References

[1] D. Przeworska-Rolewicz, *On equation with reflection*, Studia Math. 33 (1969), p. 191–200.
[2] — and S. Rolewicz, *Equations in linear spaces*, Monografie Matematyczne 47, Warszawa 1968.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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On conditional bases in non-nuclear Fréchet spaces

by

W. WOJTYŃSKI (Warszawa)

In the present paper we give some criteria for the nuclearity of Fréchet spaces with bases. Our main result is the following:

A. Let X be a Fréchet space with a basis. Then X is nuclear if and only if every basis of X is absolute (the basis $\{e_n\}$ is *absolute* if $\sum_{n=1}^{\infty} \|t_n e_n\| < \infty$ for each $x = \sum_{n=1}^{\infty} t_n e_n$ and each pseudonorm $\|\cdot\|$ on X).

For countably Hilbert spaces this result is strengthened as follows:

B. A Hilbertian Fréchet space X with a basis is nuclear if and only if every basis $\{e_n\}$ of X is unconditional (i.e. $\sum_{n=1}^{\infty} |x^*(t_n e_n)| < \infty$ for each $x = \sum_{n=1}^{\infty} t_n e_n \in X$, and each linear functional $x^* \in X^*$).

Observe that the part "only if" of our results is a consequence of the Dynin–Mitiagin theorem [3] which asserts that in a nuclear space each basis is unconditional. We do not know whether the converse is true, however, we believe the following holds:

CONJECTURE (see [9]). *A Fréchet space X with a basis is nuclear provided each basis in X is unconditional.*

The conjecture is already established for Banach spaces, because the class of nuclear Banach spaces coincides with the class of finite-dimensional spaces, and, by result of Pełczyński and Singer [9], in every infinite-dimensional Banach space with a basis there exists a conditional basis.

Statement B can be regarded as a generalization of a result due to Babenko asserting that in a Hilbert space there exists a conditional basis; [1], cf. also [4], [5] and [7].

Statement A is a generalization of an unpublished result of professor J. Rutherford (presented on the conference on functional analysis in Sopot 1968) that a Fréchet space satisfying the assumption of A is a Schwartz space.

The present paper consists of five sections. The first two have preliminary character. In section 3 we compute the "degree of conditionality" of the Babenko basis in l^2 and of some conditional bases in l^p-spaces. Section 4 is devoted to a construction of non-nuclear "regular" subspaces of non-nuclear Köthe spaces. The proof of the main result is completed in section 5.

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1. Notation and terminology. A topological linear space X is called *Fréchet* iff the topology of X is given by a denumerable system of pseudonormes $(\|\cdot\|_n,\ n = 1, 2, \ldots)$ and if X is complete.

If $(\|\cdot\|_n',\ n = 1, 2, \ldots)$ is another denumerable system of pseudonorms on X which induces the topology of X, then those two systems are *equivalent* in the following sense: for each $n \in \mathcal{N}$ ($\mathcal{N}$ – the set of natural numbers) there exist an m and a constant $C = C(n, m)$ such that

$$\|x\|_n \leqslant C \max_{i \leqslant m} \|x\|_i'$$

and

$$\|x\|_n' \leqslant C \max_{i \leqslant m} \|x\|_i.$$

For each system $(\|\cdot\|_i,\ i \in \mathcal{N})$ of pseudonorms on X there exists an equivalent system $(\|\cdot\|_n',\ n \in \mathcal{N})$ which is monotone, i.e. $\|x\|_n' \leqslant \|x\|_m'$ for each $n \leqslant m$ and $x \in X$.

A sequence $\{e_n\}_{n\in\mathcal{N}}$ of elements of a Fréchet space X is a *basis* of X iff for each $x \in X$ there exists a unique sequence of scalars $\{\lambda_n(x)\}_{n\in\mathcal{N}}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n(x) e_n. \tag{1.1}$$

A series $\sum_{n=1}^{\infty} x_n$, $x_n \in X$, is said to be *unconditionally convergent* iff for any permutation σ of the set $\mathcal{N}$ the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is convergent.

A basis $\{e_n\}_{n\in\mathcal{N}}$ of a Fréchet space X is *unconditional* if the series (1.1) is unconditionally convergent for each $x \in X$. The basis is *conditional* if it is not unconditional.

The basis $\{e_n\}_{n\in\mathcal{N}}$ of X is *absolute* if the series (1.1) is absolutely convergent for $x \in X$, i.e.

$$\sum_{n=1}^{\infty} |\lambda_n(x)|\, \|e_n\|_a < +\infty$$

for each pseudonorm $\|\cdot\|_a$ on X.

The functions $\lambda_n(\cdot)$ will be called the *coefficient functionals* of the basis $\{e_n\}_{n\in\mathcal{N}}$. They are linear and continuous.

Let X and Y be Banach spaces. A linear operator $A: X \to Y$ is *nuclear* if $A = \sum_{n=1}^{\infty} A_n$, where $A_n: X \to Y$ are 1-dimensional, and $\sum_{n=1}^{\infty} \|A_n\| < +\infty$.

If X is a Fréchet space with the topology given by monotone sequence of pseudonorms $(\|\cdot\|_n,\ n \in \mathcal{N})$, then

$$Y_n = \{x \in X: \|x\|_n = 0\}$$

are closed linear subspaces of X. Let X_n denote the completion of the space X/Y_n with respect to the norm induced by pseudonorm $\|\cdot\|_n$. Since the sequence $(\|\cdot\|_n,\ n \in \mathcal{N})$ is monotone, there are defined the canonical homomorphisms $B_n: X_{n+1} \to X_n$.

The Fréchet space X is *nuclear* if a monotone system of pseudonorms can be chosen such that all the operators B_n are nuclear.

Let Γ be an abstract set and $1 \leqslant p \leqslant \infty$. By l_Γ^p we denote the space of all complex-valued functions on Γ for which the quantity

$$\|f\| = \begin{cases} \left(\sum_{\gamma\in\Gamma} |f(\gamma)|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup_{\gamma\in\Gamma} |f(\gamma)| & \text{if } p = \infty \end{cases}$$

is finite, with the topology given by the norm $\|\cdot\|$.

In particular, if $\Gamma = \mathcal{N}$, we denote the space l_Γ^p by l^p, and if Γ is a finite set of n elements, we denote l_Γ^p by l_n^p.

Let $[a_{m,n}]_{m,n\in\mathcal{N}}$ be a real, non-negative-valued infinite matrix (for the sake of brevity we call it a *Köthe matrix*).

The *Köthe space* $l^p[a_{m,n}]$ is the space of all complex sequences $\{\xi_n\}_{n\in\mathcal{N}}$ for which

$$\|\{\xi_n\}\|_m = \Big(\sum_{n=1}^{\infty} a_{n,m} |\xi_n|^p\Big)^{1/p} < +\infty \tag{1.2}$$

with a topology given by the system of pseudonorms (1.2).

A Köthe matrix is said to be *monotone* (we denote it by M.K.M.) if $a_{m+1} \geqslant a_{m,n}$ for $m, n = 1, 2, \ldots$

More generally, let $[a_{m,n}]$ be a Köthe matrix, and X_n be a sequence of Banach spaces. Then by $l^p([a_{m,n}]; X_n)$ we denote the linear space of all sequences $\{x_n\}_{n\in\mathcal{N}}$ such that:

(a) $x_n \in X_n$ for $n = 1, 2, \ldots,$

(b) for any $m \in \mathcal{N}$

$$\|\{x_n\}\|_m = \Big(\sum_{n=1}^{\infty} a_{m,n} \|x_n\|^p\Big)^{1/p} < +\infty. \tag{1.3}$$

The topology on $l^p([a_{m,n}]; X_n)$ is given by the system of pseudo-norms (1.3).

The Köthe space $l^p[a_{m,n}]$ as well as $l^p([a_{m,n}]; X_n)$ are Fréchet spaces.

2. Some facts concerning the nuclearity of the spaces. Let $[a_{m,n}]$ be an M.K.M. Then ([6], p. 71-72) the Köthe space $l^p[a_{m,n}]$ is nuclear iff for each $m \in \mathcal{N}$ there exists $k \in \mathcal{N}$ such that

$$\sum_{n\in\mathcal{N}} \frac{a_{m,n}}{a_{m+k,n}} < +\infty \tag{2.1}$$

(here by 0/0 we mean 0).

Let $l_0^p[a_{m,n}]$ denote the dense linear subspace of $l^p[a_{m,n}]$ consisting of all sequences with finite number of non-zero terms. For a real a let us define an operator $A^a\colon l_0^p[a_{m,n}] \to l_0^p[a_{m,n}]$ by

$$A^a(x) = \{n^a\xi_n\}_{n\in\mathcal{N}} \quad \text{for } x = \{\xi_n\}_{n\in\mathcal{N}} \in l_0^p[a_{m,n}].$$

In the sequel we shall use the following

LEMMA 2.1. *Let $[a_{m,n}]$ be an M.K.M. If the operator $A^a\colon l_0^p[a_{m,n}] \to l_0^p[a_{m,n}]$ is continuous for some positive a, then the space $l^p[a_{m,n}]$ is nuclear.*

Proof. Assume that A^a is continuous for some $a > 0$. Since $A^a \circ A^a = A^{2a}$, we get that A^{ka} is continuous for $k \in \mathcal{N}$. Thus without lost of generality we may assume that $a > 1$. By continuity of A^a, for any $m \in \mathcal{N}$ there exists $k \in \mathcal{N}$ and a positive constant C such that, for $x \in l_0^p[a_{m,n}]$,

$$\|A(x)\|_m \leqslant C\|x\|_{m+k}.$$

Hence, putting $x = \{\delta_n^i\}_{n\in\mathcal{N}}$, we obtain

$$(a_{m,i})^{1/p} i^a \leqslant C(a_{m+k,i})^{1/p} \quad \text{for } p < \infty$$

and

$$a_{m,i} i^a \leqslant C a_{m+n,i} \quad \text{for } p = \infty.$$

Hence

$$\sum_{n=1}^{\infty} \frac{a_{m,n}}{a_{m+k,n}} < +\infty.$$

Thus the space $l^p[a_{m,n}]$ is nuclear.

Let $[a_{m,n}]$ be an M.K.M., and let $\{X_n\}_{n\in\mathcal{N}}$ be a sequence of finite-dimensional Banach spaces. By $l_0^p([a_{m,n}]; X_n)$ we denote the dense linear subspace of $l^p([a_{m,n}]; X_n)$ consisting of all sequences $\{x_n\}_{n\in\mathcal{N}}$ with finite number of non-zero terms.

Let $\{A_n\}_{n\in\mathcal{N}}$ be a sequence of linear operators $A_n\colon X_n \to X_n$. Then by $\bigoplus_{n=1}^{\infty} A_n$ we denote the cartesian product of A_n, i.e. for $x = \{x_n\}_{n\in\mathcal{N}}$ we put $\bigoplus_{n=1}^{\infty} A_n(x) = \{A_n(x_n)\}_{n\in\mathcal{N}}$. In the special case, $A_n = [n \dim X_n]^a I$ (I — the identity operator in X_n), we shall abbreviate $\bigoplus_{n=1}^{\infty} A_n$ by B^a.

The following proposition is a generalization of Lemma 2.1:

PROPOSITION 2.2. *Let $[a_{m,n}]$ be an M.K.M., $\{X_n\}$, $n \in \mathcal{N}$, be a sequence of finite-dimensional Banach spaces, and let $A_n\colon X_n \to X_n$ be a sequence of linear operators such that for some $a > 0$*

$$\|A_n\| \geqslant [n\cdot\dim X_n]^a. \tag{2.2}$$

If

$$\dim X_{n+1} \geqslant \dim X_n \tag{2.3}$$

and the operator $\bigoplus_{n=1}^{\infty} A_n\colon l_0^p([a_{m,n}]; X_n) \to l_0^p([a_{m,n}]; X_n)$ is continuous, then the space $l^p([a_{m,n}]; X_n)$ is nuclear.

Proof. Similarly as before we assume that $a > 1$. By continuity of $\bigoplus_{n=1}^{\infty} A_n$, we obtain that for each m there exist a k and a positive constant C such that $\|\bigoplus_n A_n(x)\|_m \leqslant C\|x\|_{m+k}$ for any x in $l^p([a_{m,n}]; X_n)$.

Assume that $p < \infty$. (The proof for $p = \infty$ is similar.)

Take $x_n^0 \in X_n$ such that $\|A_n(x_n^0)\| = \|A_n\|$ and $\|x_n^0\| = 1$. Hence putting $x = \{\xi_n x_n^0\}$, where $\{\xi_n\}$ belongs to $l_0^p[a_{m,n}]$, we have

$$\sum_{n=1}^{\infty} a_{m,n}\|A_n\|^p|\xi_n|^p \leqslant C^p \sum_{n=1}^{\infty} a_{m+k,n}|\xi_n|^p,$$

and, by (2.2),

$$\sum_{n=1}^{\infty} a_{m,n}[\dim X_n\cdot n]^{pa}|\xi_n|^p \leqslant C^p \sum_{n=1}^{\infty} a_{m+k,n}|\xi_n|^p. \tag{2.4}$$

Hence we get that the operator $A^a\colon l_0^p[a_{m,n}] \to l_0^p[a_{m,n}]$ is continuous, and so, by Lemma 2.1, the space $l^p[a_{m,n}]$ is nuclear.

It is known ([6], p. 71-72) that in this case the identity operator is an isomorphism of $l_0^1[a_{m,n}]$ onto $l_0^p[a_{m,n}]$ for any p. It is easy to see that the same holds for $l_0^1([a_{m,n}]; X_n)$ and $l_0^p([a_{m,n}]; X_n)$. Therefore we can assume without loss of generality that $p = 1$.

It is known ([10], p. 120) that in a finite-dimensional Banach space E with a norm $\|\cdot\|$ there exists a basis $\{e_i\}_{i=1}^{\dim E}$ with coefficient functionals $\{f_i\}_{i=1}^{\dim E}$ for which $\|e_i\| = \|f_i\| = 1$ ($i = 1, 2, \ldots, \dim E$). Such a basis we shall call *Auerbach basis*. Having Auerbach basis in E we can define two new norms on E, putting $\|x\|_1 = \sum_i |f_i(x)|$ and $\|x\|_\infty = \sup_i |f_i(x)|$. We get

$$\|x\|_\infty \leqslant \|x\| \leqslant \|x\|_1 \tag{2.5}$$

and

$$\|x\|_1 \leqslant \dim E \|x\|_\infty. \tag{2.6}$$

Let us denote by $\tilde{X}_n$ the space X_n equipped with the norm $\|\cdot\|_1$.

We shall prove that the spaces $l^1([a_{m,n}]; \tilde{X}_n)$ and $l^1([a_{m,n}]; X_n)$ are isomorphic. Indeed, putting in (2.4) (with $p = 1$) $\xi_n = \delta_n^i$, we obtain

$$a_{m,i}[i \dim X_i]^a \leqslant C a_{m+k,i} \quad \text{for } i, m \in \mathcal{N},$$

and hence, by (2.5) and (2.6),

$$a_{m,i}\|x_i\| \leqslant a_{m,i}\|x_i\|_1 \leqslant a_{m,i} \dim X_i \|x_i\| \leqslant C a_{m+k,i}\|x_i\|.$$

Summing these inequalities, we get

$$\sum_{n=1}^{\infty} a_{m,n}\|x_n\| \leqslant \sum_{n=1}^{\infty} a_{m,n}\|x_n\|_1 \leqslant \sum_{n=1}^{\infty} a_{m+k,n}\|x_n\|.$$

Hence the identity is an isomorphism of $l^1([a_{m,n}]; X_n)$ and $l^1([a_{m,n}]; \tilde{X}_n)$.

It is easy to see that the space $\tilde{X}_n$ is isometric to $l^1_{\dim X_n}$, and so the spaces $l^1([a_{m,n}]; \tilde{X}_n)$ and $l^1([a_{m,n}]; l^1_{\dim X_n})$ are isomorphic.

The space $l^1_0([a_{m,n}]; l^1_{\dim X_n})$ is isomorphic to $l^1[b_{m,n}]$, where $b_{m,k} = a_{m,n}$ for $\sum_{i=1}^{n-1} \dim X_i < k \leqslant \sum_{i=1}^{n} \dim X_i$.

Putting in (2.4) (with $p = 1$) $\xi_n = \|x_n\|$, we get that the operator $B^a\colon l^1_0([a_{m,n}]; X_n) \to l^1_0([a_{m,n}]; X_n)$ is continuous. The same remains true for the space $l^1([a_{m,n}]; l^1_{\dim X_n})$.

Since $k \leqslant \sum_{i=1}^{n} \dim X_i$ implies $k \leqslant n \dim X_n$, the continuity of B^a: $l^1([a_{m,n}]; l^1_{\dim X_n}) \to l^1([a_{m,n}]; l^1_{\dim X_i})$ implies that of $A^a\colon l^1_0[b_{m,n}] \to l^1_0[b_{m,n}]$. Hence, by Lemma 2.1, the space $l^1[b_{m,n}]$, and so the space $l^p([a_{m,n}]; X_n)$, are nuclear.

Remark. In Proposition 2.2 we may assume instead of (2.2) and (2.3),

$$\|A_n\| \geqslant [\dim X_n]^\beta \quad \text{for some } \beta > 0. \tag{2.2'}$$

$$\dim X_{n+1} \geqslant \dim X_n \geqslant n, \tag{2.3'}$$

Indeed, assuming (2.2′) and (2.3′) we have $[\dim X_n]^2 \geqslant n \dim X_n$ and so $\dim X_n \geqslant [n \dim X_n]^{1/2}$. Thus $\|A_n\| \geqslant [\dim X_n]^\beta$ implies $\|A_n\| \geqslant [n \dim X_n]^{\beta/2}$, and (2.3) holds.

3. Bases in the spaces l^p. Let X be a Banach space with a basis $\{e_k\}_{k\in I}$, where either $I = \{1, 2, \ldots, n\}$ for $\dim X < \infty$ or $I = \mathcal{N}$ for $\dim X = \infty$.

For a finite set of indices σ the projection P_σ in X is defined as follows:

$$P_\sigma(x) = \sum_{k\in\sigma} \xi_k e_k \quad \text{for } x = \sum_{k\in I} \xi_k e_k.$$

If $\sigma_r = \{1, 2, \ldots, r\}$, we write P_r instead of P_{σ_r}. For a basis $\{e_k\}_{k\in I}$ let us put

$$K\{e_k\} = \sup_{r\in I} \|P_r\|$$

and

$$K_u^n\{e_k\} = \sup_{\sigma\subset\sigma_n} \|P_\sigma\| \quad \text{for } n = 1, 2, \ldots$$

Of course, $K_u^{n+1}\{e_k\} \geqslant K_u^n\{e_k\}$, so there exists the limit $\lim_n K_u^n\{e_k\}$. We shall denote it by $K_u\{e_k\}$.

We call K *basis constant*, and K_u *unconditional basis constant*. It is known that $K\{e_k\}$ is finite, and $K_u\{e_k\}$ is finite if and only if $\{e_k\}_{k\in I}$ is an unconditional basis.

LEMMA 3.1. *There exists a basis $\{d_k\}_{k=1}^{\infty}$ in l^1 such that*

1° *For any $n \in \mathcal{N}$ the space $E_n = \operatorname{span}\{d_1, \ldots, d_n\}$ is isometric with l^1_n;*

2° *$K_u^n\{d_k\} \geqslant D(\varepsilon) n^{1/2-\varepsilon}$, $n \in \mathcal{N}$, for each $\varepsilon > 0$ and for some $D(\varepsilon)$.*

Proof. Let $\{d_k\}_{k=1}^{\infty}$ be the sequence

$$d_1 = (1, 0, 0, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$d_k = (\underbrace{0, 0, \ldots, -1}_{k-1}, 1, 0, \ldots), \quad k = 2, 3, \ldots$$

By $\{e_k\}_{k=1}^{\infty}$ let us denote the usual basis of unit vectors, i.e. $e_k = \{\delta_i^k\}_{i=1}^{\infty}$. Since $e_n = \sum_{k=1}^{n} d_k$, 1° holds.

Let $\{\xi_k\}_{k=1}^{\infty}$ be the sequence of coefficient functionals of the basis $\{e_k\}_{k=1}^{\infty}$, and let $\{a_k\}_{k=1}^{\infty}$ be the sequence of functionals biorthogonal to $\{d_k\}_{k=1}^{\infty}$ (i.e. $a_j(d_k) = \delta_j^k$).

Let P_r be the projections in l^1, defined at the beginning of this section, corresponding to the sequence $\{d_k\}$. Since, for $k \in \mathcal{N}$, $\xi_k = a_k - a_{k+1}$, and $d_1 = e_1$, $d_k = e_k - e_{k-1}$ for $k = 2, 3, \ldots$, we have

$$P_r(x) = \sum_{k=1}^{r} a_k(x) d_k = a_1(x) e_1 + \sum_{k=2}^{r} a_k(x)(e_k - e_{k-1})$$
$$= \sum_{k=1}^{r-1} \big(a_k(x) - a_{k+1}(x)\big) e_k + a_r(x) e_r$$
$$= \sum_{k=1}^{r} \big(a_k(x) - a_{k+1}(x)\big) e_k + a_{r+1}(x) e_r = \sum_{k=1}^{r} \xi_k(x) e_k + a_{r+1}(x) e_r.$$

Since $a_k(x) = \sum_{i=1}^{k} \xi_i(x)$, we have $|a_k(x)| \leqslant \|x\|$, $k \in \mathcal{N}$. Therefore

$$\|P_r(x)\| \leqslant \Big\| \sum_{k=1}^{r} \xi_k(x)\, e_k \Big\| + |a_{r+1}| \leqslant 2\|x\|.$$

It follows that $\{d_k\}_{k=1}^{\infty}$ is a basis in l^1.

Let γ be a fixed real number, $\gamma > \frac{1}{2}$. For

$$x_0 = \{k^{-\gamma} - (k+1)^{-\gamma}\}$$

we have

$$x_0 \in l^1 \text{ and } a_k(x_0) = 1/k^{\gamma}, \quad k \in \mathcal{N}.$$

Let σ be the set of indices $\{1, 3, 5, \ldots, 2m-1\}$. Consider the projection

$$P_\sigma(x) = \sum_{k=1}^{m} a_{2k-1}(x)\, d_{2k-1}.$$

Since

$$a_k\big(P_\sigma(x_0)\big) = \begin{cases} k^{-\gamma} & \text{for } k \leqslant 2m-1 \text{ and } k \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}$$

we obtain

$$\xi_k\big(P_\sigma(x_0)\big) = \begin{cases} k^{-\gamma} & \text{for } k \leqslant 2m-1 \text{ and } k \text{ odd}, \\ -(k+1)^{-\gamma} & \text{for } k \leqslant 2m-1 \text{ and } k \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus

$$\|P_\sigma(x_0)\| \geqslant \sum_{k=2}^{2m} k^{-\gamma} \geqslant \int_2^{2m} x^{-\gamma}\, dx \geqslant \frac{(2m)^{1-\gamma} - 2^{1-\gamma}}{1-\gamma}.$$

On the other hand,

$$\|P_{2m}(x_0)\| = \sum_{k=1}^{2m} \frac{(k+1)^{\gamma} - k^{\gamma}}{k^{\gamma}(k+1)^{\gamma}}$$

$$\leqslant \sum_{k=1}^{2m} \frac{1}{k^{2\gamma}} \leqslant \int_1^{2m} x^{-2\gamma}\, dx \leqslant \frac{(2m)^{1-2\gamma} - 1}{1-2\gamma} \leqslant \frac{1}{2\gamma - 1}.$$

Since $P_\sigma(x_0) = P_\sigma\big(P_{2m}(x_0)\big)$, we have

$$\|P_\sigma\| \geqslant \frac{\|P_\sigma\big(P_{2m}(x_0)\big)\|}{\|P_{2m}(x_0)\|} = \frac{\|P_\sigma(x_0)\|}{\|P_{2m}(x_0)\|}.$$

Thus if γ is close to $\frac{1}{2}$, we obtain

$$\|P_\sigma\| \geqslant \frac{(2m)^{1-\gamma} - 2^{1-\gamma}}{1-\gamma}(2\gamma - 1).$$

It follows that for any $\varepsilon > 0$ there exists a positive constant $D(\varepsilon)$ independent of m such that

$$\|P_\sigma\| \geqslant D(\varepsilon)\, m^{1/2-\varepsilon}.$$

Thus

$$K_u^n\{d_k\} \geqslant D(\varepsilon)\, n^{1/2-\varepsilon}, \quad n = 1, 2, 3, \ldots,$$

q.e.d.

Lemma 3.2. *Let $\{e_k\}_{k\in\mathcal{N}}$ be a basis in l^p, $1 \leqslant p \leqslant 2$, with $\|e_k\| = 1$. Let $\{t_k\}_{k\in\mathcal{N}}$ denote the sequence of coefficient functionals of the basis $\{e_k\}$. Then there exists a positive constant D_p such that*

$$K_u^n\{e_k\} \geqslant D_p \sup_{x \in l^p} \frac{\big(\sum_{k=1}^{n} |t_k(x)|^2\big)^{1/2}}{\|x\|}, \quad n = 1, 2, \ldots \tag{3.1}$$

Proof. Let us denote by G the set of all infinite sequences $g = \{\varepsilon_k\}_{k=1}^{\infty}$ with terms equal $+1$ or -1, and with all but finite number of terms equal $+1$. To each sequence $g \in G$ we assign the linear involution A_g in l^p defined by

$$A_g(x) = \sum_{k=1}^{\infty} \varepsilon_k t_k(x)\, e_k.$$

Let us put $P_g = \frac{1}{2}(I - A_g)$ (I is the identity operator). Obviously, P_g is a finite-dimensional projection. Denote by G_n a subset of G consisting of all sequences $\{\varepsilon_k\}$ such that $\varepsilon_k = 1$ for $k > n$.

We have

$$K_u^n\{e_k\} = \sup_{g\in G_n} \|P_g\|.$$

By the definition of P_g,

$$K_u^n\{e_k\} = \sup_{g\in G_n} \frac{1}{2}\|I - A_g\| \geqslant \frac{1}{2}\big(\sup_{g\in G_n} \|A_g\| - 1\big).$$

Now we use the following inequality due to Orlicz [8]: Let $\{x_k\}_{k=1}^{\infty}$ be any sequence in l^p, $1 \leqslant p \leqslant 2$; then there exists a positive constant C_p such that

$$\sup_G \Big\| \sum_{k=1}^{\infty} \varepsilon_k x_k \Big\| \geqslant C_p \Big(\sum_{k=1}^{\infty} \|x_k\|^2 \Big)^{1/2}.$$

In the special case, putting $x_k = t_k(x)\, e_k$ for $k = 1, 2, \ldots, n$ and $x_k = 0$ for $k > n$, we obtain

$$\sup_{g\in G_n} \|A_g(x)\| \geqslant C_p \Big(\sum_{k=1}^{n} |t_k(x)|^2 \Big)^{1/2}.$$

Therefore

$$\sup_{g\in G_n}\|A_g\| = \sup_{g\in G_n}\sup_{x\in l_n^p}\frac{\|A_g(x)\|}{\|x\|} \geqslant \sup_{x\in l^p}\frac{\left(\sum_{k=1}^{n}|t_k(x)|^2\right)^{1/2}}{\|x\|}.$$

So we have

$$K_u^n\{e_k\} \geqslant \frac{1}{2}\left(C_p\sup_{x\in l^p}\frac{\left(\sum_{k=1}^{n}|t_k(x)|^2\right)^{1/2}}{\|x\|}-1\right) \geqslant D_p\sup_{x\in l^p}\frac{\left(\sum_{k=1}^{n}|t_k(x)|^2\right)^{1/2}}{\|x\|}$$

for $n\in\mathcal{N}$, where D_p is a sufficiently small positive constant, q.e.d.

The first example of a conditional basis in a Hilbert space has been given by Babenko [1]. He has proved that the sequence $\{f_{a,k}\}$, $k=0, +1, -1, +2, -2, \ldots$, where $f_{a,k}(s) = |s|^a e^{iks}$ with fixed a, $-\frac{1}{2}<a<\frac{1}{2}$, is a conditional basis in the space $L^2(-\pi,\pi)$.

Using inequality (3.1) we shall prove the following

LEMMA 3.3. *Let $\frac{1}{4}<a<\frac{1}{2}$ and let $\{f_{a,k}\}$ be the Babenko basis in $L^2(-\pi,\pi)$. Then for any n*

$$K_u^n\{f_{a,k}\}\geqslant D_a n^{2a-1/2}, \tag{3.2}$$

where D_a is a positive constant independent of n.

Proof. Let $\{t_{a,k}\}$ be the sequence of coefficient functionals for the basis $\{f_{a,k}\}$. It is easy to see that for $x=x(s)$, $x\in L^2(-\pi,\pi)$, we have

$$f_{a,k}(x) = \int_{-\pi}^{\pi} x(s)e^{-iks}|s|^{-a}ds.$$

Let us put $x_a(s) = |s|^{-a}$. We obtain

$$t_{a,k}(x_a) = \int_{-\pi}^{\pi} e^{-iks}|s|^{-2a}ds = 2\int_0^{\pi}\cos ks\, s^{-2a}ds$$
$$= 2\int_0^{k\pi}\frac{\cos w}{(w/k)^{2a}}\frac{dw}{k} = 2k^{(-1+2a)}\int_0^{k\pi}\frac{\cos w}{w^{2a}}dw.$$

Since

$$\int_0^{\infty}\frac{\cos w}{w^{2a}}dw = \frac{\pi}{2\Gamma(2a)\sin\pi a}>0,$$

there exists a positive constant A_a such that

$$f_{a,k}(x_a)\geqslant A_a k^{(-1+2a)}.$$

By (3.1), we have

$$K_u^n\{f_{a,k}\}\geqslant\frac{D_2}{\|x_a\|}\left(\sum_{k=1}^{n}|t_{a,k}(x)|^2\right)^{1/2}. \tag{3.3}$$

But

$$\left(\sum_{k=1}^{n}|t_{a,k}(x_a)|^2\right)^{1/2}\geqslant A_a\left(\sum_{k=1}^{n}k^{(2a-1)\cdot 2}\right)^{1/2}\geqslant A_a\left(\sum_{k=1}^{n}k^{4a-2}\right)^{1/2}\geqslant A_a\left(\int_2^n s^{4a-2}ds\right)^{1/2}$$
$$= A_a\left(\frac{n^{4a-1}}{4a-1}-\frac{2^{4a-1}}{4a-1}\right)^{1/2}.$$

Therefore for $n\geqslant 3$ and $a>\frac{1}{4}$ there exists a positive constant $\bar{A}_a$ such that

$$\left(\sum_{k=1}^{n}|t_{a,k}(x_a)|^2\right)^{1/2}\geqslant\bar{A}_a n^{2a-1/2}. \tag{3.4}$$

Since $K_u^n\{f_{a,k}\}\geqslant 1$ for any $n\in\mathcal{N}$, from (3.3) and (3.4) we infer that (3.2) holds, q.e.d.

Let T be the unit circle on the complex plane, e.g. the set $\{z\colon |z|=1\}$, and let $L^p(T)$ be the space of all complex-valued functions, p-integrable with respect to the Lebesgue measure on T. Denote by T_n the set $\{\varepsilon_{n,1}, \varepsilon_{n,2}, \ldots, \varepsilon_{n,n}\}$, where $\varepsilon_{n,j} = \exp\{j\cdot 2\pi i n^{-1}\}$. The space l_k^p is isometric to l_k^{*p} — the space of functions on T_k, with norm defined by

$$\|f\|_p = \left(\frac{1}{k}\sum_{j=1}^{k}|f(\varepsilon_{k,j})|^p\right)^{1/p}.$$

Let A_k be a linear operator from the subspace of $L^p(T)$, spanned by the functions z^n for $-k\leqslant n\leqslant k$, into l_{2k+1}^{*p}, defined for the functions z^n by the formula

$$A_k(z^n) = \{(\varepsilon_{2k+1,j})^n\}_{j=1}^{2k+1}.$$

The theorem of Marcinkiewicz [11], p. 46, states that $\|A_k\|\leqslant C_p$ and $\|A_k^{-1}\|\leqslant C_p$ for the some constant C_p independent of k. Write $e_n^{2k+1} = \{(\varepsilon_{2k+1,j})^n\}_{j=1}^{2k+1}$. It is known that for each $1<p<\infty$ the functions $f^n(z) = z^n$, $n=0,1,-1,2,-2,\ldots$, form a basis in $L^p(T)$. Hence it follows by the Marcinkiewicz theorem that all bases $\{e_n^{2k+1}\}_{n=1}^{2k+1}$ have basis constants uniformly bounded in k.

LEMMA 3.4. *For each $1<p<2$ there exists a constant $C(p)$ such that for the basis $\{e_n^{2k+1}\}_{n=1}^{2k+1}$ in l_{2k+1}^{*p},*

$$K_u\{e_n^{2k+1}\}_{n=1}^{2k+1}\geqslant(2k+1)^{(1/p-1/2)}.$$

Proof. Let $f_0\in l_{2k+1}^{*p}$ be defined by

$$f_0 = \left(\frac{1}{2k+1}\right)^{1-1/p}\sum_{n=1}^{2k+1}e_n^{2k+1}.$$

It is easy to see that

$$f_0(\varepsilon_{2k+1,j}) = \delta^j_{2k+1}(2k+1)^{1/p},$$

and hence $\|f_0\|_p = 1$. Therefore, by (3.1), we obtain

$$K_u\{e_n^{2k+1}\} \geqslant D_p\left((2k+1)\left[\left(\frac{1}{2k+1}\right)^{1-1/p}\right]^2\right)^{1/2} = D_p(2k+1)^{(1/p-1/2)}.$$

Remark. The analogous result for $2 < p \leqslant \infty$ can be obtained from Lemma 3.4 by the standard dual argument.

For further application we summarize the results of this section in the following

COROLLARY 3.5. *For each $1 \leqslant p \leqslant \infty$ there exists positive constants $C'(p)$, $C''(p)$, $a(p)$, and bases $\{e_k^n\}_{k=1}^n$ in the spaces l_n^p such that*

$$K\{e_k^n\} \leqslant C'(p) \quad \textit{and} \quad K_u\{e_k^n\} \geqslant C''(p)n^{a(p)}.$$

4. A basic lemma. Let $[a_{m,n}]$ be M.K.M. Any matrix of a form $[a_{m_k,n_j}]$, where $\{m_k\}_{k\in\mathcal{N}}$ and $\{n_j\}_{j\in\mathcal{N}}$ are increasing sequences of indices, is called a *submatrix* of $[a_{m,n}]$. We call M.K.M. $[a_{m,n}]$ *nuclear* (*non-nuclear*), if the space $l^1[a_{m,n}]$ is nuclear (non-nuclear).

The main result of this section is

THEOREM 4.1. *Let $[a_{m,n}]$ be a non-nuclear M.K.M. Then there is a non-nuclear submatrix $[a_{m_k,n_j}]$ such that for each $p \geqslant 1$ the space $l^p[a_{m_k,n_j}]$ is isomorphic with the space $l^p([d_{m,n}]; l^p_{q}(n))$ for some M.K.M. $[d_{m,n}]$ and for a sequence of indices $\{q(n)\}$ with $q(n+1) \geqslant q(n) \geqslant n$.*

To prove this theorem we shall need several lemmas. We omit simple proofs of the first two lemmas.

LEMMA 4.2. *For each non-nuclear M.K.M. there exists a non-nuclear submatrix $[b_{m,n}]$ such that $b_{m,n} \neq 0$ for each $m, n \in \mathcal{N}$.*

LEMMA 4.3. *Let $[a_{m,n}]$ be M.K.M. such that $a_{m,n} \neq 0$ for each m and n. Then for each $k \in \mathcal{N}$ the spaces $l^p[a_{m,n}]$ and $l^p[b_{m,n}]$ (where $b_{m,n} = a_{m+k,n}/a_{k,n}$) are isomorphic.*

LEMMA 4.4. *Let $[a_{m,n}]$ be an M.K.M. with $a_{1,n} = 1$ such that $\liminf_{n\to\infty} a_{m,n} < +\infty$ for $m = 2, 3, \ldots$; then either*

1° *there exists a submatrix $[a_{m_k,n_j}]$ which is non-nuclear and $\lim_{j\to\infty} a_{m_k,n_j} = +\infty$ for any k*

or

2° *there exists a submatrix $[a_{m,n_j}]$ such that $\limsup_{j\to\infty} a_{m,n_j} < +\infty$ for any m,*

Proof ([1]). Let us denote by $\mathcal{T}$ the family of all finite subsets of $\mathcal{N}$, by $\mathcal{B}_m$ the family of all subsets $\sigma \subset \mathcal{N}$ such that

$$\sup_{n\in\sigma} a_{m,n} < +\infty,$$

by $\mathcal{Z}_m$ the family of all subsets σ such that

$$\sum_{n\in\sigma} \frac{1}{a_{m,n}} < +\infty,$$

and let $\varrho_{k,m} = \{n \in \mathcal{N}: a_{m,n} < k\}$. Since $a_{m,n} \leqslant a_{m+1\,n}$ for $m, n \in \mathcal{N}$,

$$(4.1)\qquad \begin{cases} \mathcal{B}_1 \supset \mathcal{B}_2 \supset \mathcal{B}_3 \supset \ldots, \\ \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \mathcal{Z}_3 \subset \ldots, \\ \mathcal{T} \subset \mathcal{B}_m, \quad m = 1, 2, \ldots, \\ \mathcal{T} \subset \mathcal{Z}_m, \quad m = 1, 2, \ldots, \\ \mathcal{B}_{m_1} \cap \mathcal{Z}_{m_2} \subset \mathcal{T} \quad \text{for } m_1 \geqslant m_2 \end{cases}$$

and, moreover, the families $\mathcal{T}$, $\mathcal{B}_m$, $\mathcal{Z}_m$, $\mathcal{Z} = \bigcap_{m=1}^{\infty} \mathcal{Z}_m$ are ideals of subsets of $\mathcal{N}$.

It is easy to see that 2° holds iff $\bigcap_{m=1}^{\infty} \mathcal{B}_m \neq \mathcal{T}$. So let us suppose that $\bigcap_{m=1}^{\infty} \mathcal{B}_m = \mathcal{T}$. We shall prove that 1° holds. For some m_0 the ideal $\overline{\mathcal{Z} \cup \mathcal{B}_{m_0}}$ generated by the sum $\mathcal{Z} \cup \mathcal{B}_{m_0}$ is proper. Indeed, let us suppose on the contrary that, for each m, $\mathcal{N}$ is a member of the ideal $\overline{\mathcal{Z} \cup \mathcal{B}_m}$. Then for each m we have $\mathcal{N} = z_{m'} \cup \sigma_m$ for some $\sigma_m \in \mathcal{B}_{m'}$ and $z_{m'} \in \mathcal{Z}_{m'}$ (without loss of generality we may assume that $m' > m$).

Thus, by induction, a monotone sequence of indices $\{k_n\}_{n\in\mathcal{N}}$ and two sequences $\{\sigma_n\}_{n\in\mathcal{N}}$, $\{z_{n'}\}_{n\in\mathcal{N}}$ of subsets of $\mathcal{N}$ may be constructed such that $\sigma_n \in \mathcal{B}_{k_n}$, $z_{n'} \in \mathcal{Z}_{k_{n+1}}$ and $\sigma_n \cup z_{n'} = \mathcal{N}$. Since $\liminf_{n\to\infty} a_{m,n} < \infty$, σ_n are infinite.

By (4.1), $z_{n'} \cap \sigma_{n+1} \in \mathcal{T}$. Hence $\sigma_{n+1} \setminus \sigma_n$ is finite. Now, by the standard diagonal procedure, an infinite set σ may be constructed such that $\sigma \setminus \sigma_m$ is finite for each m. Hence $\sigma \in \bigcap_{m=1}^{\infty} \mathcal{B}_m \setminus \mathcal{T}$, and this contradicts the supposition. Thus, there are σ_0 and m_0 such that $\sigma_0 \notin \overline{\mathcal{Z} \cap \mathcal{B}_{m0}}$; therefore $\sigma_0 \setminus \varrho_{k,m_0} \notin \mathcal{Z}$ for each k. This makes it possible to find for each k a finite set $\sigma_k \subset \sigma_0 \setminus \varrho_{k,m_0}$ such that

$$\sum_{n\in\sigma_k} \frac{1}{a_{k,n}} > k.$$

([1]) The author is indebted to S. Kwapień for the present form of the proof.

Let $\delta = \bigcup_{k=1}^{\infty} \sigma_k$. Then the sequence of those numbers which are enumerated in an increasing order, has the desired properties of 1°. Indeed, by the construction, $\delta \cap \varrho_{k,m_0} \in \mathcal{T}$ for each k, and so $\lim_j a_{m_0,n_j} = \infty$. Hence $\lim_j a_{m,n_j} = \infty$ for $m > m_0$. Moreover,

$$\sum_{j=1}^{\infty} \frac{1}{a_{m,n_j}} \geqslant \sum_{n\in\delta} \frac{1}{a_{k,n}} > k \quad \text{for each } k > m > m_0.$$

This implies

$$\sum_{j=1}^{\infty} \frac{1}{a_{m,n_j}} = \infty,$$

and thus the matrix $[a_{m_0+k,n_j}]$ is non-nuclear. This completes the proof.

LEMMA 4.5. *Let $\{a_n\}_{n\in\mathcal{N}}$ and $\{b_n\}_{n\in\mathcal{N}}$ be two sequences such that*

(i) $$0 < b_n \leqslant a_n,$$

(ii) $$\lim_n a_n = 0.$$

Moreover, let $\{\varrho_i\}_{i\in\mathcal{N}}$ be a sequence of pairwise disjoint subsets of $\mathcal{N}$ such that

(iii) $$\sum_i \sum_{n\in\varrho_i} b_n = \infty$$

and

(iv) $$b_n < \frac{C}{2^{i^\alpha}} \quad \text{for } n\in\varrho_i,$$

where $C > 0$ and $0 < \alpha \leqslant 1$ are constants independent of i.

Then there exists a sequence $\{\sigma_j\}_{j\in\mathcal{N}}$ of finite, pairwise disjoint subsets of $\mathcal{N}$ such that

(1) *each σ_j is contained in some ϱ_i,*

(2) $\sum_j \sum_{n\in\sigma_j} b_n = \infty$,

(3) $\operatorname{card} \sigma_j \geqslant j$,

(4) $b_n \leqslant D/2^{j^\beta}$ *for $n\in\sigma_j$, where $D > 0$ and $0 < \beta \leqslant 1$ are constants independent of j,*

(5) $a_n/a_m \leqslant 2$ *for each j and $n, m\in\sigma_j$.*

Proof. Let us put $C_1 = \sup_n a_n$ and define the sets

$$\delta_k = \left\{n\in\mathcal{N}: \frac{C_1}{2^k} < a_n \leqslant \frac{C_1}{2^{k-1}}\right\} \quad \text{for } k = 1, 2, \ldots$$

Let us put $\mathcal{L}_{i,k} = \varrho_i \cap \delta_k$. The sets $\{\mathcal{L}_{i,k}\}_{i,k\in\mathcal{N}}$ are finite and pairwise disjoint.

For $n\in\mathcal{L}_{i,k}$ we have $b_n < \max(C, 2C_1)/2^{[\max(i,k)]^\alpha}$. Indeed, if $\max(i,k) = i$, then the desired inequality is an immediate consequence of (iv), if $\max(i,k) = k$, then, by the definition of the sets δ_k and $\mathcal{L}_{i,k}$,

$$b_n \leqslant \frac{2C_1}{2^k} \leqslant \frac{\max(C, 2C_1)}{2^{[\max(i,k)]^\alpha}}.$$

Let us rearrange the double sequence $\{\mathcal{L}_{i,k}\}_{i,k\in\mathcal{N}}$ into a single one putting $\mathcal{L}_{i,k} = \mathcal{V}_s$, where

$$s(i,k) = \frac{(i+k-1)(i+k-2)}{2} + k.$$

Then $\max(i,k) \geqslant \frac{1}{2}[s(i,k)]^{1/2}$ and hence for $n\in\mathcal{V}_s$

$$b_n \leqslant \frac{\max(C, 2C_1)}{(2^{1/2})^{s^{\alpha/2}}} \leqslant \frac{D}{2^{s^\beta}}, \tag{4.2}$$

where $\beta = \frac{1}{3}\alpha$, and D is a positive constant.

Putting $R = \{s\in\mathcal{N}: \operatorname{card}\mathcal{V}_s < 1\}$ we obtain

$$\sum_{s\in R} \sum_{n\in\mathcal{V}_s} b_n \leqslant \sum_{s\in R} \operatorname{card}\mathcal{V}_s \cdot \frac{D}{2^{s^\beta}} \leqslant D \sum_{s\in\mathcal{N}} s\left(\frac{1}{2}\right)^{s^\beta} < +\infty. \tag{4.3}$$

Let us denote by $\{\sigma_j\}_{j\in\mathcal{N}}$ the sequence $\{\mathcal{V}_s\}_{s\in\mathcal{N}\setminus R}$ enumerated in the same order.

The sequence $\{\sigma_j\}_{j\in\mathcal{N}}$ satisfies all assumptions of the lemma. Indeed, (1) and (2) follow by the construction of the sets $\mathcal{L}_{i,k}$. Condition (3) is a consequence of (iii) and (4.3). Since $\mathcal{V}_s = \sigma_j$ implies $s \geqslant j$, we have $\operatorname{card}\sigma_j = \operatorname{card}\mathcal{V}_s \geqslant s \geqslant j$, and this implies (4). Moreover, by (4.2), we get for $n\in\sigma_j = \mathcal{V}_s$

$$b_n \leqslant \frac{D}{2^{s^\beta}} \leqslant \frac{D}{2^{j^\beta}}.$$

This proves that (5) is also satisfied.

LEMMA 4.6. *Let $[a_{m,n}]$ be an M.K.M. such that $a_{1,n} = 1$ for each n, $\lim_{n\to\infty} a_{m,n} = \infty$ and $\sum_{n=1}^{\infty} 1/a_{m,n} = \infty$ for each m.*

Then there exists a sequence $\{\sigma_k\}_{k\in\mathcal{N}}$ of finite, pairwise disjoint subsets of $\mathcal{N}$ with the following properties:

(a) $\operatorname{card}\sigma_{k+1} \geqslant \operatorname{card}\sigma_k \geqslant k$;

(b) *for each m there exists a constant $K(m)$ such that for any k and $n_1, n_2 \in \sigma_k$*

$$\frac{a_{m,n_1}}{a_{m,n_2}} \leqslant K(m);$$

(c) $\sum_{k=1}^{\infty} \sum_{n \in \sigma_k} 1/a_{m,n} = \infty$ *for each m.*

Proof. Let us write $d_{m,n} = 1/a_{m,n}$. Let $\{n_m\}_{m \in \mathcal{N}}$ be a monotone sequence of natural numbers such that $n_1 = 1$ and

$$\sum_{n=n_m}^{n_{m+1}-1} d_{m,n} > 1.$$

Define the sequence $\{c_n\}_{n \in \mathcal{N}}$ by putting $c_n = d_{m,n}$ for $n_m \leqslant n < n_{m+1}$. Since the matrix $[a_{m,n}]$ is monotone, $c_n \leqslant d_{m,n}$ for $n > n_m$.

Now, we apply Lemma 4.5 to the sequences $\{d_{2,n}\}_{n \in \mathcal{N}}$, $\{c_n\}_{n \in \mathcal{N}}$ and the sequence of sets $\{D_{1,j}\}_{j \in N}$, where $D_{1,1} = \mathcal{N}$ and $D_{1,j} = \emptyset$ for $j > 1$. We obtain the sequence $\{D_{2,j}\}_{j \in \mathcal{N}}$ of finite pairwise disjoint sets satisfying conditions (1)-(5) of Lemma 4.5.

Let us choose a number j_1 such that

$$\sum_{j=1}^{j_1} \sum_{n \in D_{2,j}} c_n > 1,$$

and $d_{3,n} \geqslant c_n$ for $j > j_1$ and $n \in D_{2,j}$.

Applying Lemma 4.5 to the sequences $\{d_{3,n}\}_{n \in \mathcal{N}}$, $\{c_n\}_{n \in \mathcal{N}}$ and the sequence of sets $\{\tilde{D}_{2,j}\}_{j \in \mathcal{N}}$, where $\tilde{D}_{2,j} = D_{2,j_1+j}$, we get the sequence $\{D_{3,j}\}_{j \in \mathcal{N}}$ of the sets satisfying conditions (1)-(5) of Lemma 4.5.

Let us write $k_2 = \operatorname{card} D_{2,j_1}$ and let j_2 be a natural number such that

$$\sum_{j=k_2}^{j_2} \sum_{n \in D_{3,j}} c_n > 1,$$

and $d_{4,n} > c_n$ for $j > j_2$ and $n \in D_{3,j}$.

Continuing this process by induction, we get sequences of finite sets $\{D_{m,j}\}_{j \in \mathcal{N}}$, $m = 2, 3, \ldots$, and the sequence of natural numbers $1 = k_1 < j_1 < k_2 < j_2 < \ldots$ such that

(I) $d_{m+2,n} > c_n$ for $j > j_m$ and $n \in D_{m,j}$;

(II) the sets $\{D_{m,j}\}$ for $k_m < j \leqslant j_m$ and $m = 1, 2, \ldots$ are pairwise disjoint;

(III) $a_{k,n_1}/a_{k,n_2} \leqslant 2$ for $k \geqslant m$, $j \geqslant k_m$ and $n_1, n_2 \in D_{k,j}$;

(IV) $\sum_{j=k_m}^{j_m} \sum_{n \in D_{m+1,j}} c_n > 1.$

Let $\sigma_k = D_{m,j}$ for $k = \sum_{p=1}^{m-1} (j_p - k_p) + j - k_m$.

The sequence $\{\sigma_k\}_{k \in \mathcal{N}}$ has properties (a)-(c) of the Lemma. Indeed, property (b) is a consequence of (II) and (III), property (c) follows from (I) and (IV).

Since $\sigma_k = D_{m,j}$ implies $k \leqslant j$, we have $\operatorname{card} \sigma_k = \operatorname{card} D_{m,j} \geqslant j \geqslant k$. Having the sequence $\{\sigma_k\}_{k \in \mathcal{N}}$ of finite sets for which $\operatorname{card} \sigma_k \geqslant k$, we can reorder it so that the new sequence satisfies condition (c).

Corollary 4.7. *Under the assumption of Lemma 4.6 there exists a non-nuclear submatrix $[a'_{m,n}]$ of the matrix $[a_{m,n}]$, a sequence of indices k_n for which $k_{n+1} \geqslant k_n \geqslant n$, and M.K.M $[d_{m,n}]_{m,n \in \mathcal{N}}$ such that $l^p[a'_{m,n}]$ is isomorphic to $l^p([d_{m,n}]; l^p_{k_n})$.*

Proof. Let us put $d_{m,k} = \inf_{n \in \sigma_k} a_{m,n}$ and $k_n = \operatorname{card} \sigma_n$. Define $[a'_{m,n}]$ as a submatrix consisting of all $a_{m,n}$ with $m \in \sigma_k$ for some k. The required isomorphism is provided by the properties (a), (b) and (c) of Lemma 4.6.

Now we are ready to prove Theorem 4.1.

Let $[a_{m,n}]$ be a non-nuclear M.K.M. By Lemma 4.2, there exists a non-nuclear submatrix $[a'_{m,n}]$ such that $a'_{m,n} \neq 0$ for $m, n \in \mathcal{N}$. Since $[a'_{m,n}]$ is non-nuclear, there exists an m_0 such that

$$\sum_n \frac{a'_{m,n}}{a_{m,n}} = \infty \quad \text{for } m > m_0.$$

Hence, by Lemma 4.3, it is enough to prove Theorem 4.1 under the assumption of Lemma 4.4. But then $[a_{m,n}]$ satisfies 1° or 2°.

First, assume that $[a_{m,n}]$ has a submatrix $[b_{m,n}]$ such that $b_{m,n} \leqslant M_m$ for $m = 1, 2, \ldots$ If so, then $l^p[b_{m,n}]$ is isomorphic to l^p. On the other hand, $l^p([c_{m,n}]; l^p_n) \approx l^p$, where $c_{m,n} = 1$ for $m, n \in \mathcal{N}$. This implies the assertion of Theorem 4.1.

Now, assume that $[a_{m,n}]$ satisfies 2°. Then $[a_{m,n}]$ has a submatrix with properties of Lemma 4.6. Applying Corollary 4.7 we get the proof.

5. The main results. The considerations of the preceding sections lead to the following theorem:

Theorem 5.1. *There exists a conditional basis in each non-nuclear Köthe space $l^p[a_{m,n}]$.*

Proof. By Theorem 4.1, the problem is reduced to the case of non-nuclear space $l^p([b_{m,n}]; l^p_{k(n)})$, where $[b_{m,n}]$ is M.K.M and $\{k(n)\}_{n \in \mathcal{N}}$ is a sequence of indices such that $k(n+1) \geqslant k(n) \geqslant n$.

Indeed, Theorem 4.1 implies that the space $l^p[a_{m,n}]$ is a direct sum of a space isomorphic to $l^p([b_{m,n}]; l^p_{k(n)})$ and the space $l^p[d_{m,n}]$, where $[d_{m,n}]$ is M.K.M.

For each k let $\{e_i^k\}_{i=1}^k$ be a basis in the space l_k^p with the basis constant less than C_1 and the unconditional basis constant greater than $C_2 k^a$. The existence of positive constants C_1, C_2 and a universal for all k is provided by Corollary 3.5.

Now, define a basis in the space $l^p([b_{m,n}]; l_{k(n)}^p)$ by

$$j = \sum_{n=1}^{m} k(n)+i, \quad \text{where } i \leqslant k(m+1),$$

and let

$$d_j = (\underbrace{0, 0, \ldots, 0}_{m}, e_i^{k(m+1)}, 0, 0, \ldots).$$

The sequence $\{d_j\}_{j\in\mathcal{N}}$ is a basis, since all the basis constants of $\{e_i^{k(n)}\}_{i=1}^{k(n)}$ are uniformly bounded by C_1.

Let P_{σ_n} be a projection in the space $l_{k(n)}^p$ with $\|P_{\sigma_n}\| \geqslant \frac{1}{2} C_2 [k(n)]^a$ (compare the definition of unconditional basis constant). If the basis $\{d_j\}_{j\in\mathcal{N}}$ is unconditional, then the operator $\bigoplus_{n=1}^{\infty} P_{\sigma_n}$ is continuous in $l^p([b_{m,n}]; l_{k(n)}^p)$, and this, by Proposition 2.2, contradicts the non-nuclearity of $l^p([b_{m,n}]; l_k^p)$. This completes the proof.

As a consequence of Theorem 5.1 we get two further results.

COROLLARY 5.2. *If all bases of a Fréchet space X with a basis are absolute, then X is nuclear.*

Proof. If a basis $\{e_n\}_{n\in\mathcal{N}}$ in a Fréchet space X is absolute, then X is isomorphic with some Köthe space $l^1[a_{m,n}]$. Now, Corollary 4 follows from Theorem 5.1.

COROLLARY 5.3. *If all bases of a countably-Hilbert space X with a basis are unconditional, then X is nuclear.*

Proof. Let $\{e_n\}_{n\in\mathcal{N}}$ be an unconditional basis of X and let $\{\|\cdot\|_m\}_{m\in\mathcal{N}}$ be a monotone system of Hilbertian pseudonorms on X.

Let us denote by T^∞ the group of all complex sequences $\varepsilon = \{\varepsilon_n\}_{n\in\mathcal{N}}$, $|\varepsilon_n| = 1$, with coordinatewise multiplication as a group operation and Tychonoff product topology. Then T^∞ is a compact topological group.

Since the basis $\{e_n\}$ is unconditional, for each m there exist $m_1 > m$ and positive constant K such that

$$x = \sum_{n=1}^{\infty} t_n(x) e_n \quad \text{for each } x \in X,$$

and for each sequence $\{\varepsilon_n\} \in T$ the inequality

$$\Big\|\sum_{n=1}^{\infty} \varepsilon_n t_n(x) e_n\Big\|_m \leqslant K \|x\|_{m_1} \tag{5.1}$$

holds.

To each $\varepsilon \in T$ we can now assign a linear bounded operator A_ε on X defined by

$$A_\varepsilon(x) = \sum_{n=1}^{\infty} \varepsilon_n t_n(x) e_n.$$

Since T^∞ is a group, by (5.1) we obtain that for each m there exist m_1 and m_2 $(m \leqslant m_1 \leqslant m_2)$ and positive constants K_1 and K_2 such that

$$\|x\|_m \leqslant K_1 \|A_\varepsilon(x)\|_{m_1} \leqslant K_2 \|x\|_{m_2} \tag{5.2}$$

for each $x \in X$ and $\varepsilon \in T^\infty$.

It is not difficult to verify that the correspondence $(\varepsilon, x) \to A_\varepsilon(x)$ is a continuous function of ε and x. Therefore we can define a new system of pseudonorms $\{|||\cdot|||_m\}_{m\in\mathcal{N}}$ putting

$$|||x|||_m^2 = \int_{T^\infty} \|A_\varepsilon(x)\|_m^2 \, d\varepsilon \tag{5.3}$$

(integration with respect to the normalized Haar measure on T^∞).

By 5.3 we obtain

$$\frac{1}{K_1} \|x\|_m \leqslant |||x|||_{m_1} \leqslant \frac{K_2}{K_1} \|x\|_{m_2}.$$

Therefore the new system of pseudonorms is equivalent to the previous one. It is also evident that for the new pseudonorms the parallelogram equality is valid, and hence they are Hilbertian.

By invariancy of the Haar measure, we have $|||A_\varepsilon x|||_m = |||x|||_m$ for each x, ε and m.

Hence $|||e_k + e_j|||_m = |||e_k - e_j|||_m$ for each j, k and m. Therefore for $j \neq k$ and each m

$$(e_k, e_j)_m = \frac{1}{4} (|||e_k + e_j|||_m^2 - |||e_k - e_j|||_m^2) = 0,$$

and so we have

$$|||x|||_m^2 = \sum_{n=1}^{\infty} |t_n(x)|^2 |||e_n|||_m^2.$$

Thus the correspondence $x \to \{t_k(x)\}_{k\in\mathcal{N}}$ gives an isomorphism of X onto $l^2[a_{m,n}]$, where $a_{m,n} = |||e_n|||_m^2$.

Since X is isomorphic to $l^2[a_{m,n}]$ by Theorem 5.1, it is nuclear.

References

[1] К. Бабенко, *О сопряженных функциях*, ДАН СССР 62, № 2 (1948), p. 157–160.

[2] M. M. Day, *Normed linear spaces*, Berlin – Göttingen – Heidelberg 1959.

[3] A. Dynin and B. Mitiagin, *Criterion for nuclearity in terms of approximative dimension*, Bull. Acad. Pol. Sc. 8 (1960), p. 535–540.

[4] В. Ф. Гапошкин, *Одно обобщение теоремы М. Рисса о сопряженных функциях*, Матем. сб. 46 (88) (1958), p. 359–372.
[5] C. A. McCarthy and J. Schwartz, *On the norm of a finite Boolean algebra of projections, and applications to theorem of Kreiss and Morton*, Comm. Pure Appl. Math. 18 (1965), p. 191–201.
[6] Б. С. Митягин, *Аппроксимативная размерность и базисы в ядерных пространствах*, ДАН 16 (4) (1961), p. 63–132.
[7] Л. А. Муравей, *Базисы Рисса в $L_2(-1, 1)$*, Труды мат. инст. им. Стеклова 91 (1967), p. 113–131.
[8] W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen I*, Studia Math. 4 (1933), p. 33–37.
[9] A. Pełczyński and I. Singer, *On non-equivalent bases and unconditional bases in Banach spaces*, ibidem 25 (1964), p. 5–25.
[10] A. Pietsch, *Nukleare lokalkonvexe Räume*, Berlin 1965.
[11] A. Zygmund, *Trigonometric series*, Vol. II, Cambridge 1965.

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A remark on (s, t)-absolutely summing operators in L_p-spaces

by

NICOLE TOMCZAK (Warszawa)

In this paper we prove a theorem on the composition of the p-absolutely summing and (s, t)-absolutely summing operators which is a generalization of a theorem proved by Pietsch (see [7]) concerning the composition of p-absolutely summing operators. The proof of the theorem follows Pietsch's proof.

As an application of this theorem we prove that for some class of spaces the ideals of (s, t)-absolutely summing operators have properties quite analogous to those of ideals of (s, t)-absolutely summing operators in a Hilbert space provided $1/t - 1/s = \frac{1}{2}$ and $t \leqslant 2$. The proof is quite analogous to that of the theorem stating that $A_{11}(l_r, X) \in A_{12}(l_r, X)$ if $r \leqslant 2$ (see [5]).

Definition. Let X and Y be Banach spaces, let $T \in B(X, Y)$ and let $1 \leqslant q \leqslant p < \infty$. Put

$$a_{p,q}(T) = \inf\{C: \Big(\sum_i \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup_{\|x^*\|\leqslant 1} \Big(\sum_i |x^*(x_i)|^q\Big)^{1/q}$$
$$\text{for } x_i \in X,\ i = 1, \ldots, n \text{ and } n = 1, 2, \ldots\}.$$

An operator T is said to be (p, q)-*absolutely summing* ($T \in A_{p,q}(X, Y)$) if $a_{p,q}(T) < \infty$.

It turns out that $A_{p,q}(X, Y)$ with the norm $a_{p,q}(\cdot)$ is the Banach ideal.

PROPOSITION. *Let X, Y and Z be Banach spaces, $T \in A_{p,p}(X, Y)$ and $S \in A_{s,t}(Y, Z)$. Then the operator $ST \in B(X, Z)$ is (r, q)-absolutely summing, where*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s} \leqslant 1, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{t} \leqslant 1$$

and $a_{r,q}(ST) \leqslant a_{s,t}(S)\, a_{p,p}(T)$.