_1 \in \nu^e$, $\mu_1 \perp \nu_1$, and we will therefore write $\mu^e \perp \nu^e$.

(7) If there is a finite number of μ_i's, we may write

$$\sum_{i=1}^{n}{}^{\perp} \mu_i = \mu_1 \pm \mu_3 \pm \cdots \pm \mu_n.$$

For any two measures μ, ν there exist unique decompositions

(III.2.1) $\mu = \mu_0 \pm \mu_1, \quad \nu = \nu_0 \pm \nu_1, \quad$ where $\mu_0 \sim \nu_0$, $\mu_1 \perp \nu$, and $\nu_1 \perp \mu$.

If $\nu = \Lambda$, then, in the corresponding decomposition $\mu = \mu_0 \pm \mu_1$ of μ, μ_0 is called the *absolutely continuous part* and μ_1 the *singular part* of μ.

The order relation $\mu^e \ll \nu^e$ induced in $\mathscr{M}^e$ by the order relation $\ll$ in $\mathscr{M}$ makes $\mathscr{M}^e$ into a Boolean σ-ring. The operations $\vee$ (*union*), $\wedge$ (*intersection*), and $\setminus$ (*relative complementation*) are defined as follows: for two measures μ, ν we consider the decomposition (III.2.1). Then

$$\mu^e \vee \nu^e = (\mu+\nu_1)^e = (\mu_1+\nu)^e = (\mu+\nu)^e = \big(\sup(\mu, \nu)\big)^e,$$

$$\mu^e \wedge \nu^e = \mu_0^e = \nu_0^e = \big(\inf(\mu, \nu)\big)^e,$$

and

$$\mu^e \setminus \nu^e = \mu_1^e.$$

Consider a standard self-adjoint system $[\mathfrak{D}(A), \mathscr{H}, I, A]$ and the resolution of the identity $\{E(\lambda)\}$ corresponding to A. For each vector $x \in \mathscr{H}$ consider the measure μ_x defined by $d\mu_x(\lambda) = d\|E(\lambda)x\|^2$. If $\mu \in \mathscr{M}$ and $\mu \ll \mu_x$, then we shall say that x *majorates* μ. Two vectors x, y are called *strongly orthogonal* (or, following Halmos, *very orthogonal*) if and only if $\big(E(\lambda)x, E(\lambda)y\big) = 0$ for every $\lambda \in R^1$, from which it follows that $(x, y) = 0$.

LEMMA III.2.1. *Let $0 \neq \mu \in \mathscr{M}$. Then all maximal families $\{x_i\}$ of mutually strongly orthogonal vectors x_i majorating μ have the same cardinality $\mathfrak{M}$.*

Definition III.2.1. The cardinal number $\mathfrak{M}$ of Lemma III.2.1 will be called the *multiplicity* of μ (rel. A) and denoted by $m(\mu) \equiv m^{(A)}(\mu)$.

The cardinal-valued function $m(\mu)$ satisfies the axioms for a multiplicity function used by Halmos [15] (except that we define $m(\mu)$ only for $\mu \neq 0$):

1° If $0 \neq \nu \ll \mu$, then $m(\nu) \geqslant m(\mu)$.

2° If $\mu = \Sigma^{\perp}\mu_i$, where all μ_i's are different from zero, then $m(\mu) = \min_i\{m(\mu_i)\}$.

Since for all $\nu \sim \mu \neq 0$, $m(\nu) = m(\mu)$, we can define $m(\mu^e) = m(\mu)$.

Thus each self-adjoint operator A determines a multiplicity function (rel. A). It is well known [15] that two self-adjoint operators determine the same multiplicity function if and only if they are unitarily equivalent.

Definition III.2.2. A measure $\mu \in \mathscr{M}$, $\mu \neq 0$, is called *stationary* (rel. A) (or, in the terminology of Halmos, has *uniform multiplicity*) if and only if $m(\nu) = m(\mu)$ for every non-zero $\nu \ll \mu$.

THEOREM III.2.1. *For every non-zero $\mu \in \mathscr{M}$, there exists a unique, at most enumerable sequence $\{\mathfrak{M}_k\}_{k=0,1,\ldots}$ of cardinals $\mathfrak{M}_0 < \mathfrak{M}_1 < \ldots$ and a unique orthogonal decomposition*

(III.2.2) $$\mu = \sum_{k\geqslant 0}^{\perp} \mu_k$$

such that, for each k, μ_k is stationary and $m(\mu_k) = \mathfrak{M}_k$. (Note that $\mathfrak{M}_0 = m(\mu)$.)

This theorem shows that it is enough to know the multiplicity for all stationary measures in order to know it for all measures.

Definition III.2.3. A class of stationary (rel. A) measures is *sufficient for A* if and only if every non-zero measure $\mu \in \mathscr{M}$ can be written as $\mu = \Sigma^{\perp}\mu_k$ with μ_k's in this class.

It is clear that the knowledge of the multiplicity for all measures in a sufficient class of stationary measures determines the multiplicity for all measures. The fact that makes this notion of "sufficiency" useful is the following:

PROPOSITION III.2.1. *Let $\{[\mathfrak{D}(A_k), \mathscr{H}_k, I, A_k]\}$ be a finite set of standard self-adjoint systems. Then the class of all measures $\mu \neq 0$ which are stationary rel. every A_k is a sufficient class for each A_k.*

The proof of Proposition III.2.1 is immediate.

Let $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$. Then $\mathscr{H}_1, \mathscr{H}_2$ *reduce* A if and only if

$$\mathfrak{D}(A) = \mathfrak{D}(A) \cap \mathscr{H}_1 + \mathfrak{D}(A) \cap \mathscr{H}_2$$

and $A\big(\mathfrak{D}(A) \cap \mathscr{H}_i\big) \subset \mathscr{H}_i$, $i = 1, 2$; or, equivalently, if and only if, for every $\lambda \in R^1$, $E(\lambda)P_i = P_iE(\lambda)$, where P_i is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_i$, $i = 1, 2$. The restriction A_i of A to $\mathfrak{D}(A_i) = \mathfrak{D}(A) \cap \mathscr{H}_i$ is called the *part* of A in $\mathscr{H}_i$ and is a self-adjoint operator in $\mathscr{H}_i$, $i = 1, 2$. We have the following:

THEOREM III.2.2. *If $\mathscr{H}_1, \mathscr{H}_2$ reduce the self-adjoint operator A in $\mathscr{H}$, and A_1, A_2 are the corresponding parts of A, then every measure $\mu \neq 0$ which is stationary rel. A_1 and A_2 is stationary rel. A, and*

$$m^{(A)}(\mu) = m^{(A_1)}(\mu) + m^{(A_2)}(\mu).$$

The proof is easily obtained using the methods of [15] and Proposition III.2.1.

Let us return to a standard self-adjoint system $[\mathfrak{D}(A), \mathscr{H}, I, A]$ and consider the class of all measures $\mu \neq 0$ with $m(\mu) \geqslant 1$. If $\vee\{\mu^e\colon m(\mu) \geqslant 1\} = \mu_A^e$ exists, then every measure in μ_A^e will be called a *spectral measure* of A. The spectral measures μ_A therefore have the property that for every $\nu \perp \mu_A$ $m(\nu) = 0$, while for every $\nu \ll \mu_A$, $m(\nu) \geqslant 1$. In separable Hilbert spaces spectral measures always exist; they may or may not exist in non-separable Hilbert spaces. When a spectral measure exists, then for a complete knowledge of the multiplicity function it is sufficient to know the multiplicity for all stationary measures $\nu \ll \mu_A$.

In analogy to the definition of μ_A^e we can define $\Lambda_A^e = \bigvee\{\mu^e\colon m(\mu) \geqslant 1$ and $\mu \ll \Lambda\}$, which always exists. If a spectral measure exists, then its absolutely continuous part belongs to Λ_A^e. To give a significance to Λ_A^e in the general case we introduce the following general concepts:

For every measure $\mu \neq 0$ consider the decomposition of $\mathscr{H}$ into $\mathscr{H}_\mu = \{x \in \mathscr{H}\colon \mu_x \ll \mu\}$ and $\mathscr{H}_{\mu\perp} = \{x \in \mathscr{H}\colon \mu_x \perp \mu\}$. Then $\mathscr{H} = \mathscr{H}_\mu \pm \mathscr{H}_{\mu\perp}$, and $\mathscr{H}_\mu$, $\mathscr{H}_{\mu\perp}$ reduce A. The part A_μ of A in $\mathscr{H}_\mu$ is called the *part of A corresponding to μ*. In particular, we can consider $\mu = \Lambda$. Then A_Λ is called the *absolutely continuous part of A*. A_Λ is a self-adjoint operator in $\mathscr{H}_\Lambda$, and the elements of Λ_A^e are spectral measures for A_Λ.

3. Standard self-adjoint systems generated by a single vector. Let $[\mathfrak{D}(A), \mathscr{H}, I, A]$ be a standard self-adjoint system; let $\{E(\lambda)\}$ be the resolution of the identity for A, and suppose that $x \in \mathscr{H}$ is such that $\mathscr{H}$ is the closure of the space spanned by the set $\{E(\lambda)x\colon \lambda \text{ real}\}$ or, equivalently, by the set $\{A_\zeta^{-1}x\colon \operatorname{Im}\zeta \neq 0\}$, where $A_\zeta = A - \zeta I$. Then $\mathscr{H}$ is separable, and the finite Borel measure μ_x defined by $d\mu_x(\lambda) = d\|E(\lambda)x\|^2$ is a spectral measure corresponding to A. Moreover, it is clear from definition III.2.1 that for every $\mu \in \mathscr{M}$, $m(\mu)$ is either zero or one. In fact, $m(\mu) = 0$ unless $\mu \ll \mu_x$, in which case $m(\mu) = 1$.

Let μ be any finite Borel measure and consider the Lebesgue-Jordan decomposition

$$\mu = \mu^0 \pm \mu^1, \tag{III.3.1}$$

where μ^0 is the absolutely continuous part, and μ^1 is the singular part corresponding to μ. A classical theorem of de la Vallée-Poussin assigns supports to these measures as follows:

$$S_P^0 = \left\{\xi\colon \frac{d\mu(\xi)}{d\lambda} \text{ exists and } 0 < \frac{d\mu(\xi)}{d\lambda} < \infty\right\},$$

$$S_P^1 = \left\{\xi\colon \frac{d\mu(\xi)}{d\lambda} \text{ exists and equals } +\infty\right\}.$$

In particular, these supports may be used for the Lebesgue-Jordan decomposition of the spectral measure μ_x corresponding to A.

In what follows we shall also need a few facts concerning measures μ associated with functions $\varphi(\zeta)$ of class P; i.e., functions analytic in the upper half-plane with positive imaginary part there (see Aronszajn and Donoghue [6], [7] for details). Such functions are precisely those of the form

$$\varphi(\zeta) = \alpha\zeta + \beta + \int_{-\infty}^{\infty}\left[\frac{1}{\lambda-\zeta} - \frac{\lambda}{\lambda^2+1}\right] d\mu(\lambda), \tag{III.3.2}$$

where $\alpha \geqslant 0$, β is real, and

$$\int_{-\infty}^{\infty} \frac{d\mu(\lambda)}{1+\lambda^2} < \infty.$$

For example, the function

$$\varphi(\zeta) = (A_\zeta^{-1}x, x) = \int_{-\infty}^{\infty} \frac{1}{\lambda-\zeta}\, d\mu_x(\lambda)$$

is in class P and (III.3.2) holds with

$$\alpha = 0 \quad \text{and} \quad \beta = \int_{-\infty}^{\infty} \frac{\lambda}{1+\lambda^2}\, d\mu_x(\lambda).$$

Let the function $\varphi(\zeta)$ in P be given by (III.3.2). Then supports for the measures appearing in the decomposition (III.3.1) of μ may be given in terms of the behavior of $\varphi(\zeta)$ near the real axis; they are called the *standard supports* (see [4]) of μ and are defined as follows ([8]):

$$\begin{aligned} S^0 &= \{\xi\colon \lim\varphi(\zeta) \text{ exists, is finite, and has positive imaginary part when } \zeta \to \xi \text{ in any angle}\},\\ S^1 &= \{\xi\colon \operatorname{Im}\varphi(\zeta) \to \infty \text{ when } \zeta \to \xi \text{ in any angle}\}.\end{aligned} \tag{III.3.3}$$

Remark. The supports S^0, S^1, as well as S_P^0, S_P^1, have the property that they cannot be diminished by any set of positive Lebesgue measure without ceasing to be supports of the corresponding measures (in fact S^1 and S_P^1 are already of Lebesgue measure zero). More generally, for any measures μ, ν, a support S of ν is called *minimal rel.* μ if and only if any support $S_1 \subset S$ of ν differs from S by a set of μ measure zero. It is clear that for any two measures μ, ν, minimal supports of ν rel. μ always exist.

This notion is important for the following reason: if two measures ν, ν' are absolutely continuous with respect to μ and have a common support minimal rel. μ, then ν and ν' are equivalent.

Minimal supports rel. Λ are called simply *minimal*. With this terminology, all of the above mentioned supports S^0, S^1, S_P^0, S_P^1 are minimal.

4. Perturbations of type one. Consider two systems $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim [\mathfrak{D}(B), \mathscr{H}, I, B]$, the perturbation being of type one. By definition III.1.1.1° we have a decomposition $\mathfrak{D}(A) = \mathfrak{D}(B) = \mathfrak{D} + [f]$ and an orthogonal projection P of $\mathscr{H}$ onto a one-dimensional subspace $[x]$, $\|x\| = 1$, such that $P(\mathfrak{D}) = 0$ and $Pf = x$. Denoting $\overline{\mathfrak{D}}$ by V, we see that

([8]) "$\zeta \to \xi$ in any angle" means that for some ε, $0 < \varepsilon < \pi/2$, ζ converges to ξ, remaining inside the angle $\varepsilon < \operatorname{Arg}(\zeta - \xi) < \pi - \varepsilon$.

$\mathscr{H} = \overline{\mathfrak{D}(A)} = V+[f]$, and $P(V) = 0$. Hence $\mathscr{H} = V \pm [x]$. Thus (see definition III.1.1.1°)

$$\mathfrak{D}(A) = \mathfrak{D}(B) = \mathfrak{D}+[f],$$

where

1° $\mathscr{H} = V\pm[x]$, $V = \overline{\mathfrak{D}}$, $\|x\| = 1$, $(f, x) = 1$;

2° $A = B$ on $\mathfrak{D}$ and $Bf-Af = \lambda_0 x$, with λ_0 real.

To calculate the determinant of the perturbation, we consider the linear functional (u, x) which is continuous in $\mathscr{H}$ and therefore, a fortiori, continuous on $\mathfrak{D}(A), \mathfrak{D}(B)$ with their respective graph norms. For $u = f$ we have $(f, x) = 1$. Hence (see equation (I.1.12))

$$\begin{aligned}\varphi_A(\zeta) &= \det(A_\zeta \backslash B_\zeta) = (A_\zeta^{-1}B_\zeta f, x)\\ &= \big(A_\zeta^{-1}(A_\zeta f+\lambda_0 x), x\big)\\ &= (f+\lambda_0 A_\zeta^{-1}x, x) = 1+\lambda_0(A_\zeta^{-1}x, x).\end{aligned}$$

Similarly, we obtain

$$\varphi_B(\zeta) = \det(B_\zeta \backslash A_\zeta) = 1-\lambda_0(B_\zeta^{-1}x, x).$$

Using the measures $d\nu_A(\lambda) = d\|E_A(\lambda)x\|^2$, $d\nu_B(\lambda) = d\|E_B(\lambda)x\|^2$, where $\{E_A(\lambda)\}, \{E_B(\lambda)\}$ are the resolutions of the identity corresponding to A, B respectively, we find

$$\text{(III.4.1)}\qquad \begin{aligned}\varphi_A(\zeta) &= \lambda_0\left(\frac{1}{\lambda_0}+\int_{-\infty}^{\infty}\frac{1}{\lambda-\zeta}\,d\nu_A(\lambda)\right),\\ \varphi_B(\zeta) &= -\lambda_0\left(\frac{-1}{\lambda_0}+\int_{-\infty}^{\infty}\frac{1}{\lambda-\zeta}\,d\nu_B(\lambda)\right).\end{aligned}$$

Since, by Corollary I.1.3′, $\varphi_A(\zeta)\varphi_B(\zeta) = 1$, we immediately see, using the standard (minimal) supports (III.3.3), that in the decomposition

$$\nu_A = \nu_A^0 \pm \nu_A^1, \qquad \nu_B = \nu_B^0\pm\nu_B^1,$$

the absolutely continuous parts ν_A^0, ν_B^0 are equivalent while the singular parts ν_A^1, ν_B^1 are orthogonal.

Consider now the closed subspace M_A generated by x relative A; i.e., the closure of the space spanned by $\{E_A(\lambda)x\colon \lambda \text{ real}\}$ (or by $\{A_\zeta^{-1}x\colon \operatorname{Im}\zeta \neq 0\}$). Then $M_A, M_A^\perp$ reduce A. Similarly we introduce the closed subspace M_B generated by x relative to B. We are going to prove that $M_A = M_B$, which means that $M_A, M_A^\perp$ reduces both A and B.

For this purpose note that if $u = B_\zeta^{-1}x$, then $u = d+\alpha f$, where $d \in \mathfrak{D}$, and hence

$$x = B_\zeta u = B_\zeta d+\alpha B_\zeta f = A_\zeta u+\alpha\lambda_0 x,$$

so that $u = (1-\alpha\lambda_0)A_\zeta^{-1}x \in M_A$. Thus $M_A \subset M_B$, and, similarly, $M_B\subset M_A$.

It follows that $\mathscr{H}_1 = M_A = M_B$, $\mathscr{H}_2 = M_A^\perp = M_B^\perp$ reduce both A and B, and, since $\mathscr{H}_2 \subset V$, that $A = B$ on $\mathfrak{D}(A)\cap\mathscr{H}_2 = \mathfrak{D}(B)\cap\mathscr{H}_2$. Let A_i, B_i be the parts of A, B respectively in $\mathscr{H}_i$, $i = 1, 2$. Then $A_2 = B_2$ and ν_A, ν_B are spectral measures corresponding to A_1, B_1 respectively.

Let $\mu_0 = \nu_A^1\pm\nu_B^1$. If $\mu \neq 0$ is orthogonal to μ_0, then (see Section III.3) either $m^{(A_1)}(\mu) = 0$ or $\mu \ll \nu_A$, in which case (μ being orthogonal to ν_A^1) $\mu \ll \nu_A^0$. But $\nu_A^0 \sim \nu_B^0$; hence $\mu \ll \nu_B^0 \ll \nu_B$, and so

$$m^{(A_1)}(\mu) = m^{(B_1)}(\mu) = 1.$$

The same argument holds if the roles of A_1 and B_1 are interchanged. On the other hand, for any $\mu\neq 0$, $m^{(A_1)}(\mu)$ is either zero or one and $m^{(B_1)}(\mu)$ is either zero or one. Hence $|m^{(A_1)}(\mu)-m^{(B_1)}(\mu)| \leqslant 1$. In view of Theorem III.2.2 we therefore have, with $\mu_0 = \nu_A^1+\nu_B^1$,

LEMMA III.4.1. *Theorem* III.1.3 *holds for perturbations of type one.*

5. Perturbations of type two. Let $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim [\mathfrak{D}(B), \mathscr{H}, I, B]$, the perturbation being of type two. Then A and B are proper self-adjoint extensions of a closed symmetric operator T with dense domain $\mathfrak{D}(T)$ in $\mathscr{H}$ and deficiency indices $(1, 1)$.

Let T^* be the adjoint of T. Then (see [13])

$$T\subset A, \qquad B\subset T^*,$$

and the domain $\mathfrak{D}(T^*)$ of T^*, provided with the graph norm $\|u\|_*^2 = \|u\|^2 + \|T^*u\|^2$, becomes a Hilbert space. $\mathfrak{D}(A), \mathfrak{D}(B)$ are closed subspaces of the Hilbert space $\mathfrak{D}(T^*)$.

Let $\mathfrak{D}_+ = N[T^*-iI]$, $\mathfrak{D}_- = N[T^*+iI]$ be the deficiency spaces for T. Then

$$\mathfrak{D}(T^*) = \mathfrak{D}(T)\pm\mathfrak{D}_+\pm\mathfrak{D}_-,$$

where the sum is orthogonal in the graph norm of $\mathfrak{D}(T^*)$. Since T has deficiency indices $(1,1)$, $\dim\mathfrak{D}_+ = \dim\mathfrak{D}_- = 1$. Therefore A, B are restrictions of T^* to domains

$$\mathfrak{D}(A) = \mathfrak{D}(T)\pm[f], \qquad \mathfrak{D}(B) = \mathfrak{D}(T)\pm[g]$$

respectively, where $f = d_+ + S_A d_+$, $g = d_+ + S_B d_+$, $0 \neq d_+ \in \mathfrak{D}_+$, and S_A, S_B are isometric isomorphisms of $\mathfrak{D}_+$ onto $\mathfrak{D}_-$.

Let $d_- = S_A d_+$. Then $S_B d_+ = \lambda d_-$, and $\|d_-\|_* = \|d_+\|_* = |\lambda|\,\|d_-\|_*$, implying $\lambda = e^{2i\alpha}$ is a complex number with modulus one. Thus $f = d_+ + d_-$, $g = d_+ + e^{2i\alpha}d_-$, $Af = T^*f = id_+ - id_-$, and $Bg = T^*g = id_+ - ie^{2i\alpha}d_-$.

Therefore the perturbation $A\sim B$ is represented in the Hilbert space $\mathfrak{D}(T^*)$ by

$$\text{(III.5.1)}\qquad \mathfrak{D}(A) = \mathfrak{D}(T)\pm[f], \qquad \mathfrak{D}(B) = \mathfrak{D}(T)\pm[g],$$

where:

1° $f = d_+ + d_-$, $g = d_+ + e^{2i\alpha} d_-$, with $0 \neq d_+ \in \mathfrak{D}_+$, $0 \neq d_- \in \mathfrak{D}_-$, and α real;

2° $A = B$ on $\mathfrak{D}$, $Af = id_+ - id_-$, and $Bg = id_+ - ie^{2i\alpha} d_-$.

Without loss of generality we may assume that $\|f\|_* = \|g\|_* = 1$. Using the canonical isomorphism S_{BA} determined by (III.5.1) we find that

$$\det(A_\zeta \diagdown B_\zeta) = (A_\zeta^{-1} B_\zeta S_{BA} f, f)_* = (A_\zeta^{-1}B_\zeta g, f)_*$$
$$= (A_\zeta^{-1}B_\zeta g, f) + (AA_\zeta^{-1}B_\zeta g, Af).$$

Using the fact that A is self-adjoint and that $A = A_\zeta + \zeta I$, we find

$$\det(A_\zeta\diagdown B_\zeta) = (B_\zeta g, Af) + \zeta(B_\zeta g, f) + (1+\zeta^2)(A_\zeta^{-1}B_\zeta g, f).$$

Next we express $B_\zeta g$ in terms of f and $A_\zeta f$:

(III.5.2) $$B_\zeta g = \tfrac{1}{2} i(1+\zeta^2)(1-e^{2i\alpha})f + \tfrac{1}{2}[(1+e^{2i\alpha}) + i\zeta(1-e^{2i\alpha})]A_\zeta f$$

and note that $\|Af\|^2 = \|T^*f\|^2 = \|f\|_*^2 - \|f\|^2 = 1-\|f\|^2$. It is then a matter of direct computation to verify that

(III.5.3) $\det(A_\zeta\diagdown B_\zeta)$
$$= e^{i\alpha}\sin\alpha\{\zeta+\cot\alpha+\zeta(1+\zeta^2)\|f\|^2 + (1+\zeta^2)(Af,f) + (1+\zeta^2)^2(A_\zeta^{-1}f,f)\}.$$

Let $\{E_A(\lambda)\}$ be the resolution of the identity corresponding to A and $d\nu_A(\lambda) = d\|E_A(\lambda)f\|^2$. Then

$$1 = \|f\|_*^2 = \int_{-\infty}^{\infty}(1+\lambda^2)\,d\nu_A(\lambda), \qquad \|f\|^2 = \int_{-\infty}^{\infty} d\nu_A(\lambda),$$

$$(Af,f) = \int_{-\infty}^{\infty} \lambda\, d\nu_A(\lambda), \quad \text{and} \quad (A_\zeta^{-1}f,f) = \int_{-\infty}^{\infty} \frac{1}{\lambda-\zeta}\, d\nu_A(\lambda).$$

Substituting into equation (III.5.3) and writing $d\mu_A(\lambda) = (1+\lambda^2)^2 \times d\nu_A(\lambda)$, we obtain, after some computation,

(III.5.4) $$\varphi_A(\zeta) = \det(A_\zeta\diagdown B_\zeta) = e^{i\alpha}\sin\alpha\left\{\cot\alpha + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda-\zeta} - \frac{\lambda}{1+\lambda^2}\right) d\mu_A(\lambda)\right\},$$

where

$$\int_{-\infty}^{\infty} \frac{1}{1+\lambda^2}\, d\mu_A(\lambda) = \|f\|_*^2 = 1.$$

Entirely analogous procedures, using the resolution of the identity $\{E_B(\lambda)\}$ for B and $d\nu_B(\lambda) = d\|E_B(\lambda)g\|^2$, give

$$\varphi_B(\zeta) = \det(B_\zeta\diagdown A_\zeta) = (B_\zeta^{-1}A_\zeta S_{AB} g, g)_*$$
$$= -e^{-i\alpha}\sin\alpha\left\{-\cot\alpha + \int_{-\infty}^{\infty}\left(\frac{1}{\lambda-\zeta} - \frac{\lambda}{1+\lambda^2}\right)d\mu_B(\lambda)\right\},$$

where

$$d\mu_B(\lambda) = (1+\lambda^2)^2 d\nu_B(\lambda) \quad\text{and}\quad \int_{-\infty}^{\infty} \frac{1}{1+\lambda^2}\,d\mu_B(\lambda) = \|g\|_*^2 = 1.$$

As in Section III.4, $\varphi_A(\zeta)\varphi_B(\zeta) = 1$, and it follows that the absolutely continuous parts μ_A^0, μ_B^0 of μ_A, μ_B respectively are equivalent, while the singular parts μ_A^1, μ_B^1 are orthogonal. The same relations will hold if we multiply all these measures by the factor $(1+\lambda^2)^{-2}$, obtaining the measures $\nu_A = \nu_A^0 \pm \nu_A^1$, $\nu_B = \nu_B^0 \pm \nu_B^1$, where $\nu_A^0 \sim \nu_B^0$ and $\nu_A^1 \perp \nu_B^1$.

Let M_A, M_B be the closures in $\mathscr{H}$ of the spaces spanned by $\{E_A(\lambda)f\}$, $\{E_B(\lambda)g\}$ (or by $\{A_\zeta^{-1}f\}, \{B_\zeta^{-1}g\}$) respectively. Since

$$d_+ = \frac{1}{2i}(A+i)f = \frac{1}{2i}(B+i)g,$$

$$d_- = \frac{-1}{2i}(A-i)f = -\frac{e^{-2i\alpha}}{2i}(B-i)g,$$

d_+, d_- belong to both M_A and M_B. Thus $f, g \in M_A \cap M_B$.

Let $u = B_\zeta^{-1}g$. Then $u = d+\beta g$, where $d\in\mathfrak{D}(T)$, and, using (III.5.2), we get

$$g = B_\zeta u = B_\zeta d + \beta B_\zeta g = A_\zeta d + \frac{\beta}{2}i(1+\zeta^2)(1-e^{2i\alpha})f +$$
$$+ \frac{\beta}{2}[(1+e^{2i\alpha})+i\zeta(1-e^{2i\alpha})]A_\zeta f.$$

Thus

$$d = A_\zeta^{-1}g - \frac{\beta}{2}i(1+\zeta^2)(1-e^{2i\alpha})A_\zeta^{-1}f - \frac{\beta}{2}[(1+e^{2i\alpha})+i\zeta(1-e^{2i\alpha})]f$$

belongs to M_A and so therefore does $u = d+\beta g$. It follows that $M_B\subset M_A$, and, similarly, $M_A\subset M_B$.

Thus $\mathscr{H}_1 = M_A = M_B$, $\mathscr{H}_2 = \mathscr{H}_1^\perp$ reduce both A and B. Let A_i, B_i be the parts of A, B respectively in $\mathscr{H}_i$, $i = 1,2$. Since $\mathfrak{D}(A_2) = \mathfrak{D}(A)\cap\mathscr{H}_2$ is orthogonal to $\mathfrak{D}(A_1) = \mathfrak{D}(A)\cap\mathscr{H}_1$ in the graph norm of A, $\mathfrak{D}(A_2)\subset\mathfrak{D}(T)$, and, similarly, $\mathfrak{D}(B_2)\subset\mathfrak{D}(T)$. Hence $\mathfrak{D}(A_2) = \mathfrak{D}(B_2)\subset\mathfrak{D}(T)$ and $A_2 = B_2$.

We then find, exactly as in Section III.4, with $\mu_0 = \nu_A^1 + \nu_B^1$:

LEMMA III.5.1. *Theorem* III.1.3 *holds for perturbations of type two.*

6. Perturbations of standard self-adjoint systems. Using Theorem III.1.2 and the results of the preceding sections we can now prove Theorem III.1.3. For we can, by Theorem III.1.2, achieve the given perturbation (I): $[\mathfrak{D}(A), \mathscr{H}, I, A] \sim$ (II): $[\mathfrak{D}(B), \mathscr{H}, I, B]$ of dimension m through precisely m one-dimensional perturbations

$$(\mathrm{I}) = (I_0) \sim (I_1) \sim \ldots \sim (I_m) = (\mathrm{II}),$$

each perturbation $(I_{j-1}) \sim (I_j)$ being of type one or two, $j = 1, \ldots, m$. For each of these perturbations we have, by Lemmas III.4.1 and III.5.1, a singular measure $\mu_0^{(j)}$ which has the properties required by Theorem III.1.3. For the perturbation (I) $\sim$ (II) we take $\mu_0 = \mu_0^{(1)} + \ldots + \mu_0^{(m)}$ and Theorem III.1.3 follows.

The proof of Corollary III.1.3′ is obtained as follows: Consider the absolutely continuous part A_A of A. For any $\mu \neq 0$ the multiplicity $m^{(A_A)}(\mu)$ is either $m^{(A)}(\mu)$ or zero, depending on whether or not μ is absolutely continuous. The same is true for multiplicities rel. B_A and B. By Theorem III.1.3, therefore, $m^{(A_A)}(\mu) = m^{(B_A)}(\mu)$ for every $\mu \neq 0$ and so A_A and B_A are unitarily equivalent.

We next consider the isolated eigenvalues for standard self-adjoint systems. As already noted, for such systems the quasi-resolvent set coincides with the meromorphy domain, and the essential spectrum and the isolated eigenvalues all lie on the real axis. For any interval $\mathscr{I}$ of R^1 (open, closed, or open at either end) which lies entirely in the meromorphy domain of the system $[\mathfrak{D}(A), \mathscr{H}, I, A]$, we denote by $n_A(\mathscr{I})$ the number of isolated eigenvalues of the system (each counted as many times as its multiplicity) lying in $\mathscr{I}$. We then have:

THEOREM III.6.1. *Let* (I): $[\mathfrak{D}(A), \mathscr{H}, I, A]$, $[\mathfrak{D}(B), \mathscr{H}, I, B]$ *be standard self-adjoint systems and let* (I) $\sim$ (II) *be a perturbation of dimension* m. *Then for every interval* $\mathscr{I}$ *contained in the meromorphy domain of* (I) (*and therefore of* (II))

$$(\mathrm{III.6.1}) \qquad |n_A(\mathscr{I}) - n_B(\mathscr{I})| \leqslant m$$

(*where, as in Theorem* III.1.3, (III.6.1) *is meant to imply that* $n_A(\mathscr{I})$ *is infinite if and only if* $n_B(\mathscr{I})$ *is, in which case they are equal; otherwise both are finite and satisfy inequality* (III.6.1)).

Proof. We first note that it is sufficient to prove the theorem for a closed interval $\mathscr{I}$ contained in the meromorphy domain $\mathscr{R}$, since any interval contained in $\mathscr{R}$ can be written as the increasing limit of closed intervals contained in $\mathscr{R}$. Next note that, since A and B are self-adjoint,

the generalized eigenspace of (I), (II) corresponding to a point $\lambda \in \mathscr{R}$ is simply $N[A_\lambda]$, $N[B_\lambda]$ respectively, and the multiplicity of λ as an eigenvalue of (I), (II) is $\alpha(A_\lambda)$, $\alpha(B_\lambda)$ respectively.

By Theorem II.6.2,

$$(\mathrm{III.6.2}) \qquad \alpha(B_\lambda) - \alpha(A_\lambda) = \text{index of } \det(A_\zeta \diagdown B_\zeta) \text{ at } \zeta = \lambda.$$

For $\mathscr{I}$ a closed interval contained in $\mathscr{R}$, let $p(\mathscr{I})$ be the number of poles and $z(\mathscr{I})$ the number of zeros of $\det(A_\zeta \diagdown B_\zeta)$ lying in $\mathscr{I}$ (each pole and zero counted as many times as its order). Then from (III.6.2) it follows that

$$(\mathrm{III.6.3}) \qquad n_B(\mathscr{I}) - n_A(\mathscr{I}) = z(\mathscr{I}) - p(\mathscr{I}).$$

By Theorem III.1.2 it suffices to consider the case that the perturbation (I) $\sim$ (II) is of dimension one. In this case (see equations (III.4.1) and (III.5.4)) $\det(A_\zeta \diagdown B_\zeta)$ is a non-zero multiple of a function in class P. Thus, if $\mathscr{I}$ is a closed interval contained in $\mathscr{R}$, then (see [6]) $\det(A_\zeta \diagdown B_\zeta)$ is analytic and non-zero in $\mathscr{I}$ except for finitely many simple poles and zeros. Moreover, between any two consecutive zeros in $\mathscr{I}$ there is exactly one pole, and between any two consecutive poles there is exactly one zero. Thus, in view of (III.6.3), $|n_B(\mathscr{I}) - n_A(\mathscr{I})| \leqslant 1$ and the theorem is proved.

Remark. In case A and B are bounded we can extend the results of Theorem III.6.1 to include intervals $\mathscr{I}$ on the projective real line (including ∞) which are outside the essential spectrum(⁹). The proof follows the same lines as the proof of Theorem III.6.1, except when we consider the poles and zeros in a one dimensional perturbation we use the fact that a function in class P is transformed by the mapping

$$\zeta' = \frac{a\zeta + b}{c\zeta + d} \quad \text{with } a, b, c, d \text{ real and } \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0,$$

into a function in class P, and that, moreover, the mapping can be chosen so as to transform an interval on the projective line into a bounded interval.

APPENDIX

The determination of the isolated elementary divisors and of the character of a perturbed system. Examples arising from ordinary differential equations with boundary conditions. In this appendix examples are given to show how the theorems and techniques of Sections 6 and 7 of Chapter II can be applied to eigenvalue problems arising from differential

(⁹) Such intervals, besides the usual intervals, include those of the form $\mathscr{I} = (b, \infty] \cup [-\infty, a)$, where $a < b$.

operators with not necessarily self-adjoint boundary conditions. For this purpose we use the Hilbert space $\mathscr{H} = P^4(-1,1)$; i.e., the perfect functional completion of $C^4([-1,1])$ with respect to the norm

$$\|x\|^2 = \int_{-1}^{1}\left|\frac{d^4x(t)}{dt^4}\right|^2 dt + \sum_{k=0}^{3}[|x^{(k)}(-1)|^2 + |x^{(k)}(1)|^2]. \tag{A.1}$$

The elements of $\mathscr{H}$ are functions in $C^3([-1,1])$ and have absolutely continuous third derivatives. (For these and other properties of $\mathscr{H}$ see [1], [10]; the properties we shall use can easily be proved directly however.)

We consider the differential operator d^4/dt^4 applied to functions in a closed subspace V of $\mathscr{H}$ determined by four linearly independent boundary conditions

$$\sum_{k=0}^{3}[p_{jk}x^{(k)}(-1) + q_{jk}x^{(k)}(1)] = 0, \qquad j = 1, \ldots, 4, \tag{A.2}$$

where the coefficients are constant.

Write $\mathscr{H}$ as an orthogonal direct sum:

$$\mathscr{H} = \mathfrak{D} \pm [1, t, \ldots, t^7].$$

Then it is easily seen that $\mathfrak{D}$ is the subspace of $\mathscr{H}$ composed of functions $x \in \mathscr{H}$ such that $x^{(k)}(-1) = x^{(k)}(1) = 0$, $k = 0, \ldots, 3$. Thus $\mathfrak{D}$ is a subspace of the space V determined by the boundary conditions (A.2), and V is therefore the direct sum of $\mathfrak{D}$ with a subspace of $[1, t, \ldots, t^7]$. In fact, the spaces V determined by boundary conditions of the form (A.2) are precisely those subspaces of $\mathscr{H}$ of the form

$$V = \mathfrak{D} \pm [x_1, \ldots, x_4], \tag{A.3}$$

where $x_1, \ldots, x_4$ are linearly independent elements of $[1, t, \ldots, t^7]$.

To see that V can be characterized in this way note that there is a 1-1 correspondence between $[1, t, \ldots, t^7]$ and C^8. For if

$$x = \sum_{k=0}^{7} r_k t^k \in [1, t, \ldots, t^7],$$

then x uniquely determines the vector

$$\xi = \big(x(-1), x'(-1), \ldots, x'''(-1), x(1), \ldots, x'''(1)\big) \in C^8.$$

Conversely, x is uniquely determined by ξ according to the formula

$$r = (r_0, r_1, \ldots, r_7) = \xi Q^{-1},$$

where Q is the non-singular 8×8 matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -2 & 2\cdot 1 & 0 & 1 & 2 & 2\cdot 1 & 0 \\ -1 & 3 & -3\cdot 2 & 3\cdot 2\cdot 1 & 1 & 3 & 3\cdot 2 & 3\cdot 2\cdot 1 \\ 1 & -4 & 4\cdot 3 & -4\cdot 3\cdot 2 & 1 & 4 & 4\cdot 3 & 4\cdot 3\cdot 2 \\ -1 & 5 & -5\cdot 4 & 5\cdot 4\cdot 3 & 1 & 5 & 5\cdot 4 & 5\cdot 4\cdot 3 \\ 1 & -6 & 6\cdot 5 & -6\cdot 5\cdot 4 & 1 & 6 & 6\cdot 5 & 6\cdot 5\cdot 4 \\ -1 & 7 & -7\cdot 6 & 7\cdot 6\cdot 5 & 1 & 7 & 7\cdot 6 & 7\cdot 6\cdot 5 \end{pmatrix}.$$

Thus, if V is the subspace determined by (A.2) then the 4×8 matrix $P = (p_{jk}, q_{jk})$ has rank four and its rows determine a four-dimensional subspace of C^8. Equation (A.2) determines the orthogonal complement to this subspace; to this four-dimensional complementary subspace corresponds a four-dimensional subspace $[x_1, \ldots, x_4]$ of $[1, t, \ldots, t^7]$; and V is given by (A.3). Conversely, if V is defined by equation (A.3), then the reverse argument gives a matrix $P = (p_{jk}, q_{jk})$ of rank four, and the boundary conditions (A.2) corresponding to P determine V.

In particular, let V be determined by the self-adjoint boundary conditions $x(\pm 1) = x''(\pm 1) = 0$. Then

$$V = \mathfrak{D} \pm [y_1, \ldots, y_4], \tag{A.4}$$

where

$$y_1(t) = 5 - 6t^2 + t^4, \qquad y_3(t) = 14 - 15t^2 + t^6,$$
$$y_2(t) = 7t - 10t^3 + 3t^5, \qquad y_4(t) = 6t - 7t^3 + t^7.$$

Let $G, H\colon V \to W = L^2([-1,1])$ be defined by

$$Gx = \frac{d^4x}{dt^4}, \qquad Hx = x \quad \text{for } x \in \mathfrak{D}(G) = \mathfrak{D}(H) = V.$$

Then (I): $[V, W, H, G]$ is a Hilbert system whose quasi-resolvent set consists of one component — the entire complex plane — of character $(0, 0)$. The isolated eigenvalues of (I) are

$$\lambda_n = \frac{\pi^4 n^4}{16}, \qquad n = 1, 2, \ldots,$$

and the corresponding eigenvectors are

$$u_{2m}(t) = \sin m\pi t \quad \text{for } n = 2m,$$
$$u_{2m-1}(t) = \cos(m - \tfrac{1}{2})\pi t \quad \text{for } n = 2m-1,\ m = 1, 2, \ldots,$$

each having multiplicity one.

Let V_1 be the closed subspace of $\mathscr{H}$ determined by linearly independent boundary conditions (A.2), and define $G_1, H_1\colon V_1 \to W$ by

$$G_1 x = \frac{d^4 x}{dt^4}, \quad H_1 x = x \quad \text{for } x \in \mathfrak{D}(G_1) = \mathfrak{D}(H_1) = V_1.$$

Then

$$V_1 = \mathfrak{D} \pm [x_1, \ldots, x_4], \tag{A.5}$$

where $x_1, \ldots, x_4$ are linearly independent elements of $[1, t, \ldots, t^7]$. Clearly, (II): $[V_1, W, H_1, G_1]$ is a Hilbert system and (I) $\sim$ (II). Thus we may apply the results of Sections II.6 and II.7 and the known properties of (I) to obtain information concerning (II).

As remarked in Section II.7, one method for discovering information about system (II) is to compute explicitly the null space of $B_\lambda = G_1 - \lambda H_1$ for $\lambda \in \boldsymbol{C}$ and therefore find directly the isolated eigenvalues and the character of the quasi-resolvent set for (II). For this purpose it is useful to note that if $u \in N[B_\lambda]$, then $d^4 u/dt^4 = \lambda u \in \mathscr{H}$ is continuous. Thus u is a classical solution of the differential equation $d^4 u/dt^4 = \lambda u$; i.e., for $\lambda \neq 0$,

$$u(t) = \sum_{l=0}^{3} \eta_l e^{i^l \mu t},$$

where $\mu = \lambda^{1/4}$, and the η_l's are so chosen that u satisfies (A.2).

Using the fact that

$$u^{(k)}(t) = \sum_{l=0}^{3} (i^l \mu)^k \eta_l e^{i^l \mu t}, \quad k = 0, 1, \ldots,$$

we see that equation (A.2) implies that the vector

$$\eta = \begin{pmatrix} \eta_0 \\ \vdots \\ \eta_3 \end{pmatrix}$$

must satisfy the equation $C(\lambda)\eta = 0$, where $C(\lambda) = \{c_{jl}(\lambda)\}$ and

$$c_{jl}(\lambda) = \sum_{k=0}^{3} (i^l \mu)^k [p_{jk} e^{-i^l \mu} + q_{jk} e^{i^l \mu}], \quad j = 1, \ldots, 4, \ l = 0, \ldots, 3.$$

The dimension n of $N[B_\lambda]$ is equal to the number of linearly independent solutions of $C(\lambda)\eta = 0$; i.e.

$$n = 4 - r(\lambda), \tag{A.6}$$

where $r(\lambda)$ is the rank of $C(\lambda)$. Thus investigation of the rank of $C(\lambda)$ for $\lambda \in \boldsymbol{C}$ will give directly the isolated eigenvalues and the character of the quasi-resolvent set for (II).

In order to apply Theorem II.6.1, however, the type of a matrix representation of the perturbation (I) $\sim$ (II) must be used. Accordingly, let $A_\lambda = G - \lambda H$, $B_\lambda = G_1 - \lambda H_1$, and S_{BA} be the canonical isomorphism of V onto V_1 determined by the decompositions (A.4) and (A.5). For $j = 1, \ldots, 4$, let

$$\tilde{y}_j = A_\lambda^{-1} B_\lambda S_{BA} y_j = A_\lambda^{-1} B_\lambda x_j. \tag{A.7}$$

Then

$$\tilde{y}_j = d_j + \sum_{k=1}^{4} y_k s_{kj},$$

where $d_j \in \mathfrak{D}$, and the matrix representation $S = (s_{kj})$ of $A_\lambda \sim B_\lambda$ is given by

$$S = \{(y_k, y_l)\}^{-1} \{(\tilde{y}_k, y_j)\}.$$

Since $\det\{(y_k, y_l)\} \neq 0$, S has the same type at λ as the matrix $\tilde{D}(\lambda) = \{(\tilde{y}_k, y_j)\}$, and it is therefore sufficient to compute $\tilde{D}(\lambda)$.

From (A.7), $A_\lambda \tilde{y}_j = B_\lambda x_j$ for $j = 1, \ldots, 4$. Hence $\tilde{y}_j = x_j + z_j$, where z_j satisfies the differential equation

$$\frac{d^4 z_j}{dt^4} - \lambda z_j = 0$$

and the boundary conditions $z_j(\pm 1) = -x_j(\pm 1)$, $z_j''(\pm 1) = -x_j''(\pm 1)$. Thus z_j, and therefore $\tilde{y}_j$, can be found explicitly in terms of x_j and its derivatives. Using equation (A.1) we find

$$\tilde{D}(\lambda) = \{(\tilde{y}_j, y_k)\} = XM(\lambda) + YN, \tag{A.8}$$

where:

1. X, Y are the 4×4 matrices (a_{jk}), (b_{jk}) respectively with components

$$\begin{aligned}
a_{j1} &= x_j(1) + x_j(-1), & b_{j1} &= x_j'(1) + x_j'(-1), \\
a_{j2} &= x_j''(1) + x_j''(-1), & b_{j2} &= x_j'''(1) + x_j'''(-1), \\
a_{j3} &= x_j(1) - x_j(-1), & b_{j3} &= x_j'(1) - x_j'(-1), \\
a_{j4} &= x_j''(1) - x_j''(-1), & b_{j4} &= x_j'''(1) - x_j'''(-1).
\end{aligned}$$

2. N is the non-singular matrix

$$N = \begin{pmatrix} 0 & -8 & 0 & 5032 \\ 0 & 480 & 0 & 1008 \\ -8 & 0 & 696 & 0 \\ 48 & 0 & 480 & 0 \end{pmatrix}.$$

3. $M(\lambda)$ is the 4×4 matrix

$$\begin{pmatrix}
A(\mu) + A(i\mu) & 0 & C(\mu) + C(i\mu) & 0 \\
\dfrac{1}{\mu^2}[A(\mu) - A(i\mu)] & 0 & \dfrac{1}{\mu^2}[C(\mu) - C(i\mu)] & 0 \\
0 & B(\mu) + B(i\mu) & 0 & D(\mu) + D(i\mu) \\
0 & \dfrac{1}{\mu^2}[B(\mu) - B(i\mu)] & 0 & \dfrac{1}{\mu^2}[D(\mu) - D(i\mu)]
\end{pmatrix}$$

with $\mu = \lambda^{1/4}$ and

$$A(\mu) = -4\mu(6\mu^2-1)\tanh\mu, \qquad C(\mu) = -12\mu(20\mu^2+29)\tanh\mu,$$
$$B(\mu) = -4\mu(60\mu^2-1)\coth\mu, \qquad D(\mu) = -4\mu(126\mu^2+629)\coth\mu.$$

Note that system (II) is uniquely determined by the 4×8 matrix (X, Y) of rank four. For, as seen above, system (II) is uniquely determined by the 4×8 matrix Z with linearly independent rows $\xi_1, \ldots, \xi_4$, where $\xi_j = \big(x_j(-1), \ldots, x_j'''(-1), x_j(1), \ldots, x_j'''(1)\big)$, and (X, Y) is the product of Z with an 8×8 non-singular matrix which is independent of $x_1, \ldots, x_4$. In the examples that follow we shall determine system (II) either by giving the boundary conditions (A.2) or the matrix (X, Y) explicitly.

By Theorems I.2.1 and II.4.1 the quasi-resolvent set of (I) always consists of one component — the entire complex plane — of character (n, n), where $0 \leqslant n \leqslant 4$. We investigate various possibilities for n.

Case A. From equation (A.6), $n = 4$ if and only if all components of $C(\lambda)$ are identically zero. But each such component is a linear combination of linearly independent exponentials of the form $e^{(a+ib)\mu}$ with polynomial coefficients, hence vanishes identically if and only if all the polynomial coefficients are identically zero. Since this is the case if and only if $p_{jk} = q_{jk} = 0$ for $j = 1, \ldots, 4$, $k = 0, \ldots, 3$, the quasi-resolvent set of (II) cannot have character $(4, 4)$ for any choice of linearly independent boundary conditions (A.2).

Case B. $n = 3$ if and only if all 2×2 minors of $C(\lambda)$ are identically zero. But as in case A, each such minor is a linear combination of linearly independent exponentials with polynomial coefficients. Expanding the 2×2 minors of $C(\lambda)$ and setting the proper coefficients equal to zero, we find, after simplification, that $n = 3$ if and only if all the 2×2 minors of $P = (p_{jk}, q_{jk})$ are zero. Since then the boundary conditions (A.2) cannot be linearly independent, (II) can never have character $(3, 3)$.

Case C. $n = 2$ if and only if all 3×3 minors of $C(\lambda)$ vanish identically. As in the preceding cases we compute all 3×3 minors of $C(\lambda)$ and set the appropriate coefficients equal to zero to obtain necessary and sufficient conditions on P that all 3×3 minors vanish identically. If we write

$$D(r_{i_1} r_{i_2} r_{i_3}) = 0$$

as abbreviation for

$$\begin{vmatrix} r_{ji_1} & r_{ji_2} & r_{ji_3} \\ r_{ki_1} & r_{ki_2} & r_{ki_3} \\ r_{li_1} & r_{li_2} & r_{li_3} \end{vmatrix} = 0 \quad \text{for all } j, k, l = 1, \ldots, 4,$$

then these necessary and sufficient conditions are:

$$\begin{aligned}
D(p_0p_1q_0) &= 0,\\
D(p_0p_1q_1) &= 0,\\
D(p_0p_1q_2) &= D(p_0p_3q_0)+D(p_1p_2q_0),\\
D(p_0p_2q_0) &= 0,\\
D(p_0p_2q_1) &= D(p_0p_3q_0)+D(p_1p_2q_0),\\
2D(p_0p_2q_2) &= D(p_0p_3q_1)+D(p_1p_2q_1)+D(p_0p_1q_3),\\
2D(p_0p_2q_3) &= D(p_0p_3q_2)+D(p_1p_2q_2)-D(p_2p_3q_0),\\
2D(p_1p_3q_0) &= D(p_0p_3q_1)+D(p_1p_2q_1)-D(p_0p_1q_3),\\
2D(p_1p_3q_1) &= D(p_0p_3q_2)+D(p_1p_2q_2)+D(p_2p_3q_0),\\
D(p_1p_3q_2) &= D(p_0p_3q_3)+D(p_1p_2q_3),\\
D(p_1p_3q_3) &= 0,\\
D(p_2p_3q_1) &= D(p_0p_3q_3)+D(p_1p_2q_3),\\
D(p_2p_3q_2) &= 0,\\
D(p_2p_3q_3) &= 0,
\end{aligned}$$

$$\begin{aligned}
D(q_0q_1p_0) &= 0,\\
D(q_0q_1p_1) &= 0,\\
D(q_0q_1p_2) &= D(q_0q_3p_0)+D(q_1q_2p_0),\\
D(q_0q_2p_0) &= 0,\\
D(q_0q_2p_1) &= D(q_0q_3p_0)+D(q_1q_2p_0),\\
2D(q_0q_2p_2) &= D(q_0q_3p_1)+D(q_1q_2p_1)+D(q_0q_1p_3),\\
2D(q_0q_2p_3) &= D(q_0q_3p_2)+D(q_1q_2p_2)-D(q_2q_3p_0),\\
2D(q_1q_3p_0) &= D(q_0q_3p_1)+D(q_1q_2p_1)-D(q_0q_1p_3),\\
2D(q_1q_3p_1) &= D(q_0q_3p_2)+D(q_1q_2p_2)+D(q_2q_3p_0),\\
D(q_1q_3p_2) &= D(q_0q_3p_3)+D(q_1q_2p_3),\\
D(q_1q_3p_3) &= 0,\\
D(q_2q_3p_1) &= D(q_0q_3p_3)+D(q_1q_2p_3),\\
D(q_2q_3p_2) &= 0,\\
D(q_2q_3p_3) &= 0,\\
D(q_0q_1q_2) &= -D(p_1p_2q_0),\\
2D(q_0q_1q_3) &= D(p_0p_3q_1)-D(p_1p_2q_1)-D(p_0p_1q_3),\\
2D(q_0q_2q_3) &= D(p_0p_3q_2)-D(p_1p_2q_2)-D(p_2p_3q_0),\\
D(q_1q_2q_3) &= -D(p_1p_2q_3),
\end{aligned}$$

$$\begin{aligned}
D(p_0p_1p_2) &= -D(q_1q_2p_0),\\
2D(p_0p_1p_3) &= D(q_0q_3p_1)-D(q_1q_2p_1)-D(q_0q_1p_3),\\
2D(p_0p_2p_3) &= D(q_0q_3p_2)-D(q_1q_2p_2)-D(q_2q_3p_0),\\
D(p_1p_2p_3) &= -D(q_1q_2p_3).
\end{aligned}$$

Thus $n = 2$ if and only if P is of rank four and its coefficients satisfy the above equations.

Example 1. Let (II) be defined by the boundary conditions

$$x(1)-x(-1) = 0, \qquad x''(1)-x''(-1) = 0,$$
$$x'(1)+x'(-1) = 0, \qquad x'''(1)+x'''(-1) = 0.$$

Then

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is of rank four and satisfies the above conditions; system (II) therefore has a quasi-resolvent set of character $(2, 2)$. Solving the differential equation $d^4x/dt^4-\lambda x = 0$ subject to the above boundary conditions, we find that for $\lambda \neq 0$

$$N[B_\lambda] = [\cos\mu t, \cosh\mu t], \qquad \mu = \lambda^{1/4},$$

while, for $\lambda = 0$,

$$N[B_\lambda] = N[G_1] = [1, t^2].$$

Thus $a(B_\lambda) = 2$ for all $\lambda \in C$ and (II) has no isolated eigenvalues.

Case D. $n = 1$ if and only if $\det C(\lambda) \equiv 0$ but all 3×3 minors of $C(\lambda)$ do not vanish identically. Proceeding as in the preceding cases, and writing $D(r_{i_1}r_{i_2}r_{i_3}r_{i_4}) = 0$ as an abbreviation for

$$\begin{vmatrix} r_{1i_1} & \dots & r_{1i_4} \\ \vdots & & \vdots \\ r_{4i_1} & \dots & r_{4i_4} \end{vmatrix} = 0,$$

we find the following necessary and sufficient conditions that P must satisfy in order that $\det C(\lambda) = 0$:

$$D(p_0p_1q_0q_1) = 0,$$
$$D(p_2p_3q_2q_3) = 0,$$
$$D(p_0p_1q_0q_2)-D(p_0p_2q_0q_1) = 0,$$
$$D(p_2p_3q_1q_3)-D(p_1p_3q_2q_3) = 0,$$
$$2D(p_0p_2q_0q_2)-D(p_0p_3q_0q_1)-D(p_0p_1q_0q_3)-D(p_0p_1q_1q_2)- \\ -D(p_1p_2q_0q_1) = 0,$$
$$2D(p_1p_3q_1q_3)-D(p_2p_3q_0q_3)-D(p_0p_3q_2q_3)-D(p_1p_2q_2q_3)- \\ -D(p_2p_3q_1q_2) = 0,$$
$$D(p_0p_3q_0q_2)+D(p_0p_1q_1q_3)+D(p_1p_2q_0q_2)-D(p_0p_2q_0q_3)- \\ -D(p_1p_3q_0q_1)-D(p_0p_2q_1q_2) = 0,$$

$$D(p_1p_3q_0q_3)+D(p_0p_2q_2q_3)+D(p_1p_3q_1q_2)-D(p_0p_3q_1q_3)- \\ -D(p_2p_3q_0q_2)-D(p_1p_2q_1q_3) = 0,$$
$$2D(p_1p_3q_0q_2)+2D(p_0p_2q_1q_3)-D(p_0p_3q_0q_3)-D(p_1p_2q_1q_2)- \\ -D(p_1p_2q_0q_3)-D(p_0p_1q_2q_3)-D(p_2p_3q_0q_1)-D(p_0p_3q_1q_2) = 0,$$
$$D(p_0p_1p_2q_0)+D(p_0q_0q_1q_2) = 0,$$
$$D(p_1p_2p_3q_3)+D(p_3q_1q_2q_3) = 0,$$
$$D(p_0p_1p_3q_0)+D(p_1q_0q_1q_2)-D(p_0p_1p_2q_1)-D(p_0q_0q_1q_3) = 0,$$
$$D(p_0p_2p_3q_3)+D(p_2q_1q_2q_3)-D(p_1p_2p_3q_2)-D(p_3q_0q_2q_3) = 0,$$
$$D(p_0p_2p_3q_0)+D(p_0q_0q_2q_3)-D(p_0p_1p_3q_1)-D(p_1q_0q_1q_3)+ \\ +D(p_0p_1p_2q_2)+D(p_2q_0q_1q_2) = 0,$$
$$D(p_0p_1p_3q_3)+D(p_3q_0q_1q_3)-D(p_0p_2p_3q_2)-D(p_2q_0q_2q_3)+ \\ +D(p_1p_2p_3q_1)+D(p_1q_1q_2q_3) = 0,$$
$$D(p_0p_1p_3q_2)-D(p_2q_0q_1q_3)-D(p_0p_1p_2q_3)+D(p_3q_0q_1q_2)- \\ -D(p_0p_2p_3q_1)+D(p_1q_0q_2q_3)+D(p_1p_2p_3q_0)-D(p_0q_1q_2q_3) = 0,$$
$$D(p_1p_3q_0q_2)+D(p_0p_2q_1q_3)-D(p_0p_3q_0q_3)-D(p_1p_2q_1q_2)+ \\ +2D(p_0p_1p_2p_3)+2D(q_0q_1q_2q_3) = 0.$$

Thus $n = 1$ if and only if P is of rank four, satisfies all the above equations, but does not satisfy all the equations of case C.

Example 2. Let

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{696} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & -\frac{10}{776} & \frac{29}{1552} \\ 0 & 0 & \frac{1}{776} & \frac{1}{4656} \\ 0 & 0 & 0 & 0 \\ \frac{-11}{50488} & \frac{105}{50488} & 0 & 0 \end{pmatrix}.$$

Then it is easily seen that

$$P = \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & \frac{50488}{11} & 0 & 0 & 1 & \frac{50488}{11} & 0 & 0 \\ -1 & 0 & 0 & -\frac{50488}{105} & 1 & 0 & 0 & -\frac{0\text{g}488}{105} \\ 0 & -\frac{29}{20} & -\frac{29}{20} & -1 & 0 & \frac{29}{20} & -\frac{29}{20} & 1 \end{pmatrix}$$

has rank four and satisfies the equations of case D but not those of case C. Hence the corresponding system (II) has character $(1, 1)$. In fact, direct computation shows that, for λ near 0 but $\neq 0$,

$$N[B_\lambda] = \left[\left(\frac{29}{20}\mu^2\cosh\mu - \left(\frac{29}{20}+\mu^2\right)\mu\sinh\mu\right)\cos\mu t + \right.$$
$$\left. + \left(\frac{29}{20}\mu^2\cos\mu - \left(\frac{29}{20}-\mu^2\right)\mu\sin\mu\right)\cosh\mu t\right], \qquad \mu = \lambda^{1/4}.$$

Since for $\lambda = 0$, however,

$$N[B_\lambda] = [1, t^2],$$

$\lambda = 0$ is an isolated eigenvalue of (II).

We shall use Theorem II.6.1 to investigate the isolated eigenvalue $\lambda = 0$. From equation (A.8)

$$\tilde{D}(\lambda) = \begin{pmatrix} d_{11}(\lambda) & 0 & 0 & 0 \\ d_{21}(\lambda) & 0 & d_{23}(\lambda) & 0 \\ d_{31}(\lambda) & 0 & d_{33}(\lambda) & 0 \\ 0 & d_{42}(\lambda) & 0 & d_{44}(\lambda) \end{pmatrix},$$

where

$$d_{11}(\lambda) = 1,$$
$$d_{21}(\lambda) = \frac{1}{696\mu^2}[A(\mu)-A(i\mu)],$$
$$d_{23}(\lambda) = \frac{1}{696\mu^2}[C(\mu)-C(i\mu)]+1,$$
$$d_{31}(\lambda) = A(\mu)+A(i\mu),$$
$$d_{33}(\lambda) = C(\mu)+C(i\mu),$$
$$d_{42}(\lambda) = [B(\mu)+B(i\mu)]+\frac{1}{\mu^2}[B(\mu)-B(i\mu)]+1,$$
$$d_{44}(\lambda) = [D(\mu)+D(i\mu)]+\frac{1}{\mu^2}[D(\mu)-D(i\mu)]+1.$$

Since $d_{11}(0) = 1 \neq 0$ and $d_{42}(0) = 9-\frac{1}{3}(1432) \neq 0$, $\tilde{D}(\lambda)$ can be transformed, without changing its type at $\lambda = 0$, into the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d_{23}(\lambda) & 0 \\ 0 & 0 & d_{33}(\lambda) & 0 \end{pmatrix}.$$

Since $d_{23}(\lambda) = \frac{14}{145}\lambda+\ldots$ and $d_{33}(\lambda) = -248\lambda+\ldots$, it follows immediately that $D(\lambda)$ has type $(1, \infty)$ at $\lambda = 0$, and thus, from Theorem II.6.1, that $\lambda = 0$ is an isolated eigenvalue of (II) to which there corresponds a single elementary divisor of order one.

Case E. $n = 0$ if and only if $\det C(\lambda)$ does not vanish identically; i.e., if and only if P is of rank four but does not satisfy the identities of case D.

Example 3. Let X be the 4×4 identity matrix and

$$Y = \begin{pmatrix} 1 & 0 & 63/16 & 0 \\ -8 & 0 & 696 & 0 \\ 0 & -7 & 0 & \frac{3175886}{631} \\ 0 & \frac{1432}{3} & 0 & \frac{8056}{3} \end{pmatrix} N^{-1}.$$

Then, since $\det X \neq 0$, the matrix (X, Y) has rank four and uniquely determines a system (II) $\sim$ (I). We shall investigate the point $\lambda = 0$ in the quasi-resolvent set of (II).

Equation (A.8) gives

$$D(\lambda) = \begin{pmatrix} d_{11}(\lambda) & 0 & d_{13}(\lambda) & 0 \\ d_{21}(\lambda) & 0 & d_{23}(\lambda) & 0 \\ 0 & d_{32}(\lambda) & 0 & d_{34}(\lambda) \\ 0 & d_{42}(\lambda) & 0 & d_{44}(\lambda) \end{pmatrix},$$

where

$$d_{11}(\lambda) = A(\mu)+A(i\mu)+1,$$
$$d_{21}(\lambda) = \frac{1}{\mu^2}[A(\mu)-A(i\mu)]-8,$$
$$d_{13}(\lambda) = C(\mu)+C(i\mu)+63/16,$$
$$d_{23}(\lambda) = \frac{1}{\mu^2}[C(\mu)-C(i\mu)]+696,$$
$$d_{32}(\lambda) = B(\mu)+B(i\mu)-7,$$
$$d_{42}(\lambda) = \frac{1}{\mu^2}[B(\mu)-B(i\mu)]+\frac{932}{3},$$
$$d_{34}(\lambda) = D(\mu)+D(i\mu)+\frac{3175886}{631},$$
$$d_{44}(\lambda) = \frac{1}{\mu^2}[D(\mu)-D(i\mu)]+\frac{8056}{3}.$$

Since $d_{11}(0) = d_{32}(0) = 1 \neq 0$, $D(\lambda)$ can be transformed, without changing its type for λ near 0, into the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Delta_1(\lambda) & 0 \\ 0 & 0 & 0 & \Delta_2(\lambda) \end{pmatrix},$$

where

$$\Delta_1(\lambda) = d_{11}(\lambda)d_{23}(\lambda) - d_{21}(\lambda)d_{13}(\lambda) = \frac{22330}{27}\lambda^2 + \ldots,$$

$$\Delta_2(\lambda) = d_{32}(\lambda)d_{44}(\lambda) - d_{42}(\lambda)d_{34}(\lambda)$$

$$= \frac{64\cdot 128\cdot 4\,730\,845\,509}{3\cdot 631\cdot 11!}\lambda^2 + \ldots$$

Thus $D(\lambda)$ has type $(2, 2)$ at $\lambda = 0$ and is without type for all λ near but not equal to 0. By Theorem II.6.1, therefore, the quasi-resolvent set of (II) has character $(0, 0)$, and $\lambda = 0$ is an isolated eigenvalue to which correspond two elementary divisors, each of order two.

Note that we could have proceeded here as in Example 2; i.e., we could have used X, Y to find a matrix P giving boundary conditions (A.2) for system (II). We could then have checked P against the equations of case D to see that the quasi-resolvent set of (II) has character $(0, 0)$, and computed $N[B_\lambda]$ directly in order to see that $\alpha(B_\lambda) = 2$ at $\lambda = 0$. Using Theorem II.6.2 we could then have discovered, without using Theorem II.6.1, that $\lambda = 0$ is an isolated eigenvalue corresponding to which there are two elementary divisors, whose orders sum to four. But we could not have known without Theorem II.6.1 that both elementary divisors are of order two; i.e., that there is not one elementary divisor of order one, the other being of order three.

Example 4. Let X be the 4×4 identity matrix and

$$Y = \begin{pmatrix} 0 & 0 & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ b_{31} & b_{32} & 0 & 0 \\ b_{41} & b_{42} & 0 & 0 \end{pmatrix},$$

where

$$b_{13} = \frac{1}{2} + \frac{\pi}{4}\tanh\frac{\pi}{2} - \frac{k_1}{4}\left[\frac{3\pi^3}{2}\left(15 - \frac{\pi^4}{8}\right)\tanh\frac{\pi}{2} + \frac{\pi^8}{2^5} + 15\pi^4 - \frac{9\pi^6}{8} - 45\pi^2\right],$$

$$b_{14} = -\frac{3\pi^2}{8} + \frac{\pi^3}{16}\tanh\frac{\pi}{2} + \\ + \frac{k_1\pi^2}{16}\left[\frac{3\pi^3}{2}\left(15 - \frac{\pi^4}{8}\right)\tanh\frac{\pi}{2} + \frac{\pi^8}{2^5} + 15\pi^4 - \frac{9\pi^6}{8} - 45\pi^2\right],$$

$$b_{23} = \frac{4}{\pi^2}\left\{-\frac{1}{2} + \frac{\pi}{4}\tanh\frac{\pi}{2} - \\ - \frac{k_1}{4}\left[\frac{3\pi^3}{2}\left(15 - \frac{\pi^4}{8}\right)\tanh\frac{\pi}{2} - \frac{3\pi^8}{2^6} + \frac{15\pi^4}{4} - \frac{3\pi^6}{4} + 45\pi^2\right]\right\},$$

$$b_{24} = \frac{4}{\pi^2}\left\{\frac{3\pi^2}{8} + \frac{\pi^3}{16}\tanh\frac{\pi}{2} + \\ + \frac{k_1\pi^2}{16}\left[\frac{3\pi^3}{2}\left(15 - \frac{\pi^4}{8}\right)\tanh\frac{\pi}{2} - \frac{3\pi^8}{2^6} + \frac{15\pi^4}{4} - \frac{3\pi^6}{4} + 45\pi^2\right]\right\},$$

$$b_{31} = \frac{\pi}{4}\coth\frac{\pi}{2} + \frac{5}{4\cdot 6311}k_2\Delta_{32}^{(1)},$$

$$b_{32} = \frac{\pi^3}{16}\coth\frac{\pi}{2} + \frac{1}{48\cdot 6311}k_2\Delta_{32}^{(1)},$$

$$b_{41} = \frac{1}{\pi}\coth\frac{\pi}{2} + \frac{5}{4\cdot 6311}k_2\Delta_{42}^{(1)},$$

$$b_{42} = \frac{\pi}{4}\coth\frac{\pi}{2} + \frac{1}{48\cdot 6311}k_2\Delta_{42}^{(1)},$$

with

$$k_1 = \left(\frac{\pi^8}{16^2} - \frac{3\pi^6}{16} + \frac{15\pi^4}{4} - \frac{45\pi^2}{2}\right)^{-1},$$

$$k_2 = -\frac{\Delta_{32}^{(1)}\Delta_{44}^{(1)} - \Delta_{42}^{(1)}\Delta_{34}^{(1)}}{\Delta_{42}^{(1)}\Delta_{32}^{(2)} - \Delta_{32}^{(1)}\Delta_{42}^{(2)}},$$

and

$$\Delta_{32}^{(1)} = -\frac{8}{\pi}\left(45 - \frac{1}{\pi^2}\right)\coth\frac{\pi}{2} + 4\left(15 - \frac{1}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2} - 4\left(15 + \frac{1}{\pi^2}\right),$$

$$\Delta_{32}^{(2)} = \frac{1}{2}\left\{\frac{96}{\pi^5}\left(15 - \frac{1}{\pi^2}\right)\coth\frac{\pi}{2} + \frac{2^4}{\pi^4}\left(45 + \frac{1}{\pi^4}\right)\operatorname{csch}^2\frac{\pi}{2} - \frac{2^4}{\pi^4}\left(45 - \frac{1}{\pi^2}\right) - \\ - \frac{2^4}{\pi^3}\left(15 - \frac{1}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2}\coth\frac{\pi}{2}\right\},$$

$$\Delta_{42}^{(1)} = -\frac{2^5}{\pi^3}\left(15+\frac{1}{\pi^2}\right)\coth\frac{\pi}{2}+\frac{16}{\pi^2}\left(15-\frac{1}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2}+\frac{16}{\pi^2}\left(15+\frac{1}{\pi^2}\right),$$

$$\Delta_{42}^{(2)} = \frac{1}{2}\left\{\frac{5\cdot 2^7}{\pi^7}\left(9+\frac{1}{\pi^2}\right)\coth\frac{\pi}{2}-\frac{5\cdot 2^6}{\pi^6}\left(3-\frac{1}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2}-\right.$$
$$\left.-\frac{5\cdot 2^6}{\pi^6}\left(3+\frac{1}{\pi^2}\right)-\frac{2^6}{\pi^5}\left(15-\frac{1}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2}\coth\frac{\pi}{2}\right\},$$

$$\Delta_{34}^{(1)} = -\frac{4}{\pi}\left(189+\frac{1258}{\pi^2}\right)\coth\frac{\pi}{2}+2\left(63+\frac{1258}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2}-$$
$$-2\left(63-\frac{1258}{\pi^2}\right),$$

$$\Delta_{44}^{(1)} = -\frac{16}{\pi^3}\left(63-\frac{1258}{\pi^2}\right)\coth\frac{\pi}{2}+\frac{8}{\pi^2}\left(63+\frac{1258}{\pi^2}\right)\operatorname{csch}^2\frac{\pi}{2}+\frac{8}{\pi^2}\left(63-\frac{1258}{\pi^2}\right).$$

(Note that $\Delta_{42}^{(1)}\Delta_{32}^{(2)}-\Delta_{32}^{(1)}\Delta_{42}^{(2)}\neq 0$, so k_2 is finite.)

Then (X, Y) has rank four and X, Y therefore uniquely defines a system (II) $\sim$ (I). We shall use Theorem II.6.1 to investigate the point $\lambda_1 = \pi^4/16$ as an isolated eigenvalue of (II). Since $\lambda_1 = \pi^4/16$ is an isolated eigenvalue of (I), however, Theorem II.6.1 does not apply directly to the perturbation (I) $\sim$ (II) and we must use the procedure outlined in case B of Section II.7.

Corresponding to the isolated eigenvalue $\lambda_1 = \pi^4/16$ of (I) is the simple elementary divisor $[y_5]$, where $y_5(t) = \cos \pi t/2$. Let

$$p(t) = \frac{\pi}{32}\left(5-\frac{\pi^2}{4}\right)y_1(t)-\frac{\pi}{32}\left(1-\frac{\pi^2}{12}\right)y_3(t),$$
$$x_5(t) = y_5(t)-p(t).$$

It is easily seen that $x_5\in\mathfrak{D}$ and thus, for $j = 1, \ldots, 4$, $(x_5, x_j) = (x_5, y_j) = 0$, while

$$(x_5, y_5) = (x_5, x_5) = \frac{\pi^8}{16^2}-\frac{3\pi^6}{16}+\frac{15\pi^4}{4}-\frac{45\pi^2}{2} = \frac{1}{k_1}.$$

Write $\mathfrak{D}$ as an orthogonal direct sum

$$\mathfrak{D} = \mathfrak{D}'\pm[x_5].$$

Then

$$V = \mathfrak{D}'\pm[y_1, \ldots, y_5], \quad V_1 = \mathfrak{D}'\pm[x_1, \ldots, x_5], \tag{A.9}$$

and we can define a new operator $\hat{G}$: $V\to W$ by

$$\hat{G} = \begin{cases} G & \text{on } \mathfrak{D}'\pm[y_1, \ldots, y_4], \\ G-\varepsilon H & \text{on } [y_5], \end{cases}$$

$\varepsilon\neq 0$ being arbitrary but fixed. Then $(\hat{\mathrm{I}})$: $[V, W, H, \hat{G}]$ is a Banach system, and it is clear from (A.9) that $(\hat{\mathrm{I}})\sim$ (II). However, $\lambda_1 = \pi^4/16$ is not an isolated eigenvalue of $(\hat{\mathrm{I}})$ so we may use Theorem II.6.1 and a matrix representation of $(\hat{\mathrm{I}})\sim$ (II) to investigate $\lambda_1 = \pi^4/16$ as an eigenvalue of (II).

Just as in the case of the perturbation (I) $\sim$ (II) we see that it is sufficient to examine the type at $\lambda_1 = \pi^4/16$ of the 5×5 matrix $\hat{D}(\lambda) = \{(\hat{y}_j, y_k)\}$, where $\hat{y}_j = \hat{A}_\lambda^{-1}B_\lambda x_j$, $j = 1, \ldots, 5$, and

$$\hat{A}_\lambda = \hat{G}-\lambda H = \begin{cases} A_\lambda & \text{on } \mathfrak{D}'\pm[y_1, \ldots, y_4], \\ A_{\lambda+\varepsilon} & \text{on } [y_5]. \end{cases}$$

For $y\in V$, we see that

$$y = \frac{(y, x_5)}{(x_5, x_5)}x_5+d,$$

with $d\in\mathfrak{D}'\pm[y_1, \ldots, y_4]$. Since $\hat{A}_\lambda x_5 = \hat{A}_\lambda y_5-A_\lambda p = A_\lambda x_5-\varepsilon y_5$, therefore

$$\hat{A}_\lambda y = A_\lambda y-\varepsilon\frac{(y, x_5)}{(x_5, x_5)}y_5.$$

Hence for $j = 1, \ldots, 5$,

$$B_\lambda x_j = \hat{A}_\lambda\hat{y}_j = A_\lambda\hat{y}_j-\varepsilon\frac{(\hat{y}_j, x_5)}{(x_5, x_5)}y_5.$$

It follows that $\hat{y}_j = x_j+\hat{z}_j$, where $\hat{z}_j$ satisfies the differential equation

$$\frac{d^4\hat{z}_j}{dt^4}-\lambda\hat{z}_j = \varepsilon\frac{(\hat{y}_j, x_5)}{(x_5, x_5)}y_5,$$

and the boundary conditions $\hat{z}_j(\pm 1) = -x_j(\pm 1)$, $\hat{z}_j''(\pm 1) = -x_j''(\pm 1)$. Thus

$$\hat{y}_j = x_j+z_j+\frac{\varepsilon(\hat{y}_j, x_5)}{(\lambda_1-\lambda)(x_5, x_5)}y_5, \tag{A.10}$$

where z_j satisfies the homogeneous differential equation $d^4z_j/dt^4-\lambda z_j = 0$ and the boundary conditions $z_j(\pm 1) = -x_j(\pm 1)$, $z_j''(\pm 1) = -x_j''(\pm 1)$. In particular, $z_5(t)\equiv 0$ and $z_1, \ldots, z_4$ are the same functions used earlier to compute $\tilde{D}(\lambda)$.

Since for $j = 1, \ldots, 4$, $(x_j, x_5) = 0$, and $(y_5, x_5) = (x_5, x_5)$, equation (A.10) gives

$$(\hat{y}_j, x_5) = (z_j, x_5)+\frac{\varepsilon}{\lambda_1-\lambda}(\hat{y}_j, x_5),$$

or

$$(\hat{y}_j, x_5) = \frac{\lambda_1 - \lambda}{\lambda_1 - (\lambda + \varepsilon)} (z_j, x_5), \quad j = 1, \ldots, 4.$$

Similarly,

$$(\hat{y}_5, x_5) = \frac{\lambda_1 - \lambda}{\lambda_1 - (\lambda + \varepsilon)} (x_5, x_5).$$

Thus

$$\hat{y}_j = \begin{cases} \tilde{y}_j + \dfrac{\varepsilon}{\lambda_1 - (\lambda + \varepsilon)} \dfrac{(z_j, x_5)}{(x_5, x_5)} y_5, & j = 1, \ldots, 4, \\ x_5 + \dfrac{\varepsilon}{\lambda_1 - (\lambda + \varepsilon)} y_5, & j = 5, \end{cases}$$

and we may compute $\hat{D}(\lambda)$ explicitly to obtain

$$\hat{D}(\lambda) = \begin{pmatrix} \hat{d}_{11}(\lambda) & 0 & \hat{d}_{13}(\lambda) & 0 & \hat{d}_{15}(\lambda) \\ \hat{d}_{21}(\lambda) & 0 & \hat{d}_{23}(\lambda) & 0 & \hat{d}_{25}(\lambda) \\ 0 & \hat{d}_{32}(\lambda) & 0 & \hat{d}_{34}(\lambda) & 0 \\ 0 & \hat{d}_{42}(\lambda) & 0 & \hat{d}_{44}(\lambda) & 0 \\ \hat{d}_{51}(\lambda) & 0 & \hat{d}_{53}(\lambda) & 0 & \hat{d}_{55}(\lambda) \end{pmatrix},$$

where, using the above formulas and the values for $(\tilde{y}_j, y_k)$, $j, k = 1, \ldots, 4$, computed earlier, we can obtain explicit expressions for the $\hat{d}_{jk}(\lambda)$'s as analytic functions of λ. We are interested in the behavior of $\hat{D}(\lambda)$ in a neighborhood of λ_1.

We notice that by interchanging rows and columns we can transform $\hat{D}(\lambda)$ into the form

$$\begin{pmatrix} \hat{d}_{32}(\lambda) & \hat{d}_{34}(\lambda) & 0 & 0 & 0 \\ \hat{d}_{42}(\lambda) & \hat{d}_{44}(\lambda) & 0 & 0 & 0 \\ 0 & 0 & \hat{d}_{11}(\lambda) & \hat{d}_{13}(\lambda) & \hat{d}_{15}(\lambda) \\ 0 & 0 & \hat{d}_{21}(\lambda) & \hat{d}_{23}(\lambda) & \hat{d}_{25}(\lambda) \\ 0 & 0 & \hat{d}_{51}(\lambda) & \hat{d}_{53}(\lambda) & \hat{d}_{55}(\lambda) \end{pmatrix}.$$

It follows that to find the type of $\hat{D}(\lambda)$ at λ_1 it is sufficient to find the types of the two minors

$$\Delta_1(\lambda) = \begin{pmatrix} \hat{d}_{32}(\lambda) & \hat{d}_{34}(\lambda) \\ \hat{d}_{42}(\lambda) & \hat{d}_{44}(\lambda) \end{pmatrix},$$

$$\Delta_2(\lambda) = \begin{pmatrix} \hat{d}_{11}(\lambda) & \hat{d}_{13}(\lambda) & \hat{d}_{15}(\lambda) \\ \hat{d}_{21}(\lambda) & \hat{d}_{23}(\lambda) & \hat{d}_{25}(\lambda) \\ \hat{d}_{51}(\lambda) & \hat{d}_{53}(\lambda) & \hat{d}_{55}(\lambda) \end{pmatrix}.$$

By developing the elements in power series about λ_1 and calculating the determinants, we check that $d_{34}(\lambda_1) \neq 0$ and that $\det \Delta_1(\lambda) = k_3(\lambda - \lambda_1)^3 + \ldots$, $k_3 \neq 0$. Thus the degrees of the matrix $\Delta_1(\lambda)$ at λ_1 are 0 and 3, and $\Delta_1(\lambda)$ has only one type exponent at λ_1; namely, 3.

Next we find $\det \Delta_2(\lambda) = k_4(\lambda - \lambda_1) + \ldots$, $k_4 \neq 0$, from which it follows that the degrees of $\Delta_2(\lambda)$ at λ_1 are 0, 0, and 1. Hence $\Delta_2(\lambda)$ has only one type exponent at λ_1; namely, 1.

It is now easy to see that the type of $\hat{D}(\lambda)$ at λ_1 is $(1, 3)$. Hence system (II) has an isolated eigenvalue at λ_1, to which there correspond two elementary divisors — one of order one, and one of order three.

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UNIVERSITY OF KANSAS
DEPARTMENT OF MATHEMATICS

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On subharmonicity inequalities involving solutions of generalized Cauchy-Riemann equations

by

R. R. COIFMAN and GUIDO WEISS (St. Louis, Mo.)

Suppose $F = (F_1, F_2, \ldots, F_k)$ is a system of C^1 real-valued functions defined in a domain $U \subset R^n$ ($=$ n-dimensional Euclidean space) satisfying the partial differential equations

$$\sum_{=1}^{n} A_j \frac{\partial F}{\partial x_j} = 0, \tag{1.1}$$

where A_j is a $l \times k$ constant matrix and $\partial F/\partial x_j$ is the (column) vector having components $\partial F_i/\partial x_j$, $i = 1, 2, \ldots, k$. We say that the system of partial differential equations (1.1) is a *generalized Cauchy-Riemann* (GCR) *system* if each solution $F = (F_1, F_2, \ldots, F_k)$ has harmonic components F_i, $i = 1, 2, \ldots, k$. When $k = l = n = 2$, a linear change of variables reduces such a system to the ordinary Cauchy-Riemann equations.

Several systems of partial differential equations that generalize, in one way or the other, the Cauchy-Riemann equations have been studied by Stein and Weiss [4], [5] and Calderón and Zygmund [2] in connection with various extensions of the theory of H^p-spaces. Each of these systems is a particular example of a GCR-system. The basic fact, common to all solutions of these equations, enabling one to develop the theory of H^p-spaces is the existence of a positive $p < 1$ such that

$$|F|^p = \Big(\sum_{i=1}^{k} |F_i|^2\Big)^{p/2}$$

is subharmonic (see [4]). A. P. Calderón observed that the existence of such a p is the consequence of the ellipticity of system (1.1). More precisely, system (1.1) is called *elliptic* provided

$$\sum_{j=1}^{n} \lambda_j A_j v = 0 \tag{1.2}$$