COROLLARY 1. *Let X be an $\mathscr{L}_p$-space* (see [6]). *Let $1 \leqslant r \leqslant 2$, $1 \leqslant p \leqslant 2$. Then for every Banach space Y we have*

$$A_{r,1}(X, Y) = A_{r_1,2}(X, Y), \quad \textit{where } 1/r_1 = 1/r - 1/2.$$

This corollary is a special case of the Theorem, since $\mathscr{L}_p$ is a subspace of $\mathscr{L}_1(\mu)$ for some measure μ (see [6], Section 7).

COROLLARY 2. *Let $1 \leqslant r \leqslant 2$ and $1 \leqslant p \leqslant 2$. Then for every Banach space Y we have*

$$A_{r,1}(l_p, Y) = A_{r_1,2}(l_p, Y),$$
$$A_{r,1}(L_p(0,1), Y) = A_{r_1,2}(L_p(0,1), Y),$$

where $1/r_1 = 1/r - \frac{1}{2}$.

Definition. We denote by H_1 the space of Lebesgue-integrable functions on the circle such that

$$\int_{-\pi}^{\pi} e^{int} f(t)\,dt = 0 \quad \text{for } n = 1, 2, \ldots$$

(see [3]).

COROLLARY 3. *Let $1 \leqslant r \leqslant 2$ and let Y be an arbitrary Banach space. Then*

$$A_{r,1}(H_1, Y) = A_{r_1,2}(H_1, Y), \quad \textit{where } \frac{1}{r_1} = \frac{1}{r} - \frac{1}{2}.$$

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The estimation of an integral arising in multiplier transformations

by

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The aim of this note is to prove the following general estimate:

THEOREM. *Let $a_1 < a_2 < \ldots < a_n$ be fixed non-negative real numbers and let $b_1, \ldots, b_n$ be real numbers. Then*

$$\left| \int_{-\infty}^{\infty} \exp\{i(b_1[x]^{a_1} + b_2[x]^{a_2} + \ldots + b_n[x]^{a_n}\} \frac{dx}{x} \right| \leqslant K(a_1, a_2, \ldots, a_n),$$

where K does not depend on $b_1, b_2, \ldots, b_n$.

(The integral is defined by integrating over $\varepsilon \leqslant |x| \leqslant R$ and then letting $R \to \infty$ and $\varepsilon \to 0$.)

For fixed real a the symbol $[x]^a$ may stand for either $|x|^a$ or $\operatorname{sgn} x\,|x|^a$.

The proof of the Theorem is based on the following Lemma of Van der Corput:

LEMMA 1. *Let $f(t)$ be a real-valued differentiable function on $u \leqslant t \leqslant v$. Suppose $f'(t)$ is monotonic and that $|f'(t)| > \lambda > 0$ for $u \leqslant t \leqslant v$. Then*

$$\left| \int_u^v \exp[if(t)]\,dt \right| < 1/\lambda.$$

For the proof of Lemma 1, see [3], p. 197.

To apply Van Der Corput's Lemma, it is necessary to obtain estimates on the measure of the set on which an expression of the form

$$g(x) = d_1 x^{c_1} + d_2 x^{c_2} + \ldots + d_{m-1} x^{c_{m-1}} + x^{c_m} \tag{1.1}$$

is small.

LEMMA 2. *Let $g(x)$ be defined by (1.1) with d_i real and $c_i \geqslant 0$. Assume further that $c_j \geqslant c_{j-1} + 1$, $2 \leqslant j \leqslant m$, and that $c_1 \geqslant 1$. Then the graph of $g(x)$ for $1 \leqslant x \leqslant \infty$ consists of ν intervals $\{I_k\}$ on each side of which $g(x)$ is monotonic. On each of the intervals I_k, $k = 1, \ldots, \nu$, $|g(x)| \geqslant 1$ except on a subinterval of length at most μ_k; and what is most important ν and the numbers μ_k may be chosen so as not to depend on the numbers $d_1, d_2, \ldots, d_{m-1}$.*

We shall first show that the Theorem follows from Lemmas 1 and 2. It is easy to show that the integral of our theorem exists for any particular choice of $b_1, \ldots, b_n$. Hence it suffices to show

$$(1.2) \qquad \Big| \int_{\varepsilon \leqslant |x| \leqslant R} \exp\{i(b_1[x]^{a_1} + \ldots + b_n[x]^{a_n})\} \frac{dx}{x} \Big| \leqslant k(a_1, \ldots, a_n),$$

where k does not depend on the b's, ε, or R. We shall show (1.2) by induction on n. For $n = 1$ this is done by a change of variables. We now assume (1.2) holds for all integers less than n, and we shall prove (1.2). For all combinations of the b's for which $b_n = 0$, we have (1.2) by the inductive hypothesis. If $b_n \neq 0$, we may assume $b_n = 1$, and $a_j \geqslant a_{j-1} + 1$ for $j = 2, \ldots, n$ and $a_1 \geqslant 2$ (perhaps changing ε and R) by making a change of variables $|y|^\beta = |b_n|\,|x|^{a_n}$ with β large. Now Lemma 1, Lemma 2, and an integration by parts show that the contribution from the intervals $1 \leqslant |x| \leqslant R$ to the integral of (1.2) is uniformly bounded. Since $b_n = 1$, we have $\exp i b_n |x|^{a_n} = 1 + o(|x|^{a_n})$. Thus the interval $\varepsilon \leqslant x \leqslant 1$ contributes to the integral of (1.2) a term which can be handled by the inductive hypothesis plus a bounded term.

To prove Lemma 2, we shall need an additional lemma.

LEMMA 3. *Let $g(x)$ be defined by* (1.1), *with some of the c_i possibly negative. Then $g(x)$ has at most n zeros for* $0 < x < \infty$.

Proof. The proof is by induction on n. If $n = 1$, the Lemma is obvious. We wish to show that

$$g(x) = d_1 x^{c_1} + \ldots + d_{m-1} x^{c_{m-1}} + x^{c_m}$$

has at most n zeros. If $g(x)$ had more than n zeros, so would $x^{-c_1} g(x)$. Hence by Rolle's Theorem

$$\frac{d}{dx}\big(x^{-c_1} g(x)\big)$$

would have more than $n-1$ zeros contradicting the inductive hypothesis.

Proof of Lemma 2. By Lemma 3 the graph of $g'(x)$ has at most n zeros. Therefore the graph of $g(x)$, $1 \leqslant x < \infty$, can be divided into m intervals I_k on each of which $g(x)$ is monotonic. On each I_k we consider the operators

$$D^a g = \frac{d}{dx}\{x^{-a} g(x)\}.$$

We know that

$$D^{c_m - 1 - c_{m-1}} D^{c_{m-1} - 1 - c_{m-2}} \ldots D^{c_2 - 1 - c_1} g(x) = \gamma,$$

where γ is a non-zero constant depending only on $c_1, c_2, \ldots, c_m$. Let

$$h(x) = D^{c_{m-1} - 1 - c_{m-2}} \ldots D^{c_2 - 1 - c_1} D^{c_1} g(x).$$

Thus $D^{c_m - 1 - c_{m-1}} h(x) = \gamma$, which we shall assume without loss of generality to be positive. Hence to finish the proof of Lemma 2 we must show that if $l(x) = e_1 x^{c_1} + \ldots + e_m x^{c_m}$ and if $|D^a l(x)| \geqslant \gamma$, $a \geqslant 0$, on an interval $J \subseteq [1, \infty]$, then the following conclusion holds: $|l(x)| < 1$ for only a finite union of subintervals of J of finite length, and the number of intervals and their lengths do not depend on $e_1, e_2, \ldots, e_m$. We know by Lemma 3 that J consists of a finite number of intervals on each of which $x^{-a} l(x)$ is monotonic. Suppose we take an interval I on which $x^{-a} l(x)$ is increasing. Let α and β be points in I such that $l(\alpha) \geqslant 0$ and $\beta > \alpha$. Then

$$\begin{aligned} l(\beta)\beta^{-a} &\geqslant l(\beta)\beta^{-a} - l(\alpha)\alpha^{-a} \\ &\geqslant \int_\alpha^\beta \frac{d}{dx}\{l(x)x^{-a}\}\,dx \\ &\geqslant \gamma(\beta - \alpha). \end{aligned}$$

So $l(\beta) > 1$ if $\beta - \alpha \geqslant 1/\gamma$, since $a \geqslant 0$ and $\beta \geqslant 1$. Similarly, if $\delta < \alpha$, and $l(\alpha) \leqslant 0$, $l(\delta) \leqslant -1$ if $\alpha - \delta \leqslant 1/\gamma$. A similar argument applies if $l(x)$ is decreasing on I. Hence $|l(x)| \geqslant 1$ except for a finite union of intervals of bounded length.

We would like to thank N. Rivière and W. Rudin for helpful discussions.

REMARK

The above was written in August 1967. The result presented probably now deserves some further consideration because of its relation to certain singular integrals which have recently attracted attention. The connection of the estimate with which this paper deals to singular integrals was pointed out to us by N. Rivière. Applications to singular integrals require two related additional estimates which may also be obtained by our method. They are:

$$\text{(a)} \qquad \Big| \int_{-\infty}^{\infty} e\big(ip^*(x)\,\mathrm{sgn}\,x\big) \frac{dx}{x} \Big| \leqslant K,$$

where $p^*(x) = [x]^{a_1} P_1(\log|x|) + \ldots + [x]^{a_k} P_k(\log|x|)$, $a_1, \ldots, a_k$ are as in our theorem, and the P_k are polynomials. K depends on $a_1, \ldots, a_k$ and the degrees of $P_1, \ldots, P_k$, but K does not depend on the coefficients of the P_k;

$$\text{(b)} \qquad \Big| \int_{-\infty}^{\infty} e\big(ip^*(x)\big) \frac{dx}{x} \Big| \leqslant K \log(1/|\beta|) \quad \text{as } \beta \to 0,$$

where β is the highest order coefficient in P_k.

To prove (a) one needs an appropriate analogue of Lemma 2. This may be obtained by our proof of Lemma 2 if we use not only our $D^{(a)}$ but also D_a^* and xD_a^*, where

$$D_a^* g = \frac{d}{dx} x \frac{d}{dx} \ldots x \frac{d}{dx} \frac{g(x)}{x^a}$$

for a sufficiently large. This proves a generalized form of Lemma 2 if $a_j - a_{j-1}$ is sufficiently large. Hence we have (a) if $a_j - a_{j-1}$ is sufficiently large, and this is no loss of generality.

(b) follows by a change of variables, Lemma 1, and the generalized form of Lemma 2.

Rivière proved the following theorem from these estimates:

Let $k(x)$ be a function on E^n such that

$$k(T_\lambda x) = \lambda^{(-trp)} k(x),$$

where $T_\lambda = e^{ln\lambda p}$ is a one-parameter semi-group of $n \times n$ matrices with infinitesimal generator p. Assume the eigenvalues of p are real and positive. Suppose k is odd and locally in L^1 away from 0 or

$$\int_\Sigma k(x)\, d\sigma = 0 \quad \text{and} \quad \int_\Sigma |k(x)| \log^+ |k(x)|\, d\sigma < \infty,$$

where Σ is a "sphere" related to the symmetry T_λ. (See [2], section 3, for the precise definition.)

Then for f in $L^2(E^n)$ and

$$U_{\varepsilon,R} f = \int_{\varepsilon \leqslant |x| \leqslant R} k(x) f(y-x)\, dx, \qquad \|U_{\varepsilon,R} f\|_2 \leqslant A \|f\|_2$$

and $U_{\varepsilon,R}f$ tends in L_2 to a function Uf. Moreover,

$$\|Uf\|_2 \leqslant A \|f\|_2.$$

For odd k this theorem was proved by taking the Fourier Transform of the kernel k, introducing an appropriate generalization of polar coordinates and following the general approach of Calderón and Zygmund [1]. The case of even k is then proved by using generalized Riesz transforms.

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