[4] В. Ф. Гапошкин, *Одно обобщение теоремы М. Рисса о сопряженных функциях*, Матем. сб. 46 (88) (1958), p. 359–372.

[5] C. A. McCarthy and J. Schwartz, *On the norm of a finite Boolean algebra of projections, and applications to theorem of Kreiss and Morton*, Comm. Pure Appl. Math. 18 (1965), p. 191–201.

[6] Б. С. Митягин, *Аппроксимативная размерность и базисы в ядерных пространствах*, ДАН 16 (4) (1961), p. 63–132.

[7] Л. А. Муравей, *Базисы Рисса в* $L_2(-1, 1)$, Труды мат. инст. им. Стеклова 91 (1967), p. 113–131.

[8] W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen I*, Studia Math. 4 (1933), p. 33–37.

[9] A. Pełczyński and I. Singer, *On non-equivalent bases and unconditional bases in Banach spaces*, ibidem 25 (1964), p. 5–25.

[10] A. Pietsch, *Nukleare lokalkonvexe Räume*, Berlin 1965.

[11] A. Zygmund, *Trigonometric series*, Vol. II, Cambridge 1965.

Reçu par la Rédaction le 31. 3. 1969

STUDIA MATHEMATICA, T. XXXV. (1970)

A remark on (s, t)-absolutely summing operators in L_p-spaces

by

NICOLE TOMCZAK (Warszawa)

In this paper we prove a theorem on the composition of the p-absolutely summing and (s, t)-absolutely summing operators which is a generalization of a theorem proved by Pietsch (see [7]) concerning the composition of p-absolutely summing operators. The proof of the theorem follows Pietsch's proof.

As an application of this theorem we prove that for some class of spaces the ideals of (s, t)-absolutely summing operators have properties quite analogous to those of ideals of (s, t)-absolutely summing operators in a Hilbert space provided $1/t - 1/s = \frac{1}{2}$ and $t \leqslant 2$. The proof is quite analogous to that of the theorem stating that $A_{11}(l_r, X) \in A_{12}(l_r, X)$ if $r \leqslant 2$ (see [5]).

Definition. Let X and Y be Banach spaces, let $T \in B(X, Y)$ and let $1 \leqslant q \leqslant p < \infty$. Put

$$a_{p,q}(T) = \inf\{C : \Big(\sum_i \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup_{\|x^*\| \leqslant 1} \Big(\sum_i |x^*(x_i)|^q\Big)^{1/q} \quad \text{for } x_i \in X,\ i = 1, \ldots, n \text{ and } n = 1, 2, \ldots\}.$$

An operator T is said to be *(p, q)-absolutely summing* ($T \in A_{p,q}(X, Y)$) if $a_{p,q}(T) < \infty$.

It turns out that $A_{p,q}(X, Y)$ with the norm $a_{p,q}(\cdot)$ is the Banach ideal.

PROPOSITION. *Let X, Y and Z be Banach spaces, $T \in A_{p,p}(X, Y)$ and $S \in A_{s,t}(Y, Z)$. Then the operator $ST \in B(X, Z)$ is (r, q)-absolutely summing, where*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s} \leqslant 1, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{t} \leqslant 1$$

and $a_{r,q}(ST) \leqslant a_{s,t}(S)\, a_{p,p}(T)$.

Proof. Since the operator T is p-absolutely summing, there is a regular positive Borel measure μ on the unit ball K^* of X^* such that $\mu(K^*) = 1$ and

$$\|Tx\| \leqslant a_{p,p}(T)\Big(\int_{K^*}|x^*(x)|^p\,d\mu(x^*)\Big)^{1/p} \qquad \text{for } x \in X$$

(see [6] and [7]).

Let $(x_n)_1^N \in X$ be an arbitrary finite sequence. Put

$$x_n^0 = \Big(\int_{K^*}|x^*(x)|^q\,d\mu(x^*)\Big)^{-1/p} x_n \qquad \text{for } n = 1, \ldots, N.$$

Applying the Hölder inequality and the fact that S is (s, t)-absolutely summing, we obtain

$$\Big(\sum_n\|STx_n\|^r\Big)^{1/r} \leqslant \Big(\sum_n \|STx_n^0\|^s\Big)^{1/s}\Big(\sum_n \int_{K^*}|x^*(x_n)|^q\,d\mu(x^*)\Big)^{1/p}$$

$$\leqslant a_{s,t}(S) \sup_{\|y^*\|\leqslant 1} \Big(\sum_n|y^*(Tx_n^0)|^t\Big)^{1/t}\Big(\sum_n \int_{K^*}S|x^*(x_n)|^q\,d\mu(x^*)\Big)^{1/p}.$$

Since T is p-absolutely summing, the diagram

$$\begin{array}{ccccc} C(K^*) & \xrightarrow{I} & L^p(K^*, \mu) & \longrightarrow & \downarrow \\ i\uparrow & & & & Z \\ X & \xrightarrow[T]{} & Y & \longrightarrow & \uparrow \end{array}$$

is commutative where Z is a Banach space, $i\colon X \to C(K^*)$ is the canonical isometry $x \to x(x^*)$, and $I\colon C(K^*) \to L\ (K^*, \mu)$ is the identity map $f \to f$. Let E denote the closure of $Ii(X)$. Consider an arbitrary functional $y^* \in Y^*$. Then the formula

$$\tilde{\beta}_{y^*}(ix) = y^*(Tx)$$

determines a functional $\tilde{\beta}_{y^*}$ on E. It follows from the Hahn-Banach theorem and from the fact that $[L_p(K^*, \mu)]^*$ is isometrically isomorphic to $L_{p^*}(K^*, \mu)$ that there is an element $f \in L_{p^*}(K^*, \mu)$ such that

$$y^*(Tx) = \int_{K^*} x^*(x)f(x^*)\,d\mu(x^*)$$

and

$$\Big(\int_{K^*}|f(x^*)|^{p^*}\,d\mu\Big)^{1/p^*} \leqslant a_{p,p}(T)\cdot\|y^*\|.$$

By Hölder's inequality, we obtain

$$|y^*(Tx)| \leqslant \int_{K^*} |x^*(x)|\cdot|f(x^*)|\,d\mu(x^*)$$

$$= \int_{K^*}|x^*(x)^{q/p}\cdot\big(Tx^*(x)|^q\cdot|f(x^*)|^{p^*}\big)^{1/t}\cdot|f(x^*)|^{p^*/q^*}\,d\mu(x^*)$$

$$\leqslant \Big(\int_{K^*}|x^*(x)|^q\,d\mu(x^*)\Big)^{1/p}\Big(\int_{K^*}|x^*(x)|^q|f(x^*)|^{p^*}\,d\mu(x^*)\Big)^{1/t}\Big(\int_{K^*}|f(x^*)|^{p^*}\,d\mu(x^*)\Big)^{1/q^*}.$$

Hence for arbitrary $y^* \in Y^*$, $\|y^*\| \leqslant 1$ and for $n = 1, \ldots, N$

$$|y^*(Tx_n^0)|^t \leqslant \int_{K^*} |x^*(x_n)|^q|f(x^*)|^{p^*}\,d\mu(x^*)\Big(\int_{K^*}|f(x^*)|^{p^*}\,d\mu(x^*)\Big)^{t/q^*}.$$

Finally, we get

$$\Big(\sum_n |y^*(Tx_n^0)|^t\Big)^{1/t} \leqslant \sup_{\|x^*\|\leqslant 1}\Big(\sum_n |x^*(x_n)|^q\Big)^{1/t}\cdot\Big(\int_{K^*}|f(x^*)|^{p^*}\,d\mu(x^*)\Big)^{1/p^*}$$

$$\leqslant a_{p,p}(T)\sup_{\|x^*\|\leqslant 1}\Big(\sum_n|x^*(x_n)|^q\Big)^{1/t}.$$

Consequently,

$$\Big(\sum_n\|STx_n\|^r\Big)^{1/r} \leqslant a_{s,t}(S)\cdot a_{p,p}(T)\sup_{\|x^*\|\leqslant 1}\Big(\sum_n|x^*(x_n)|^q\Big)^{1/q}.$$

Thus, by the definition of the norm $a_{r,q}(S; T)$, we have

$$a_{r,q}(S; T) \leqslant a_{s,t}(S)\,a_{p,p}(T).$$

This completes the proof.

THEOREM. *Let X be a Banach space isomorphic to a subspace of an $L_1(\mu)$-space for some measure μ, and let Y be an arbitrary Banach space. Then for $1 \leqslant r \leqslant 2$*

$$A_{r,1}(X, Y) = A_{r_1,2}(X, Y), \qquad \textit{where } 1/r_1 = 1/r - 1/2.$$

Proof. First, observe that $A_{r,1}(X, Y) \subset A_{r_1,2}(X, Y)$ since $1 - 1/r = \frac{1}{2} - 1/r_1$ (see [4], 0.7).

The inclusion $A_{r_1,2}(X, Y) \subset A_{r,1}(X, Y)$ results from the Proposition and from the following facts:

(a) If X is isomorphic to a subspace of an $L_1(\mu)$-space, then every operator $S \in B(l_\infty, X)$ is 2-absolutely summing (see [2] and [6]).

(b) Let $T\colon X \to Y$ be a linear operator from a Banach space X into a Banach space Y. Then $T \in A_{r,1}(X, Y)$ if and only if $TS \in A_{r,1}(l_\infty, Y)$ for every $S \in B(l_\infty, X)$.

To prove (b), assume that $T \notin A_{r,1}(X, Y)$. Then there is a sequence $(x_n) \subset X$ such that the series $\sum_n x_n$ is unconditionally convergent, but $\sum_n \|Tx_n\|^r = \infty$.

Put $S(a_n) = \sum_n a_n x_n$ for $(a_n) \in l_\infty$. Since the series $\sum x_n$ is unconditionally convergent, $S \in B(l_\infty, X)$ (see [1]).

Since $\sum_n \|Tx_n\|^r = \infty$, there exists a sequence of real numbers η_n such that $\lim \eta_n = 0$ and $\sum_n (\eta_n\|Tx_n\|)^r = \infty$. Since

$$\sum \|ST(\eta_n e_n)\|^r = \sum \|T\eta_n x_n\|^r = \infty,$$

where $e_n(0, \ldots, 0, 1, 0, \ldots)$, $TS \notin A_{r,1}(l_\infty, Y)$, and this completes the proof of (b).

COROLLARY 1. *Let X be an $\mathscr{L}_p$-space* (see [6]). *Let $1 \leqslant r \leqslant 2$, $1 \leqslant p \leqslant 2$. Then for every Banach space Y we have*

$$A_{r,1}(X, Y) = A_{r_1,2}(X, Y), \quad \text{where } 1/r_1 = 1/r - 1/2.$$

This corollary is a special case of the Theorem, since $\mathscr{L}_p$ is a subspace of $\mathscr{L}_1(\mu)$ for some measure μ (see [6], Section 7).

COROLLARY 2. *Let $1 \leqslant r \leqslant 2$ and $1 \leqslant p \leqslant 2$. Then for every Banach space Y we have*

$$A_{r,1}(l_p, Y) = A_{r_1,2}(l_p, Y),$$
$$A_{r,1}(L_p(0,1), Y) = A_{r_1,2}(L_p(0,1), Y),$$

where $1/r_1 = 1/r - \frac{1}{2}$.

Definition. We denote by H_1 the space of Lebesgue-integrable functions on the circle such that

$$\int_{-\pi}^{\pi} e^{int} f(t)\,dt = 0 \quad \text{for } n = 1, 2, \ldots$$

(see [3]).

COROLLARY 3. *Let $1 \leqslant r \leqslant 2$ and let Y be an arbitrary Banach space. Then*

$$A_{r,1}(H_1, Y) = A_{r_1,2}(H_1, Y), \quad \text{where } \frac{1}{r_1} = \frac{1}{r} - \frac{1}{2}.$$

I wish to thank Professor A. Pełczyński for the inspiration of the problem and for his kind advices.

References

[1] M. M. Day, *Normed linear spaces*, Berlin 1958.
[2] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Matem. Sao Paulo 8 (1956), p. 1–79.
[3] K. Hoffman, *Banach spaces of analytic functions*, New Jersey 1962.
[4] S. Kwapień, *Some remarks on (p, q)-summing operators in l_p-spaces*, Studia Math. 28 (1968), p. 327–337.
[5] — *A remark on p-absolutely summing operators in l_r-spaces*, ibidem 34 (1969), p. 109–111.
[6] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, ibidem 29 (1968), p. 276–325.
[7] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, ibidem 28 (1967), p. 333–353.

Reçu par la Rédaction le 4. 4. 1969

STUDIA MATHEMATICA, T. XXXV. (1970)

The estimation of an integral arising in multiplier transformations

by

ELIAS M. STEIN (Princeton) and STEPHEN WAINGER (Wisconsin)

The aim of this note is to prove the following general estimate:

THEOREM. *Let $a_1 < a_2 < \ldots < a_n$ be fixed non-negative real numbers and let $b_1, \ldots, b_n$ be real numbers. Then*

$$\left| \int_{-\infty}^{\infty} \exp\{i(b_1[x]^{a_1} + b_2[x]^{a_2} + \ldots + b_n[x]^{a_n}\}\frac{dx}{x} \right| \leqslant K(a_1, a_2, \ldots, a_n),$$

where K does not depend on $b_1, b_2, \ldots, b_n$.

(The integral is defined by integrating over $\varepsilon \leqslant |x| \leqslant R$ and then letting $R \to \infty$ and $\varepsilon \to 0$.)

For fixed real a the symbol $[x]^a$ may stand for either $|x|^a$ or $\operatorname{sgn} x\,|x|^a$.

The proof of the Theorem is based on the following Lemma of Van der Corput:

LEMMA 1. *Let $f(t)$ be a real-valued differentiable function on $u \leqslant t \leqslant v$. Suppose $f'(t)$ is monotonic and that $|f'(t)| > \lambda > 0$ for $u \leqslant t \leqslant v$. Then*

$$\left|\int_u^v \exp[if(t)]\,dt\right| < 1/\lambda.$$

For the proof of Lemma 1, see [3], p. 197.

To apply Van Der Corput's Lemma, it is necessary to obtain estimates on the measure of the set on which an expression of the form

$$(1.1) \qquad g(x) = d_1 x^{c_1} + d_2 x^{c_2} + \ldots + d_{m-1} x^{c_{m-1}} + x^{c_m}$$

is small.

LEMMA 2. *Let $g(x)$ be defined by (1.1) with d_i real and $c_i \geqslant 0$. Assume further that $c_j \geqslant c_{j-1}+1$, $2 \leqslant j \leqslant m$, and that $c_1 \geqslant 1$. Then the graph of $g(x)$ for $1 \leqslant x \leqslant \infty$ consists of ν intervals $\{I_k\}$ on each side of which $g(x)$ is monotonic. On each of the intervals I_k, $k = 1, \ldots, \nu$, $|g(x)| \geqslant 1$ except on a subinterval of length at most μ_k; and what is most important ν and the numbers μ_k may be chosen so as not to depend on the numbers $d_1, d_2, \ldots, d_{m-1}$.*