References

[1] D. Przeworska-Rolewicz, *Sur les involutions d'ordre n*, Bull. Acad. Pol. Sc. 8 (1960), p. 735–739.

[2] — *On equations with reflection*, Studia Math. 33 (1969), p. 191–199.

[3] — and S. Rolewicz, *Equations in linear spaces*, Warszawa 1968.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Reçu par la Rédaction le 4. 3. 1969

STUDIA MATHEMATICA, T. XXXV. (1970)

On an equation with reflection of order n

by

BARBARA MAŻBIC-KULMA (Warszawa)

If a differential equation contains together with the unknown function $x(t)$ the function $x(-t)$, then it is called a *differential equation with reflection*.

D. Przeworska-Rolewicz gives in [1] the general solution of an equation with reflection of order 1, i.e. of the equation

$$a_0 x(t) + b_0 x(-t) + a_1 x'(t) + b_1 x'(-t) = y(t),$$

where a_0, a_1, b_0 and b_1 are scalars.

In the present paper we consider the differential equation with reflection of order n,

$$a_0 x(t) + b_0 x(-t) + \ldots + a_n x^{(n)}(t) + b_n x^{(n)}(-t) = y(t), \tag{1}$$

where the coefficients $a_0, \ldots, a_n, b_0, \ldots, b_n$ are constants. We give a general form of the solution of (1) under the following assumptions:

1° $a_n^2 - b_n^2 \neq 0$;

2° $a_{j-k} a_k - b_{j-k} b_k \neq 0$ $(k = 0, 1, \ldots, n$ and $j = k+1, \ldots, k+n)$;

3° the polynomial $\sum_{j=0}^{n} \lambda_{2j} t^j$ has single roots only for $k = 0, 1, \ldots, n$,

where

$$\lambda_j = \begin{cases} \sum_{k=0}^{j} c_{jk} & \text{for } 0 \leqslant j \leqslant n, \\ \sum_{k=j-n}^{n} c_{jk} & \text{for } n < j \leqslant 2n, \end{cases} \tag{i}$$

$$c_{jk} = (-1)^{j-k} (a_{j-k} a_k - b_{j-k} b_k)(a_n^2 - b_n^2)^{-1}. \tag{ii}$$

1. Let S be a reflection: $Sx(t) = x(-t)$. Since $S^2 = I$, where I is the identity operator, S is an involution. We write

$$Dx(t) = x'(t). \tag{2}$$

It can be proved that the operator S satisfies the following conditions:

1° S is commuting with the operator D^{2n}:

(3) $$SD^{2n}-D^{2n}S = 0;$$

2° S is anticommuting with the operator D^{2n+1}:

(4) $$SD^{2n+1}+D^{2n+1}S = 0.$$

2. Let X be a linear space over the field of complex scalars. We consider a linear equation of the form

$$(a_0 I+b_0 S)+(a_1 I+b_1 S)D+\ldots+(a_n I+b_n S)D^n = y,$$

where S is an involution on X and D is a linear operator transforming X into itself and anticommuting with S; $a_0, \ldots, a_n, b_0, \ldots, b_n$ are scalars.

Let us write

(5) $$A = \sum_{k=0}^{n} (a_k I+b_k S)D^k.$$

We prove for the operator A the following

THEOREM 1. *Let*

(6) $$B = \sum_{m=0}^{n} [(-1)^m a_m I-b_m S]D^m$$

and $R_A = (a_n^2-b_n^2)^{-1}B$. *Then*

$$AR_A = R_A A = \sum_{j=0}^{n} \lambda_{2j} D^{2j},$$

where λ_j *and* c_{jk} *are defined by* (i) *and* (ii) *respectively.*

Proof. We have

$$BA = \sum_{k=0}^{n}\sum_{m=0}^{n} [(-1)^m a_m I-b_m S]D^m(a_k I+b_k S)D^k$$
$$= \sum_{k=0}^{n}\sum_{m=0}^{n} [(-1)^m a_m I-b_m S][a_k D^m+(-1)^m b_k SD^m]D^k$$
$$= \sum_{k=0}^{n}\sum_{m=0}^{n} [(-1)^m a_m I-b_m S][a_k I+(-1)^m b_k S]D^{m+k}$$
$$= \sum_{k=0}^{n}\sum_{m=0}^{n} [(-1)^m (a_m a_k-b_m b_k)+(a_m b_k-a_k b_m)S]D^{m+k}.$$

Let us remark that

$$(a_m b_k-a_k b_m)SD^{m+k} = -(a_k b_m-a_m b_k)SD^{m+k},$$

hence

$$\sum_{k=0}^{n}\sum_{m=0}^{n} (a_m b_k-a_k b_m)SD^{m+k} = 0.$$

This implies

(7) $$BA = \sum_{k=0}^{n}\sum_{m=0}^{n} (-1)^m (a_m a_k-b_m b_k)D^{m+k}.$$

Similarly, we can show that $BA = AB$. Putting $m = j-k$ in (7), we have

$$BA = \sum_{k=0}^{n}\sum_{j=k}^{n+k} (-1)^{j-k}(a_{j-k}a_k-b_{j-k}b_k)D^j.$$

Now we write

$$\lambda_j = \begin{cases} \sum_{k=0}^{j} (-1)^{j-k}(a_{j-k}a_k-b_{j-k}b_k)(a_n^2-b_n^2)^{-1} & \text{for } 0 \leqslant j \leqslant n, \\ \sum_{k=j-n}^{n} (-1)^{j-k}(a_{j-k}a_k-b_{j-k}b_k)(a_n^2-b_n^2)^{-1} & \text{for } n < j \leqslant 2n \end{cases}$$

and

$$R_A = (a_n^2-b_n^2)^{-1}B.$$

It is easy to check that

(8) $$AR_A = R_A A = \sum_{j=0}^{2n} \lambda_j D^j$$

and that AB contains only even powers of D. Finally, we obtain

(9) $$AR_A = \sum_{j=0}^{n} \lambda_{2j} D^{2j}.$$

Let now D_T denote the domain of the operator T and Z_T the kernel of T:

$$Z_T = \{x \in D_T\colon Tx = 0\}.$$

THEOREM 2. 1° $Z_A \subset Z_T$ *and* 2° $Z_{R_A} \subset Z_T$, *where* $T = \sum_{j=0}^{n}\lambda_{2j}D^{2j}$.

Indeed, if $x \in Z_T$, then $Ax = 0$ and

$$\Big[\sum_{j=0}^{n} \lambda_{2j} D^{2j}\Big]x = R_A(Ax) = 0,$$

hence $x \in Z_T$. This implies that $Z_A \subset Z_T$. The proof of 2° is analogous.

In the following we make use of assumption 3° (p. 69) in view of which the polynomial $\sum_{j=0}^{n} \lambda_{2j} D^{2j}$, considered as a polynomial with respect to the variable D^2, has only single roots. In [1] for $n=1$ the roots are single because the corresponding polynomial is of the form $D^2-\lambda$. For $n\geqslant 2$ this polynomial may have multiple roots. Since we assume that the polynomial $\sum_{j=0}^{n}\lambda_{2j}D^{2j}$ has single roots only, we can write that

$$T=\sum_{j=0}^{n}\lambda_{2j}D^{2j}=\prod_{q=1}^{n}(D^2-u_qI),$$

where u_q denotes the q-th root.

THEOREM 3. *We have*

$$Z_T=\Big\{z\colon z=\sum_{q=1}^{n}(z_q+Sz_q')\ \text{for}\ z_q,z_q'\in Z_{D-\sqrt{u_q}I}\Big\}, \tag{10}$$

where $T=\prod_{q=1}^{n}(D^2-u_qI)$.

Proof. Let us suppose that z is of the form (10). Then

$$\begin{aligned}\Big[\prod_{q=1}^{n}(D^2-u_qI)\Big]z&=\Big[\prod_{q=1}^{n}(D^2-u_qI)\Big]\sum_{q=1}^{n}(z_q+Sz_q')\\&=\Big[\prod_{q=1}^{n}(D^2-u_qI)\Big]\Big[\sum_{q=1}^{n}z_q+S\sum_{q=1}^{n}z_q'\Big]\\&=\Big[\prod_{q=1}^{n}(D^2-u_qI)\Big]\sum_{q=1}^{n}z_q+S\Big[\prod_{q=1}^{n}(D^2-u_qI)\Big]\sum_{q=1}^{n}z_q'=0.\end{aligned}$$

Therefore $z\in Z_T$.

Conversely, let us suppose that $z\in Z_T$. We can decompose the space Z_T into a direct sum,

$$Z_T=\bigoplus_{q=1}^{n}[Z_{D-\sqrt{u_q}I}\oplus Z_{D+\sqrt{u_q}I}],$$

because D is an algebraic operator on the space Z_T with single characteristic roots (cf. [2], p. 81–82). Hence

$$z=\sum_{q=1}^{n}(z_q+z_q''),$$

where $z_q\in Z_{D-\sqrt{u_q}I}$ and $z_q''\in Z_{D+\sqrt{u_q}I}$ for $q=1,2,\ldots,n$.

We have to prove that $z_q''=Sz_q'$, where $z_q'\in Z_{D-\sqrt{u_q}I}$ for $q=1,2,\ldots,n$. But $z_q''\in Z_{D+\sqrt{u_q}I}$, hence $Dz_q''=-\sqrt{u_q}z_q''$ and $\sqrt{u_q}Sz_q''=S(\sqrt{u_q}z_q'')=-SDz_q''=DSz_q''$. Therefore

$$(D-\sqrt{u_q}I)Sz_q''=0\quad\text{and}\quad z_q'=Sz_q''\in Z_{D-\sqrt{u_q}I}$$

for $q=1,2,\ldots,n$. But $z_q''=S^2z_q''=S(Sz_q'')=Sz_q'$, which gives the required form of z.

THEOREM 4. *We have*

$$Z_A=\Big\{z\colon z=\xi\sum_{q=1}^{n}\sum_{i=0}^{n}\{[(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i]I-[(b_{2i}-b_{2i+1}\sqrt{u_q})u_q^i]S\}z_q\Big\},$$

where $z_q\in Z_{D-\sqrt{u_q}I}$, ξ *being a scalar*, $q=1,2,\ldots,n$.

Proof. Theorem 2 implies $Z_A\subset Z_T$. From Theorem 3 we infer that every $z\in Z_T$ is of the form $\sum_{q=1}^{n}(z_q+Sz_q')$, where $z_q,z_q'\in Z_{D-\sqrt{u_q}I}$.

Similarly as in the proof of Theorem 2.4 in [1] we have for $q=1,2,\ldots,n$

$$Az_q=\Big[\sum_{i=0}^{n}(a_iI+b_iS)D^i\Big]z_q=(a_0I+b_0S)z_q+\ldots+(a_nI+b_nS)D^nz_q.$$

But

$$Dz_q=\sqrt{u_q}z_q\quad\text{and}\quad DSz_q'=-\sqrt{u_q}Sz_q'$$

because $z_q\in Z_{D-\sqrt{u_q}I}$ and $Sz_q'\in Z_{D+\sqrt{u_q}I}$ for $q=1,2,\ldots,n$. Hence

$$D^iz_q=u_q^{i/2}z_q,\qquad D^iSz_q'=(-1)^iu_q^{i/2}Sz_q'$$

for $i=1,2,\ldots,2n$ and $q=1,2,\ldots,n$. Then

$$\begin{aligned}D^{2i}z_q&=u_q^iz_q, & D^{2i+1}z_q&=\sqrt{u_q}u_q^iz_q,\\ D^{2i}Sz_q'&=u_q^iSz_q', & D^{2i+1}Sz_q'&=-\sqrt{u_q}u_q^iSz_q'\end{aligned}$$

for $i=1,2,\ldots,2n$. Thus

$$Az_q=\sum_{q=1}^{n}\sum_{i=0}^{n}\{[(a_{2i}+a_{2i+1}\sqrt{u_q})u_q^i]I+[(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i]S\}z_q.$$

Similarly, we can show that

$$ASz_q'=\sum_{q=1}^{n}\sum_{i=0}^{n}\{[(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i]I+[(b_{2i}-b_{2i+1}\sqrt{u_q})u_q^i]S\}Sz_q'.$$

Hence

$$\begin{aligned}Az=&\sum_{q=1}^{n}\sum_{i=0}^{n}[(a_{2i}+a_{2i+1}\sqrt{u_q})u_q^i]z_q+\sum_{q=1}^{n}\sum_{i=0}^{n}[(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i]Sz_q+\\&+\sum_{q=1}^{n}\sum_{i=0}^{n}[(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i]Sz_q'+\sum_{q=1}^{n}\sum_{i=0}^{n}[(b_{2i}-b_{2i+1}\sqrt{u_q})u_q^i]z_q',\end{aligned}$$

but the space Z_T is a direct sum,

$$Z_T=\bigoplus_{q=1}^{n}[Z_{D-\sqrt{u_q}I}\oplus Z_{D+\sqrt{u_q}I}]$$

(see the proof of Theorem 3), and $z_q \in Z_{D-\sqrt{u_q}I}$, $Sz'_q \in Z_{D+\sqrt{u_q}I}$ and $T = \prod_{q=1}^{n}(D^2-u_qI)$, where $q = 1, 2, \ldots, n$. Thus the equality $Az = 0$ holds if and only if

$$\sum_{q=1}^{n}\sum_{i=0}^{n}[(a_{2i}+a_{2i+1}\sqrt{u_q})u_q^i z_q+(b_{2i}-b_{2i+1}\sqrt{u_q})u_q^i z_q] = 0,$$

(11)

$$\sum_{q=1}^{n}\sum_{i=0}^{n}[(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i Sz'_q+(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i Sz'_q] = 0.$$

Acting with S on both sides of the second equation of (11) and applying the property $S^2 = I$, we obtain the following system of equations:

$$\sum_{q=1}^{n}\sum_{i=0}^{n}[(a_{2i}+a_{2i+1}\sqrt{u_q})u_q^i]z_q+[(b_{2i}-b_{2i+1}\sqrt{u_q})u_q^i]z'_q = 0,$$

(12)

$$\sum_{q=1}^{n}\sum_{i=0}^{n}[(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i]z'_q+[(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i]z_q = 0.$$

From these equations it follows that z_q and z'_q are linearly dependent for $q = 1, 2, \ldots, n$. Indeed, the space X is a direct sum, which implies that (12) holds if and only if

$$[\sum_{i=0}^{n}(a_{2i}+a_{2i+1}\sqrt{u_q})u_q^i]z_q+[\sum_{i=0}^{n}(b_{2i}-b_{2i+1}\sqrt{u_q})u_q^i]z'_q = 0,$$

(13)

$$[\sum_{i=0}^{n}(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i]z'_q+[\sum_{i=0}^{n}(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i]z_q = 0.$$

This shows the linear dependence of z_q and z'_q.

We can show that the determinant of the system (13) is

$$V = \sum_{k=0}^{n}\sum_{m=0}^{n}(-1)^m(a_m a_k - b_m b_k)u_q^{(m+k)/2} = \sum_{j=0}^{2n}\lambda_j u_q^{j/2}.$$

Since u_q $(q = 1, 2, \ldots, n)$ are roots of the polynomial $\sum_{j=0}^{n}\lambda_{2j}D^2$ considered as a polynomial with respect to the variable D^2, we have

$$V = \sum_{j=0}^{2n}\lambda_j u_q^{j/2} = 0.$$

It follows that (13) has non-zero solutions for z_q and z'_q.

If we write

$$\xi_q = \sum_{i=0}^{n}(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i, \qquad \xi'_q = \sum_{i=0}^{n}(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i,$$

we obtain from the second equation of (13) that $\xi_q z_q+\xi'_q z'_q = 0$ for $q = 1, 2, \ldots, n$. Hence

$$z = \sum_{q=1}^{n}\sum_{i=0}^{n}\{[(a_{2i}-a_{2i+1}\sqrt{u_q})u_q^i]I-[(b_{2i}+b_{2i+1}\sqrt{u_q})u_q^i]S\}z_q,$$

which was to be proved.

THEOREM 5. *If $\tilde{x}$ is a solution of the equation*

(*) $$[\prod_{q=1}^{n}(D^2-u_qI)]x = y,$$

then $x = R_A\tilde{x}$ is a solution of the equation $Ax = y$.

Proof. Let $\tilde{x}$ satisfy equation (*). Then

$$Ax = AR_A\tilde{x} = [\prod_{q=1}^{n}(D^2-u_qI)]\tilde{x} = y.$$

Similarly, $t = A\tilde{x}$ is a solution of the equation $R_At = y$.

Finally, we obtain the main theorem on the general form of the solution of the equations $Ax = y$ and $R_At = y$:

THEOREM 6. *Let*

$$A = \sum_{k=0}^{n}(a_k+b_kS)D^k,$$

where S is an involution acting in a linear space X, let D be an operator transforming X into itself and anticommuting with S and let, finally, $a_0, \ldots, a_n, b_0, \ldots, b_n$ be scalars. We assume that assumptions 1°–3° (p. 69) are satisfied.

If $\tilde{x}$ is a solution of equation (*), then every solution of the equation $Ax = y$ is of the form

$$x = R_A\tilde{x}+\sum_{q=1}^{n}\sum_{i=0}^{n}\{[(a_{2i}-a_{2i+1}\sqrt{u_q})]I-[(b_{2i}+b_{2i+1}\sqrt{u_q})S]\}u_q^i z_q,$$

where $z_q \in Z_{D-\sqrt{u_q}I}$, $R_A = (a_n^2-b_n^2)^{-1}B$ and

$$B = \sum_{m=0}^{n}[(-1)^m a_m - b_mS]D^m, \qquad AR_A = R_AA = \prod_{q=1}^{n}(D^2-u_qI).$$

Similarly, any solution of the equation $R_At = y$ is of the form

$$t = Ax+\sum_{q=1}^{n}\sum_{i=0}^{n}\{[(a_{2i}+a_{2i+1}\sqrt{u_q})]I-[(b_{2i}+b_{2i+1}\sqrt{u_q})S]\}u_q^i z_q.$$

References

[1] D. Przeworska-Rolewicz, *On equation with reflection*, Studia Math. 33 (1969), p. 191–200.
[2] — and S. Rolewicz, *Equations in linear spaces*, Monografie Matematyczne 47, Warszawa 1968.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Reçu par la Rédaction le 7. 3. 1969

STUDIA MATHEMATICA, T. XXXV. (1970)

On conditional bases in non-nuclear Fréchet spaces

by

W. WOJTYŃSKI (Warszawa)

In the present paper we give some criteria for the nuclearity of Fréchet spaces with bases. Our main result is the following:

A. Let X be a Fréchet space with a basis. Then X is nuclear if and only if every basis of X is absolute (the basis $\{e_n\}$ is *absolute* if $\sum_{n=1}^{\infty} \|t_n e_n\| < \infty$ for each $x = \sum_{n=1}^{\infty} t_n e_n$ and each pseudonorm $\|\cdot\|$ on X).

For countably Hilbert spaces this result is strengthened as follows:

B. A Hilbertian Fréchet space X with a basis is nuclear if and only if every basis $\{e_n\}$ of X is unconditional (i.e. $\sum_{n=1}^{\infty} |x^*(t_n e_n)| < \infty$ for each $x = \sum_{n=1}^{\infty} t_n e_n \in X$, and each linear functional $x^* \in X^*$).

Observe that the part "only if" of our results is a consequence of the Dynin–Mitiagin theorem [3] which asserts that in a nuclear space each basis is unconditional. We do not know whether the converse is true, however, we believe the following holds:

CONJECTURE (see [9]). *A Fréchet space X with a basis is nuclear provided each basis in X is unconditional.*

The conjecture is already established for Banach spaces, because the class of nuclear Banach spaces coincides with the class of finite-dimensional spaces, and, by result of Pełczyński and Singer [9], in every infinite-dimensional Banach space with a basis there exists a conditional basis.

Statement B can be regarded as a generalization of a result due to Babenko asserting that in a Hilbert space there exists a conditional basis; [1], cf. also [4], [5] and [7].

Statement A is a generalization of an unpublished result of professor J. Rutherford (presented on the conference on functional analysis in Sopot 1968) that a Fréchet space satisfying the assumption of A is a Schwartz space.