whence after some transformulations analogous to the transmutations made in [5], we obtain

$$x^*\big(\Lambda'(u_0)\cdot u_0\big) \geqslant x^*\big(\Lambda'(u_0)\cdot u\big)$$

for all $u \in G$, q.e.d.

Theorem 2 is a generalization of Pontriagin's maximum principle. It does not give the fact that $x^*(\sigma)$ is the solution of the conjugate equation to (1), which holds for the classical Pontriagin's maximum principle [4] and which holds for the problem presented in [5].

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Banach limits in vector lattices

by

A. L. PERESSINI (Urbana, Ill.)

S. Banach ([2], p. 34) defined generalized limits (now commonly referred to as Banach limits) as positive, linear, shift-invariant functionals on l^∞ which assign the value 1 to the constant sequence with terms equal to 1. Lorentz [4] investigated the linear subspace of l^∞ consisting of those sequences in l^∞ to which all Banach limits assigned the same value; such sequences were termed almost convergent. In this paper, we consider the extension of the concepts of Banach limit and almost convergence to vector-valued sequences. In addition to seeking generalizations of the known results for the case of real-valued sequences, we also consider a number of new questions which do not arise in the classical case. Our primary objective will be to study the geometric structure of the space of almost convergent sequences and the set of Banach limits in this more general context. Notation and terminology concerning ordered vector spaces will follow Peressini [5].

1. If E is a vector lattice, then the collection $\omega(E)$ of all sequences $\vec{x} = (x_n)$ such that $x_n \in E$ for all n is a vector lattice for the usual "coordinatewise" definitions of the linear operations and order. $l^\infty(E)$ will denote the linear subspace of $\omega(E)$ consisting of all order bounded sequences, that is, all sequences $\vec{x} = (x_n)$ for which there exist y, z in E such that $y \leqslant x_n \leqslant z$ for all n. A linear mapping $L\colon l^\infty(E) \to E$ is a *Banach limit on E* if

(1) L is positive.

(2) L is shift-invariant (i.e. $L(\vec{x}) = L(\sigma\vec{x})$, where σ is the "left-shift" on $l^\infty(E)$ defined by $\sigma\big((x_n)\big) = (x_{n+1})$.

(3) If $c \in E$ and $\vec{c}$ is the constant sequence with n^{th} term c, then $L(\vec{c}) = c$.

It follows from (2) that $L(\vec{x}) = L(\sigma^k\vec{x})$ for each natural number k, where σ^k denotes the k^{th}-iterate of σ.

If E is an order complete vector lattice and if $\vec{x} = (x_n) \in l^\infty(E)$, then

$$\underline{\lim}\, x_n = \sup_k \inf_{n\geqslant k}\{x_n\} \quad \text{and} \quad \overline{\lim}\, x_n = \inf_k \sup_{n\geqslant k}\{x_n\}$$

are defined and $\vec{x} = (x_n)$ is *order convergent* to $x \in E$ (written $\operatorname{o-lim}_n x_n = x$) if $x = \overline{\lim}\, x_n = \underline{\lim}\, x_n$.

PROPOSITION 1. *If L is any Banach limit on an order complete vector lattice E, then for each $\vec{x} = (x_n) \in l^\infty(E)$*

$$\underline{\lim}\, x_n \leqslant L(\vec{x}) \leqslant \overline{\lim}\, x_n.$$

Proof. Properties (1) and (3) for L imply that

$$(*)\qquad \inf_n x_n \leqslant L(\vec{x}) \leqslant \sup_n x_n.$$

Consequently, since $L(\vec{x}) = L(\sigma^k \vec{x})$ for each natural number k and since $\inf_{n \geqslant k} x_n = \inf_n \big(\sigma^k(x_n)\big)$, $\sup_{n \geqslant k} x_n = \sup_n \big(\sigma^k(x_n)\big)$, $(*)$ implies that

$$\inf_{n \geqslant k} x_n \leqslant L\big(\sigma^k(x_n)\big) = L(\vec{x}) \leqslant \sup_{n \geqslant k} x_n$$

for each natural number k. This clearly implies the desired result.

In keeping with the terminology used by Lorentz [4] for real-valued sequences, we call a sequence $(x_n) \in l^\infty(E)$ *almost convergent to* $x \in E$ if $L\big((x_n)\big) = x$ for every Banach limit L on E; in this case, we write $x = \operatorname{a-lim} x_n$. The collection $\operatorname{ac}(E)$ of all almost convergent sequences in E is obviously a linear subspace of $l^\infty(E)$. Proposition 1 shows that if E is an order complete vector lattice, then $\operatorname{ac}(E)$ contains the linear subspace $\operatorname{oc}(E)$ of all order convergent sequences in E and $\operatorname{a-lim} x_n = \operatorname{o-lim} x_n$ for $(x_n) \in \operatorname{oc}(E)$. If $0 \neq e \in E$ and $x_n = (-1)^{n+1} e$ for all n, then $\vec{x} = (x_n) \in l^\infty(E)$ and $2L(\vec{x}) = L(\vec{x}) + L(\sigma\vec{x}) = 0$ yet $L(|\vec{x}|) = e$ for any Banach limit L. Thus, $L(|\vec{x}|) \neq |L(\vec{x})|$ for any Banach limit L, that is, no Banach limit is a lattice homomorphism; however, the restriction of any Banach limit to $\operatorname{oc}(E)$ is a lattice homomorphism.

2. In this section, we shall assume that E is an order complete vector lattice. Then $l^\infty(E)$ is also an order complete vector lattice and the linear subspace $\operatorname{ac}(E)$ of almost convergent sequences contains the vector sublattice $\operatorname{oc}(E)$ of all order convergent sequences. In this setting, the existence of Banach limits as well as Lorentz's criterion for almost convergence can be established by making use of suitable modifications of results and constructions used for the case of real sequences. For this reason, we shall not duplicate the entire argument in the proof of the following proposition, but only those parts where modifications of the technique for real sequences are required.

PROPOSITION 2. *If E is an order complete vector lattice, then Banach limits exist on E. A necessary and sufficient condition for $\vec{x} = (x_n) \in l^\infty(E)$ to be almost convergent to $x \in E$ is that*

$$x = \operatorname{o-lim}_n \Big(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+j}\Big),$$

where the convergence is uniform in j.

Proof. Suppose that $\vec{x} = (x_n) \in l^\infty(E)$. In analogy to the construction in Sucheston [7], we define

$$c_n = \sup_j \Big(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+j}\Big)$$

for each natural number n, and note as in the case of real sequences that

$$\overline{\lim_k}\, c_{r+km} \leqslant c_m$$

for each pair of natural numbers r, m. For fixed m, define a sequence $c^{(r)} \in l^\infty(E)$ for $r = 1, \ldots, m$ by

$$c_p^{(r)} = \begin{cases} c_{r+(k-1)m} & \text{for } r+(k-1)m \leqslant p < r+km \quad (k = 1, 2, \ldots), \\ c_r & \text{for } p < r. \end{cases}$$

Then

$$(*)\qquad \sup_{q \geqslant n} \{c_q\} = \sup_q \{\sigma^{n-1}(c_q)\} = \sup_{1 \leqslant r \leqslant m} \big\{\sup_q \{\sigma^{n-1}(c_q^{(r)})\}\big\}$$

for each natural number n. But

$$\overline{\lim_q}\,(c_q^{(r)}) = \overline{\lim_k}\, c_{r+km} \leqslant c_m \quad \text{for } r = 1, \ldots, m;$$

hence, $(*)$ implies that $\overline{\lim}\, c_n \leqslant c_m$ for each natural number m. Therefore $\overline{\lim}\, c_n \leqslant \underline{\lim}\, c_n$, that is, $\{c_n\}$ is an order convergent sequence.

Define a mapping $p\colon l^\infty(E) \to E$ by

$$(1)\qquad p(\vec{x}) = \operatorname{o-lim}_n \Big(\sup_j \Big[\frac{1}{n} \sum_{i=0}^{n-1} x_{i+j}\Big]\Big), \qquad \vec{x} = (x_n) \in l^\infty(E);$$

then p is a sublinear mapping and the linear mapping l defined on the linear subspace $\operatorname{oc}(E)$ of $l^\infty(E)$ by

$$l(\vec{x}) = \operatorname{o-lim}_n x_n, \qquad \vec{x} = (x_n) \in \operatorname{oc}(E)$$

satisfies $l(\vec{x}) \leqslant p(\vec{x})$ for all $\vec{x} \in \mathrm{oc}(E)$. Consequently, since the classical proof of the Hahn-Banach Theorem to be found, for example, in [2] (p. 27-29) only makes use of the fact that the real number system is an order complete vector lattice, l can be extended to a linear mapping $L\colon l^\infty(E) \to E$ such that $L(\vec{x}) \leqslant p(\vec{x})$ for all $\vec{x} \in l^\infty(E)$. L is positive since $p(\vec{x}) \leqslant 0$ whenever $\vec{x} \leqslant 0$. Since

$$-p(-\vec{x}) \leqslant L(\vec{x}) \leqslant p(\vec{x}) \quad \text{for all } x \in l^\infty(E)$$

and since

$$p(\vec{x}-\sigma(\vec{x})) \leqslant \operatorname*{o-lim}_n \left(\frac{2\sup_k |x_k|}{n}\right) = 0,$$

it follows that L is shift invariant. Therefore, L is a Banach limit on E.

To derive the necessary and sufficient condition for almost convergence stated in the proposition, we first note that the proofs of Lemma 1 and Theorem 1 in Sucheston [6] carry over immediately to the present setting. Hence, the sublinear mapping p has the following alternate description:

$$(2) \qquad p(\vec{x}) = \inf\left\{\overline{\lim_j}\left[\frac{1}{n}\sum_{k=1}^{n} x_{r_k+j}\right]\right\}, \qquad \vec{x} = (x_n) \in l^\infty(E);$$

the infimum is taken over all possible choices of natural numbers n, $r_1, \ldots, r_n$; in addition, the mapping p is shift invariant. For all natural numbers j, k, n and all $\vec{x} = (x_n) \in l^\infty(E)$ we have

$$\inf_{q\geqslant j+k} x_q \leqslant \frac{1}{n}\sum_{i=0}^{n-1} x_{i+j+k} \leqslant \sup_{q\geqslant j+k} x_q;$$

consequently,

$$\inf_{q\geqslant k} x_q \leqslant -p\big(-\sigma^k(\vec{x})\big) \leqslant p\big(\sigma^k(\vec{x})\big) \leqslant \sup_{q\geqslant k} x_q.$$

Therefore, by the shift-invariance of p,

$$(3) \qquad \underline{\lim}\, x_n \leqslant -p(-\vec{x}) \leqslant p(\vec{x}) \leqslant \overline{\lim}\, x_n, \qquad \vec{x} = (x_n) \in l^\infty(E).$$

In particular, $p(\vec{x}) = \text{o-lim}\, x_n$ for each $x = (\vec{x}_n) \in \mathrm{oc}(E)$. Using the representation (2) of p and the inequality

$$\overline{\lim}\, a_n + \underline{\lim}\, b_n \leqslant \overline{\lim}\,(a_n+b_n) \leqslant \overline{\lim}\, a_n + \overline{\lim}\, b_n,$$

one can now readily prove that

$$(4) \quad p(\vec{x}+\vec{y}) = \text{o-lim}\, x_n + p(\vec{y}), \qquad \vec{x} = (x_n) \in \mathrm{oc}(E),\ \vec{y} = (y_n) \in l^\infty(E).$$

For each Banach limit L on E, the shift-invariance of L implies that

$$-p(-\vec{x}) \leqslant L(\vec{x}) \leqslant p(\vec{x}), \qquad \vec{x} \in l^\infty(E).$$

On the other hand, if $-p(-\vec{x}) < p(\vec{x})$ for some $\vec{x} \in l^\infty(E)$, then $\vec{x} \notin \mathrm{oc}(E)$ and the limit mapping $l\colon \mathrm{oc}(E) \to E$ can be extended to the linear subspace spanned by $\mathrm{oc}(E)$ and $\vec{x}$ by defining the value at $\vec{y}+a\vec{x}$ ($\vec{y} \in \mathrm{oc}(E)$) to be $\text{o-lim}\, y_n + ac$, where c is *any* element of the order interval $[-p(-\vec{x}), p(\vec{x})]$. (This is a consequence of the proof of the Hahn-Banach Theorem and (4).) Therefore, there exist Banach limits with distinct values at $\vec{x}$, that is, $\vec{x} \notin \mathrm{ac}(E)$. Since

$$-p(-\vec{x}) = \text{o-lim}\left(\inf_j\left[\frac{1}{n}\sum_{i=0}^{n-1} x_{i+j}\right]\right)$$

for all $\vec{x} = (x_n) \in l^\infty(E)$, we conclude that $\vec{x} = (x_n)$ is almost convergent to x if and only if

$$x = \text{o-lim}\left(\frac{1}{n}\sum_{i=0}^{n-1} x_{i+j}\right)$$

uniformly in j.

PROPOSITION 3. *Suppose that M is a convex set in an order complete vector lattice E; then the set of a-limits of sequences with terms in M coincides with the set of limits of order convergent sequences with terms in M.*

Proof. If $x = \text{a-lim}\, x_n$, where $x_n \in M$ for all n, then

$$x = \text{o-lim}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$$

by Proposition 2. Since the bracketed expression is an element of M for all n, $x_i \in M$ ($i = 1, \ldots, n$), it follows that x is the limit of an order convergent sequence in M. On the other hand, if $x = \text{o-lim}\, x_n$ for $x_n \in M$ ($n = 1, 2, \ldots$), then $x = \text{a-lim}\, x_n$ by Proposition 1, so the proof is complete.

Recall that an ordered vector space F is *regularly ordered* if the set of positive linear functionals on F separates points of F (cf. [5], Chapter 2, (1.29)). If F is a regularly ordered vector lattice, the order topology $\mathscr{T}_0$ on F (that is, the locally convex topology generated by the family of all seminorms p on F with the property that $0 \leqslant x \leqslant y$ implies that $p(x) \leqslant p(y)$; see Section 1 of Chapter 3 in [5]) is a Hausdorff topology and the cone in F is normal for $\mathscr{T}_0$.

PROPOSITION 4. *Suppose that E is a regularly ordered, order complete vector lattice and that $l^\infty(E)$ is equipped with its order topology; then $\mathrm{ac}(E)$ is a closed linear subspace of $l^\infty(E)$.*

Proof. Since E is regularly ordered, it follows that the cone in E is normal for the Hausdorff locally convex topology $\sigma(E, E^+)$ and that $l^\infty(E)$ is regularly ordered. Since the sublinear mapping $p\colon l^\infty(E)\to E$ defined in the proof of Proposition 2 is positive and satisfies $p(\vec{x})\leqslant 0$ for $\vec{x}\leqslant 0$, it follows from Theorem 3 in [1] that p is continuous. Since

$$\mathrm{ac}(E) = \{\vec{x} = (x_n)\in l^\infty(E)\colon\ p(\vec{x})+p(-\vec{x}) = 0\}$$

by Proposition 2, it follows that $\mathrm{ac}(E)$ is closed.

3. In this section we shall make the added assumption that the order complete vector lattice E has an order unit e. If $\|\cdot\|$ is the corresponding "order unit norm" on E defined by

$$\|x\| = \inf\{\lambda>0\colon\ -\lambda e\leqslant x\leqslant \lambda e\},$$

then E is an abstract M-space for this norm so that E is norm and order isomorphic to the space $C(X)$ of continuous, real-valued functions on a suitable extremally disconnected, compact Hausdorff space X. Moreover, the space $l^\infty(E)$ coincides with the vector space of norm bounded sequences in E, the element $\vec{e}$ of $l^\infty(E)$ with n^{th} term equal to e is an order unit in $l^\infty(E)$ and the order unit norm corresponding to $\vec{e}$ coincides with the norm

$$\|\vec{x}\|_\infty = \sup_n\|x_n\|,\qquad \vec{x} = (x_n)\in l^\infty(E).$$

Finally, $l^\infty(E)$ is also an abstract M-space for the norm $\|\cdot\|_\infty$.

PROPOSITION 5. *Every weakly convergent sequence in E is order convergent (and hence almost convergent) to its weak limit.*

Proof. Since E is an order complete abstract M-space, E can be identified with the space $C(X)$ of continuous real-valued functions on an extremally disconnected, compact Hausdorff space. If $\{f_n\}\subset C(X)$ and (f_n) converges weakly to 0, then it is well known that $\{\|f_n\|\}$ is norm bounded and $f_n(x)$ converges to 0 for each $x\in X$. Since $C(X)$ is an order complete vector lattice, $g_n = \sup_{k\geqslant n} f_k$ exists in $C(X)$ for each natural number n. Also, if $\bar{g}_n(x) = \sup_{k\geqslant n} f_k(x)$ for each such n and each $x\in X$, then $g_n(x)\geqslant \bar{g}_n(x)$ for all $x\in X$ and $\{x\in X\colon\ g_n(x)>\bar{g}_n(x)\}$ is a set of first category in X. Since $\inf_n \bar{g}_n(x) = 0$ for all $x\in X$, it follows that $\{x\in X\colon\ (\inf_n g_n)(x)>0\}$ is empty since it is of first category and $\inf_n g_n\in C(X)$. We conclude that $\overline{\lim} f_n = 0$, and similar reasoning implies that $\underline{\lim} f_n = 0$, that is, $0 = \text{o-}\lim_n f_n$.

Order convergence in E need not imply weak convergence. In fact, if $\{u_n\}$ is a decreasing sequence in E with infimum 0 and if $\{u_n\}$ is weakly convergent to 0, then (3.4) of Chapter 2 in [5] implies that $\{u_n\}$ is norm convergent to 0. However, if we take $E = l^\infty$, then the sequence $\{u^{(n)}\}\subset l^\infty(l^\infty)$, defined by

$$u_k^{(n)} = \begin{cases}1 & \text{if } k>n,\\ 0 & \text{if } k\leqslant n,\end{cases}$$

is decreasing and has infimum 0 but does not converge in norm to 0.

Since the norms of $l^\infty(E)$ and E are both order unit norms, the space of norm continuous linear mappings of $l^\infty(E)$ into E coincides with the order complete vector lattice $L^b\big(l^\infty(E), E\big)$ of all order bounded linear mappings of $l^\infty(E)$ into E. Our next objective will be to describe the "location" of Banach limits on E within the space $L^b\big(l^\infty(E), E\big)$. We begin by discussing the space $L^0\big(l^\infty(E), E\big)$ of the order continuous mappings in $L^b\big(l^\infty(E), E\big)$. (We say that an order bounded linear mapping T is *order continuous* if $\text{o-}\lim T(x_n) = T(x)$ whenever $\text{o-}\lim x_n = x$. $L^0\big(l^\infty(E), E\big)$ is a band in $L^b\big(l^\infty(E), E\big)$ by Theorem VIII. 3.3 in [8].)

Define $l^1\langle E\rangle$ to be the vector space of all order summable sequences in $l^\infty(E)$, that is

$$l^1\langle E\rangle = \Big\{\vec{x} = (x_n)\in l^\infty(E)\colon\ \sup_n\sum_{k=1}^n |x_k|\in E\Big\}.$$

Obviously, $l^1\langle E\rangle$ is a lattice ideal in $l^\infty(E)$; however, $l^1\langle E\rangle$ is not a band in $l^\infty(E)$ since, for example the sections [1] $\vec{e}\,(\leqslant n)$ $(n = 1, 2, \ldots)$ of $\vec{e}$ are all in $l^1\langle E\rangle$, yet $\vec{e} = \sup_n\{\vec{e}\,(\leqslant n)\}\notin l^1\langle E\rangle$.

Since E is an abstract M-space with unit element e, E may be regarded as an algebra with multiplicative unit e. We make use of this induced multiplication in E in the proof of the following result:

PROPOSITION 6. *A mapping $T\in L^b\big(l^\infty(E), E\big)$ is order continuous and $T(xe^{(n)}) = xT(e^{(n)})$ for all n and $x\in E$ if and only if there is a $\vec{w} = (w_n)\in l^1\langle E\rangle$ such that*

$$(*)\qquad T(x) = \sum_{n=1}^\infty w_n x_n,\qquad \vec{x} = (x_n)\in l^\infty(E)$$

(where the series is order convergent in E). If $L^1\big(l^\infty(E), E\big)$ is the set of such order continuous linear mappings in $L^b\big(l^\infty(E), E\big)$, then the mapping $\psi\colon L^1\big(l^\infty(E), E\big)\to l^1\langle E\rangle$ defined by $\psi(T) = \vec{w}$ through $(*)$ is an order isomorphism of $L^1\big(l^\infty(E), E\big)$ onto $l^1\langle E\rangle$.

Proof. If $T\in L^1\big(l^\infty(E), E\big)$ and if $\vec{e}^{(n)}$ denotes the "n^{th} unit vector" (that is, $\vec{e}^{(n)}$ is the element of $l^\infty(E)$ with n^{th} term e and all other terms 0), define $u_n = T^+(\vec{e}^{(n)})$, $v_n = T^-(\vec{e}^{(n)})$, $w_n = T(\vec{e}^{(n)})$ for each natural num-

(1) The n^{th}-*section* $\vec{x}\,(\leqslant n)$ of a sequence $\vec{x} = (x_k)$ is the sequence whose k^{th} term is x_k for $k\leqslant n$ and 0 for $k>n$.

ber n. Since the sequence $\{\vec{e}(\leqslant n)\colon n = 1, 2, \ldots\}$ of sections of $\vec{e}$ is monotone increasing and has supremum $\vec{e}$ in $l^\infty(E)$, it follows that

$$\sum_{k=1}^{n} u_k = T^+\big(\vec{e}(\leqslant n)\big)$$

increases with n and

$$\sup_n \sum_{k=1}^{n} u_k = T^+(\vec{e}) \in E.$$

Therefore, $\vec{u} = (u_n) \in l^1\langle E\rangle$ and, similarly, $\vec{v} = (v_n) \in l^1(E)$. Consequently, since $w_n = u_n - v_n$ for all n, it follows that $\vec{w} = (w_n) \in l^1\langle E\rangle$.

For each $\vec{x} = (x_n) \in l^\infty(E)$, the sequence of sections $\{\vec{x}(\leqslant n)\}$ order converges to $\vec{x}$; consequently, $\{T(\vec{x}(\leqslant n))\}$ order converges to $T(\vec{x})$. But

$$T\big(\vec{x}(\leqslant n)\big) = \sum_{k=1}^{n} x_k w_k$$

for each natural number n, so (*) holds. Note that $\vec{w} \geqslant 0$ in $l^1\langle E\rangle$ if $T \geqslant 0$ in $L^0\big(l^\infty(E), E\big)$.

On the other hand, suppose that $0 \leqslant \vec{w} = (w_n) \in l^1\langle E\rangle$ and that $0 \leqslant \vec{x} = (x_n) \in l^\infty(E)$, then

$$\Big\{\sum_{k=1}^{n} x_k w_k\colon n = 1, 2, \ldots\Big\}$$

is a monotone increasing sequence in E which is bounded above by

$$\|\vec{x}\|_\infty \cdot \sup_n \Big\{\sum_{k=1}^{n} w_k\Big\}.$$

Hence, we can define a mapping $T_{\vec{w}}$ on the cone in $l^\infty(E)$ into E by

$$(**) \qquad T_{\vec{w}}(x) = \sum_{k=1}^{\infty} x_k w_k, \quad 0 \leqslant \vec{x} = (x_n) \in l^\infty(E).$$

It is a routine matter to verify that $T_{\vec{w}}$ is additive and positively homogeneous on the cone in $l^\infty(E)$; consequently, $T_{\vec{w}}$ has a unique linear extension to $l^\infty(E)$ which will also be denoted by $T_{\vec{w}}$. It is easy to verify that equation (**) is valid for all $\vec{x} = (x_n) \in l^\infty(E)$; moreover, $T_{\vec{w}} \geqslant 0$ in $L^b\big(l^\infty(E), E\big)$ since $\vec{w} \geqslant 0$ in $l^1\langle E\rangle$.

If $\vec{w} = (w_n)$ is an arbitrary element of $l^1\langle E\rangle$ and if $\vec{u} = (w_n^+)$, $\vec{v} = (w_n^-)$, then, for each $\vec{x} = (x_n) \in l^\infty(E)$ and each natural number k, we have

$$\Big|\sum_{n=1}^{k} x_n w_n - \big(T_{\vec{u}}(\vec{x}) - T_{\vec{v}}(\vec{x})\big)\Big| \leqslant \Big|\sum_{n=1}^{k} x_n w_n^+ - T_{\vec{u}}(\vec{x})\Big| + \Big|\sum_{n=1}^{k} x_n w_n^- - T_{\vec{v}}(\vec{x})\Big|.$$

Therefore, since (**) holds for $T_{\vec{u}}$ and $T_{\vec{v}}$, (**) defines a linear mapping of $l^\infty(E)$ into E and $T_{\vec{w}}(\vec{e}^{(n)}) = w_n$ for each n.

To prove that $T_{\vec{w}}$ is order continuous, it is sufficient to consider the case in which $\vec{w} = (w_n) \geqslant 0$. In this case, define

$$T_{\vec{w}}^{(k)}(\vec{x}) = \sum_{n=1}^{k} w_n x_n$$

for each natural number k and each $\vec{x} = (x_n) \in l^\infty(E)$. It is easy to see that $T_{\vec{w}}^{(k)} \in L^1\big(l^\infty(E), E\big)$ for each k. Moreover, (**) implies that

$$T_{\vec{w}} = \sup_k T_{\vec{w}}^{(k)} \quad \text{in } L^b\big(l^\infty(E), E\big);$$

consequently, since $L^0\big(l^\infty(E), E\big)$ is a band in $L^b\big(l^\infty(E), E\big)$, it follows that $T_{\vec{w}} \in L^1\big(l^\infty(E), E\big)$. This completes the proof of the proposition.

Since $L^0\big(l^\infty(E), E\big)$ is a band in $L^b\big(l^\infty(E), E\big)$, the space $L^b\big(l^\infty(E), E\big)$ is the order direct sum of $L^0\big(l^\infty(E), E\big)$ and the complementary band $L^0\big(l^\infty(E), E\big)^\perp$ consisting of all order bounded linear mappings $S\colon l^\infty(E) \to E$ such that $\inf(|S|, |T|) = 0$ for each order continuous linear mapping $T\colon l^\infty(E) \to E$. The mappings in the complementary band $L^0\big(l^\infty(E), E\big)^\perp$ will be called *singular mappings*.

PROPOSITION 7. *If L is a Banach limit on E, then L is a singular mapping.*

Proof. Suppose that $T \in L^0\big(l^\infty(E), E\big)$ and that $0 \leqslant T \leqslant L$. Since the sequence $\{\vec{e}(\leqslant n)\colon n = 1, 2, \ldots\}$ of sections of $\vec{e}$ increases and has supremum $\vec{e}$ in $l^\infty(E)$, it follows that $T(\vec{e}) = \sup_m T\big(\vec{e}(\leqslant n)\big)$. However, $0 \leqslant T\big(\vec{e}(\leqslant n)\big) \leqslant L\big(\vec{e}(\leqslant n)\big) = 0$ since L is shift invariant; consequently, $T(\vec{e}) = 0$, that is, $T = 0$ since $\vec{e}$ is an order unit in $l^\infty(E)$ and $T \geqslant 0$.

We have already remarked in Section 1 that a Banach limit is not a lattice homomorphism of $l^\infty(E)$ into E. In particular, it follows from (1.2) in [3] that a Banach limit is not an indecomposable mapping in $L^b\big(l^\infty(E), E\big)$. (Recall that a non-zero positive element x of an order complete vector lattice F is *indecomposable* if $x = y + z$ and $\inf\{y, z\} = 0$ imply $y = 0$ or $z = 0$.) In fact, a Banach limit L on E is not an atomic mapping, that is, L is not in the band in $L^b\big(l^\infty(E), E\big)$ generated by the indecomposable mappings in $L^b\big(l^\infty(E), E\big)$. For if T is indecomposable, then the range of T must be the 1-dimensional subspace of E spanned by some indecomposable element of E; consequently, every positive, atomic mapping in $L^b\big(l^\infty(E), E\big)$ must have its range in the band of atomic elements of E. Since every Banach limit L satisfies $L(\vec{e}) = e$, it is clear that the range of L need not be contained in the band of atomic elements of E; in fact, if $E = C(X)$, where X is an extremally disconnected compact Hausdorff space in which the isolated points are not dense in X (cf. (5.2) in [3]), then e is not atomic.

The following result makes use of Proposition 6 to characterize the order continuous indecomposable mappings in $L^1(l^\infty(E), E)$.

PROPOSITION 8. *If $0 \leqslant T \in L^1(l^\infty(E), E)$, then T is an order continuous, indecomposable mapping if and only if there exist indecomposable elements a, b in E and a natural number q such that*

$$T(\vec{x}) = \beta_q a, \quad 0 \leqslant \vec{x} = (x_n) \in l^\infty(E),$$

where $\beta_q b = \sup_m \{\inf(x_q, mb)\}$.

Proof. If T is an order continuous, indecomposable mapping, there is a $\vec{u} = (u_n) \geqslant 0$ in $l^1\langle E\rangle$ such that

$$T(\vec{x}) = \sum_{n=1}^{\infty} x_n u_n, \quad \vec{x} = (x_n) \in l^\infty(E),$$

by Proposition 6. If $\mathscr{I}$ is the set of indecomposable elements of E, then it is easy to verify that the set $\mathscr{I}^\infty$ of indecomposable elements of $l^\infty(E)$ is given by $\mathscr{I}^\infty = \{\vec{c}^{(n)}: c \in \mathscr{I}, n = 1, 2, \ldots\}$, where $\vec{c}^{(n)}$ is the element of $l^\infty(E)$ with n^{th} term c and all other terms 0. Therefore, by (1.3) in [3], there exist indecomposable elements a, b in E and a positive integer q such that

$$T\vec{x} = \beta_{\vec{x}} a, \quad \text{where } \beta_{\vec{x}} b = \sup_m \{\inf(\vec{x}, m\vec{b}^{(q)})\}.$$

If p is any positive integer such that $p \neq q$, then

$$\beta_{\vec{e}^{(p)}} \vec{b}^{(q)} = \sup_m \{\inf(\vec{e}^{(p)}, m\vec{b}^{(q)})\} = 0;$$

consequently,

$$u_p = T(\vec{e}^{(p)}) = \beta_{\vec{e}^{(p)}} a = 0.$$

Also,

$$\beta_{\vec{x}^{(q)}} \vec{b}^{(q)} = \sup_m \{\inf(\vec{x}^{(q)}, m\vec{b}^{(q)})\}$$

from which it follows that

$$\beta_{\vec{x}^{(q)}} b = \sup_m \{\inf(x, mb)\},$$

yielding the desired representation of T.

Conversely, if T has such a representation (*), then the band $[b]$ generated by b coincides with the 1-dimensional subspace spanned by b and the projection p_b of E onto $[b]$ that vanishes on the complementary band $[b]^\perp$ is order continuous and $\beta_q = p_b(x_q)$. For each $\vec{x} = (x_n) \in l^\infty(E)$, we have

$$|T(x)| = |\beta_q a| = |\beta_q| a \quad \text{and} \quad T(|\vec{x}|) = \gamma_q a,$$

where $\gamma_q b = p_b(|x_q|) = |p_b(x_q)| = |\beta_q| b$. Therefore, T is a lattice homomorphism which is order continuous since p_b is order continuous. Since the range of T is the linear hull of the indecomposable element a, it follows from (1.2) in [3] that T is indecomposable.

It should be observed that indecomposable elements in $L^b(l^\infty(E), E)$ need not be order continuous. In fact, even if E is the order complete vector lattice of real numbers, the point evaluation f_x at a point of $\beta N \sim N$ (where βN denotes the Stone-Čech compactification of N) is an indecomposable linear functional on l^∞ (regarded as the space of real-valued continuous functions on βN) which is not order continuous since it does not originate from an element of l^1.

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