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On equations with rotations

by

D. PRZEWORSKA-ROLEWICZ (Warszawa)

If an equation contains together with the unknown function $x(t)$ of a complex variable the values $x(\varepsilon_1 t - a_1), \ldots, x(\varepsilon_N t - a_N)$, where $\varepsilon_1, \ldots, \varepsilon_N$ are N-th roots of the unity, $a_1, \ldots, a_N$ are complex numbers, then it will be called *equation with rotation*.

The case $N = 2$, $a_1 = \ldots = a_N = 0$, was solved completely in paper [3] on equations with reflection. The purpose of this paper is to solve equations with rotation (for some a_k). The method is based on properties of involutions of order N (see [1]).

An ordinary differential equation with rotation will be considered as an example.

1. Let S be an involution of order N, i.e. a linear operator acting in a linear space X (over complex scalars) such that

$$S^N = I, \tag{1.1}$$

where I denotes the identity operator, $N \geqslant 2$, and there is no polynomial $P(t)$ of order less than N such that $P(S) = 0$. The following properties of involution of order N, proved in [1] (see also [3]), will be used.

Let us write

$$P_\nu = \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k, \quad \nu = 1, 2, \ldots, N, \tag{1.2}$$

where $\varepsilon = e^{2\pi i/N}$.

Since ε is the N-th root of the unity (with the smallest argument), we have $\varepsilon \neq 1$ and

$$\varepsilon^N = 1, \quad \varepsilon^{-k} = \varepsilon^{N-k}, \quad \sum_{k=0}^{N-1} \varepsilon^{km} = \begin{cases} 0 & \text{for } m = 1, 2, \ldots, N-1, \\ N & \text{for } m = N. \end{cases} \tag{1.3}$$

The operators P_ν are disjoint projectors giving a partition of unity:

$$P_\nu P_\mu = \delta_{\nu\mu} P_\nu, \quad \sum_{\nu=1}^{N} P_\nu = I, \tag{1.4}$$

where $\delta_{\nu\mu}$ is the Kronecker symbol. Moreover,

$$SP_\nu = \varepsilon^\nu P_\nu \quad (\nu = 1, 2, \ldots, N). \tag{1.5}$$

From this it follows that the space X can be decomposed in a direct sum

$$X = X_{(1)} \oplus \ldots \oplus X_{(N)} \tag{1.6}$$

of spaces $X_{(\nu)}$ such that

$$X_{(\nu)} = P_\nu X \text{ and } Sx = \varepsilon^\nu x \text{ for } x \in X_{(\nu)} \quad (\nu = 1, 2, \ldots, N) \tag{1.7}$$

and every element $x \in X$ can be written in a unique manner in the form

$$x = x_{(1)} + \ldots + x_{(N)}, \quad \text{where } x_{(\nu)} \in X_{(\nu)}, \tag{1.8}$$

if we put $x_{(\nu)} = P_\nu x$ $(\nu = 1, 2, \ldots, N)$.

A linear operator transforming X into itself is *permuting an involution* S *of order* N acting in X if both superpositions SD and DS exist and

$$DS = \varepsilon SD, \quad \text{where } \varepsilon = e^{2\pi i/N}. \tag{1.9}$$

It will be shown further that a permuting operator D and its powers permute the spaces $X_{(1)}, \ldots, X_{(N)}$ determined by decomposition (1.6).

Property 1.1. *For arbitrary positive integers k and m, if D is permuting an involution S of order N, then*

$$D^m S^k = \varepsilon^{mk} S^k D^m. \tag{1.10}$$

Proof (by induction). By assumption, (1.10) is true for $m = k = 1$. Let us suppose (1.10) be true for $m = 1$. Then $DS^{k+1} = (DS)S^{k+1}$ $= \varepsilon S(DS^k) = \varepsilon\varepsilon^k DS^{k+1} = \varepsilon^{k+1} S^{k+1} D$. Let k be arbitrarily fixed. Then, supposing (1.10) to be true, we obtain

$$\begin{aligned} D^{m+1} S^k &= D(D^m S^k) = D(\varepsilon^{km} S^k D^m) = \varepsilon^{km}(DS^k)D^m \\ &= \varepsilon^{km}\varepsilon^k S^k D^{m+1} = \varepsilon^{k(m+1)} S^k D^{m+1}. \end{aligned}$$

Property 1.2. *If D is permuting an involution S of order N, then*

$$\begin{aligned} P_\nu D &= DP_{\nu+1} \quad \text{for } \nu = 1, 2, \ldots, N-1, \\ P_N D &= DP_1. \end{aligned}$$

Proof. By definition, we have for $\nu = 1, 2, \ldots, N$

$$\begin{aligned} P_\nu D &= \left(\frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k\right) D = \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k D = \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu}\varepsilon^{-k} DS^k \\ &= D\frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-k(\nu+1)} S^k, \end{aligned}$$

and for $\nu = N$ we find $\varepsilon^{-k(N+1)} = \varepsilon^{-k}$. Hence $P_N D = DP_1$. For $\nu = 1, 2, \ldots, N-1$ we obtain $P_\nu D = DP_{\nu+1}$.

Similarly we prove the following

Property 1.3. *Under assumptions of 1.2,*

$$\begin{aligned} P_\nu D^2 &= D^2 P_{\nu+2} \quad \text{for } \nu = 1, 2, \ldots, N-2, \\ P_{N-1} D^2 &= D^2 P_1, \quad P_N D^2 = D^2 P_2. \end{aligned}$$

Property 1.4. *Under assumption of 1.2, the operator D^N is commuting with S and P_ν:*

$$D^N S = SD^N \text{ and } D^N P_\nu = P_\nu D^N \quad (\nu = 1, 2, \ldots, N).$$

Property 1.5. *Let*

$$Q(t) = \sum_{k=0}^{N-1} q_k t^k$$

be an arbitrary polynomial (of complex variable t) with constant complex coefficients. Let S be an involution of order N. Then

$$Q(S) = \sum_{\nu=1}^{N} Q(\varepsilon^\nu) P_\nu.$$

Indeed, formulae (1.4) and (1.5) imply

$$\begin{aligned} Q(S) &= \sum_{k=0}^{N-1} q_k S^k = \sum_{k=0}^{N-1} q_k S^k \Big(\sum_{\nu=1}^{N} P_\nu\Big) \\ &= \sum_{\nu=1}^{N}\Big(\sum_{k=0}^{N-1} q_k S^k P_\nu\Big) = \sum_{\nu=1}^{N}\sum_{k=0}^{N-1} q_k \varepsilon^{\nu k} P_\nu \\ &= \sum_{\nu=1}^{N}\Big[\sum_{k=0}^{N-1} q_k (\varepsilon^\nu)^k\Big] P_\nu = \sum_{\nu=1}^{N} Q(\varepsilon^\nu) P_\nu. \end{aligned}$$

Property 1.6. *Let*

$$Q(t) = \sum_{k=0}^{N-1} q_k t^k, \quad R(t) = \sum_{k=0}^{N-1} r_k t^k$$

be arbitrary polynomials (of a complex variable t) with constant complex coefficients. Let S be an involution of order N. Then

$$Q(S)R(S) = \sum_{\nu=1}^{N} Q(\varepsilon^\nu) R(\varepsilon^\nu) P_\nu.$$

Indeed, formula (1.4) and the preceding property imply

$$\begin{aligned} Q(S)R(S) &= \Big[\sum_{\nu=1}^{N} Q(\varepsilon^\nu) P_\nu\Big]\Big[\sum_{\mu=1}^{N} R(\varepsilon^\mu) P_\mu\Big] \\ &= \sum_{\nu,\mu=1}^{N} Q(\varepsilon^\nu) R(\varepsilon^\mu) P_\nu P_\mu \\ &= \sum_{\nu,\mu=1}^{N} Q(\varepsilon^\nu) R(\varepsilon^\mu) \delta_{\nu\mu} P_\nu = \sum_{\nu=1}^{N} Q(\varepsilon^\nu) R(\varepsilon^\nu) P_\nu. \end{aligned}$$

Corollary 1.1. *Under the assumption of Property* 1.6, $Q(S)R(S) = aI$, *where a is an arbitrary scalar, if and only if* $Q(\varepsilon^\nu)R(\varepsilon^\nu) = a$ *for* $\nu = 1, 2, \ldots, N$.

Corollary 1.2. *Under the assumption of Property* 1.5 *there exists* $Q^{-1}(S)$ *if and only if* $Q(\varepsilon^\nu) \neq 0$ *for* $\nu = 1, 2, \ldots, N$ *and*

$$Q^{-1}(S) = \sum_{\nu=1}^{N} Q^{-1}(\varepsilon^\nu)P_\nu.$$

This and Property 1.5 imply

$$S^k = \sum_{\nu=1}^{N} \varepsilon^{\nu k}P_\nu \quad \text{for } k = 0, \pm 1, \pm 2, \ldots \tag{1.11}$$

Property 1.7. *Under the assumption of Property* 1.5,

$$Q(\varepsilon^m S) = \sum_{\nu=1}^{N} Q(\varepsilon^{\nu+m})P_\nu \quad (m = \pm 1, \pm 2, \ldots).$$

Indeed,

$$\begin{aligned} Q(\varepsilon^m S) &= \sum_{k=0}^{N-1} q_k(\varepsilon^m S)^k = \sum_{k=0}^{N-1} q_k \varepsilon^{mk} S^k\Big(\sum_{\nu=1}^{N} P_\nu\Big) \\ &= \sum_{\nu=1}^{N}\Big(\sum_{k=0}^{N-1} q_k \varepsilon^{mk} S^k P_\nu\Big) = \sum_{\nu=1}^{N}\Big(\sum_{k=0}^{N-1} q_k \varepsilon^{mk}\varepsilon^{\nu k}P_\nu\Big) \\ &= \sum_{\nu=1}^{N}\Big(\sum_{k=0}^{N-1} q_k \varepsilon^{(\nu+m)k}\Big)P_\nu = \sum_{\nu=1}^{N} Q(\varepsilon^{\nu+m})P_\nu. \end{aligned}$$

Property 1.8. *Under the assumption of Property* 1.5, *if D is permuting S, then*

$$DQ(S) = Q(\varepsilon S)D.$$

Indeed, according to Property 1.2,

$$\begin{aligned} DQ(S) &= D\sum_{\nu=1}^{N} Q(\varepsilon^\nu)P_\nu = \sum_{\nu=1}^{N} Q(\varepsilon^\nu)DP_\nu = Q(\varepsilon^\nu)DP_1 + \sum_{\nu=2}^{N} Q(\varepsilon^\nu)DP_\nu \\ &= Q(\varepsilon)P_N D + \sum_{\nu=2}^{N} Q(\varepsilon^\nu)P_{\nu-1}D = \Big[Q(\varepsilon^{N+1})P_N + \sum_{\mu=1}^{N-1} Q(\varepsilon^{\mu+1})P_\nu\Big]D \\ &= \Big[\sum_{\nu=1}^{N} Q(\varepsilon^{\nu+1})P_\nu\Big]D = Q(\varepsilon S)D. \end{aligned}$$

Corollary 1.3. *Under the assumption of Property* 1.5, *if D is permuting S, then*

$$\begin{aligned} D^m Q(S) &= Q(\varepsilon^m S)D \quad \textit{for } m = 1, 2, \ldots, N-1, \\ D^N Q(S) &= Q(S)D^N. \end{aligned}$$

2. For any linear operator T transforming a linear space X into itself we denote by D_T the domain of T and by Z_T the kernel of T: $Z_T = \{x \in D_T\colon Tx = 0\}$.

Let S be an involution of order N acting in a linear space X and let D be permuting S. Let us consider the operator

$$A = a(S) - b(S)D,$$

where

$$a(t) = \sum_{k=0}^{N-1} a_k t^k \quad \text{and} \quad b(t) = \sum_{k=0}^{N-1} b_k t^k$$

are arbitrary polynomials with constant complex coefficients. In this section we assume that

$$a(\varepsilon^\nu) \neq 0 \text{ and } b(\varepsilon^\nu) \neq 0 \quad \text{for } \nu = 1, 2, \ldots, N. \tag{2.1}$$

Under these assumptions we shall determine the set Z_A.

Lemma 2.1. *The equation $Ax = 0$ is equivalent to the following system of equations*:

$$\begin{aligned} Dx_{(1)} &= c_N x_{(N)}, \\ Dx_{(m+1)} &= c_m x_{(m)} \quad (m = 1, 2, \ldots, N-1), \end{aligned} \tag{2.2}$$

where $x_{(m)} = P_m x$ *and* $c_m = a(\varepsilon^m)/b(\varepsilon^m) \neq 0$ (*by assumption*) *for* $m = 1, 2, \ldots, N$.

Proof. Property 1.5 and (1.4) imply

$$\begin{aligned} P_m a(S) &= P_m \sum_{\nu=1}^{N} a(\varepsilon^\nu)P_\nu = \sum_{\nu=1}^{N} a(\varepsilon^\nu)P_m P_\nu \\ &= \sum_{\nu=1}^{N} a(\varepsilon^\nu)\delta_{m\nu}P_m = a(\varepsilon^m)P_m. \end{aligned} \tag{2.3}$$

Similarly, $P_m b(S) = b(\varepsilon^m)P_m$. Hence

$$P_m A = P_m[a(S) - b(S)D] = P_m a(S) - P_m b(S)D = a(\varepsilon^m)P_m - b(\varepsilon^m)P_m D$$

and, by Property 1.2,

$$P_m A = \begin{cases} a(\varepsilon^m)P_m - b(\varepsilon^m)P_{m+1} & \text{for } m = 1, 2, \ldots, N-1, \\ a(\varepsilon^N)P_N - b(\varepsilon^N)DP_1 & \text{for } m = N. \end{cases} \tag{2.3}$$

Applying formulae (1.6), (1.7) and (1.8), we infer that the equation $Ax = 0$ is equivalent to the system of equations

$$P_m Ax = 0 \quad (m = 1, 2, \ldots, N).$$

According to (2.3′), the last system can be written as follows:

$$[a(\varepsilon^m)P_m - b(\varepsilon^m)DP_{m+1}]x = 0 \quad \text{for } m = 1, 2, \ldots, N-1,$$
$$[a(\varepsilon^N)P_N - b(\varepsilon^N)DP_1]x = 0 \quad \text{for } m = N.$$

Since $P_m x = x_{(m)}$ and $b(\varepsilon^m) \neq 0$ for $m = 1, 2, \ldots, N$, we obtain finally the system (2.2).

LEMMA 2.2. *$Z_A \subset Z_{D^N - \lambda I}$, where*

$$\lambda = c_1 c_2 \ldots c_N = \prod_{1 \leqslant \nu \leqslant N} \frac{a(\varepsilon^\nu)}{b(\varepsilon^\nu)} \neq 0.$$

Proof. Let $x \in Z_A$, i.e. $Ax = 0$. According to Lemma 2.1, the equation $Ax = 0$ is equivalent to the system (2.2). Let us consider $x_{(1)} = P_1 x$. From the system (2.2) we obtain

$$Dx_{(1)} = c_N x_{(N)},$$
$$D^2 x_{(1)} = D(Dx_{(1)}) = c_N Dx_{(N)} = c_N c_{N-1} x_{(N-1)},$$
$$D^3 x_{(1)} = D(D^2 x_{(1)}) = c_N c_{N-1} Dx_{(N-1)} = c_N c_{N-1} c_{n-2} x_{(N-2)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^N x_{(1)} = D(D^{N-1} x_{(1)}) = c_N c_{N-1} \ldots c_2 Dx_{(2)} = c_N c_{N-1} \ldots c_2 c_1 x_{(1)} = \lambda x_{(1)}.$$

Hence $(D^N - \lambda I)x_{(1)} = 0$ and $x_{(1)} \in Z_{D^N - \lambda I}$. Similarly, we can show that $x_{(m)} = P_m x \in Z_{D^N - \lambda I}$ $(m = 2, 3, \ldots, N)$. Since $x_{(m)} \in X_{(m)}$ and the space X is decomposed into a direct sum of spaces $X_{(m)}$, we obtain

$$x = \sum_{m=1}^{N} x_{(m)} \in Z_{D^N - \lambda I},$$

which was to be proved.

LEMMA 2.3. *$Z_{D^N - \lambda I} = \bigoplus_{0 \leqslant k \leqslant N-1} Z_{D - \lambda_k I}$, where λ_k are N-th roots of λ:*

$$\lambda_k = \sqrt[N]{|\lambda|}\, e^{(2\pi k + \varphi)i/N}, \tag{2.4}$$

where $\varphi = \operatorname{Arg} \lambda$ $(0 \leqslant \varphi \leqslant 2\pi)$, $k = 0, 1, \ldots, N-1$.

Proof. Let us remark that

$$D^N - \lambda I = (D - \lambda_0)(D - \lambda_1) \ldots (D - \lambda_{N-1}). \tag{2.5}$$

The operator D satisfies the polynomial identity $D^N - \lambda I = 0$ on the space $Z_{D^N - \lambda I}$. Similarly as in (1.6), we can prove (see also [3], part A, Chapter II) that $Z_{D^N - \lambda I} = \bigoplus_{0 \leqslant k \leqslant N-1} Y_k$ and $y \in Y_k$ if and only if $Dy = \lambda_k y$, because $\lambda_0, \ldots, \lambda_{N-1}$ are N-th roots of the equation $t^N - \lambda = 0$. Therefore $Y_k = Z_{D - \lambda_k I}$ for $k = 0, 1, \ldots, N-1$.

THEOREM 2.1.

$$Z_{D^N - \lambda I} = \Big\{z \in X:\ z = \sum_{k=0}^{N-1} a_k S^k z_k;\ z_k \in Z_{D - \lambda_0 I}\Big\}.$$

Proof. First we remark that

$$\lambda_k = \lambda_0 \varepsilon^k \quad \text{for } k = 1, 2, \ldots, N-1. \tag{2.6}$$

Indeed,

$$\lambda_k = \sqrt[N]{\lambda}\, e^{\frac{\varphi + 2\pi k}{N} i} = \sqrt[N]{\lambda}\, e^{\frac{\varphi}{N} i} (e^{\frac{2\pi i}{N}})^k = \lambda_0 \varepsilon^k \quad (k = 1, 2, \ldots, N-1).$$

Let us suppose that $z \in Z_{D - \lambda_k I}$. We show that $z = S^k u$, where $u \in Z_{D - \lambda_0 I}$. Indeed,

$$Dz = \lambda_k z = \lambda_0 \varepsilon^k z \quad \text{and} \quad S^{N-k} Dz = \lambda_0 \varepsilon^k S^{N-k} z.$$

But Property 1.1 implies $S^{N-k} Dz = \varepsilon^{-(N-k)} DS^{N-k} z$. Hence

$$DS^{N-k} z = \lambda_0 \varepsilon^k \varepsilon^{N-k} S^{N-k} z = \lambda_0 S^{N-k} z.$$

Therefore $u = S^{N-k} z \in Z_{D - \lambda_0 I}$. But $z = S^N z = S^k S^{N-k} z = S^k u$.

Conversely, we show that for any $z \in Z_{D - \lambda_0 I}$ we have $S^k z \in Z_{D - \lambda_k I}$. Indeed,

$$DS^k z = \varepsilon^k S^k Dz_0 = \varepsilon^k \lambda_0 z = \lambda_0 \varepsilon^k S^k z = \lambda_k S^k z.$$

Hence $S^k z \in Z_{D - \lambda_k I}$.

To find the general form of the set Z_A we shall determine first this set in a particular case.

PROPOSITION 2.2. *If $\dim Z_{D - \lambda_0 I} = 1$, then*

$$Z_A = \Big\{z \in X:\ z = d\Big[\sum_{k=0}^{N-1} d_k S^k\Big] z_0;\ z_0 \in Z_{D - \lambda_0 I} \text{ and}$$
$$\text{the scalar } d \text{ is arbitrary},\ d_k = \sum_{m=1}^{N} \lambda_0^{-m} c_1 c_2 \ldots c_m V_{k,m}\Big\},$$

where by $V_{k,m}$ we denote the subdeterminant obtained by cancelling the $(k+1)$-th column and the m-th row of the Van der Monde determinant V of numbers $\varepsilon^2, \varepsilon^3, \ldots, \varepsilon^N, \varepsilon^1$ and $c_m = a(\varepsilon^m)/b(\varepsilon^m)$.

Proof. Since $Z_A \subset Z_{D^N - \lambda I}$ (Lemma 2.2) and $\dim Z_{D - \lambda_0 I} = 1$, we have $z \in Z_A$ if and only if

$$z = \sum_{k=0}^{N-1} a_k S^k z_0, \quad z_0 \in Z_{D - \lambda_0 I},$$

is arbitrary, and the coefficients a_k are chosen suitably. Let us write

$$a(S) = \sum_{k=0}^{N-1} a_k S^k.$$

Then $z = a(S) z_0 = \sum_{\nu=1}^{N} a(\varepsilon^\nu) P_\nu z_0$ and $P_m z = a(\varepsilon^m) P_m z_0$. Hence, by formula (2.3) in Lemma 2.1,

$$[c_m a(\varepsilon^m) P_m - a(\varepsilon^{m+1}) DP_{m+1}] z_0 = 0, \quad m = 1, 2, \dots, N-1,$$
$$[c_N a(\varepsilon^N) P_N - a(\varepsilon) DP_1] z_0 = 0,$$

where $c_m = a(\varepsilon^m)/b(\varepsilon^m)$.

But

$$DP_{m+1} z_0 = P_m D z_0 = P_m \lambda_0 z_0 = \lambda_0 P_m z_0 \quad \text{for } m = 1, 2, \dots, N-1,$$
$$DP_1 z_0 = P_N D z_0 = P_m \lambda_0 z_0 = \lambda_0 P_N z_0.$$

Hence the last system can be written as follows:

$$(2.7) \qquad \begin{aligned} &[c_m a(\varepsilon^m) - \lambda_0 a(\varepsilon^{m+1})] P_m z_0 \quad (m = 1, 2, \dots, N-1),\\ &[c_N a(\varepsilon^N) - \lambda_0 a(\varepsilon)] P_N z_0 = 0. \end{aligned}$$

Let us remark that $P_m z_0 \neq 0$ for $m = 1, 2, \dots, N$, if $z_0 \neq 0$. Indeed, let us suppose that for an m we have $P_m z_0 = 0$. This means that

$$\sum_{k=0}^{N-1} \varepsilon^{-km} S^k z_0 = 0;$$

but this implies linear dependence of all elements $z_0, Sz_0, \dots, S^{N-1} z_0$. But $S^k z_0 \in Z_{D-\lambda_k I}$, and the space $Z_{D^N - \lambda I}$ is a direct sum of spaces $Z_{D-\lambda_k I}$; this implies $S^k z_0 = 0$ for $k = 0, 1, \dots, N-1$. In particular, $z_0 = 0$, a contradiction. Hence, z_0 being arbitrary, Corollary 1.1 implies that equalities (2.7) hold if and only if

$$(2.8) \qquad \begin{aligned} &c_m a(\varepsilon^m) - \lambda_0 a(\varepsilon^{m+1}) = 0 \quad (m = 1, 2, \dots, N-1),\\ &c_N a(\varepsilon^N) - \lambda_0 a(\varepsilon) = 0. \end{aligned}$$

We obtained finally the system of N homogeneous equations with N unknows $a(\varepsilon), a(\varepsilon^2), \dots, a(\varepsilon^N)$. The determinant Δ of this system is

$$\Delta = \begin{vmatrix} c_1 & -\lambda_0 & 0 & 0 & \dots & 0 & 0\\ 0 & c_2 & -\lambda_0 & 0 & \dots & 0 & 0\\ 0 & 0 & c_3 & -\lambda_0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & c_{N-1} & -\lambda_0\\ -\lambda_0 & 0 & 0 & 0 & \dots & 0 & c_N \end{vmatrix}.$$

The expansion of Δ with respect to the last row gives

$$\Delta = (-1)^{N+1}(-\lambda_0) \begin{vmatrix} -\lambda_0 & 0 & 0 & \dots & 0 & 0\\ c_2 & -\lambda_0 & 0 & \dots & 0 & 0\\ 0 & c_3 & -\lambda_0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & c_{N-1} & -\lambda_0 \end{vmatrix} +$$
$$+ (1)^{2N} c_N \begin{vmatrix} c_1 & -\lambda_0 & 0 & \dots & 0\\ 0 & c_2 & -\lambda_0 & \dots & 0\\ 0 & 0 & c_3 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & c_N \end{vmatrix}.$$

Then first determinant has zeros only above the principal diagonal and the second one, only under the principal diagonal. Therefore

$$\Delta = (-1)^{N+1}(-\lambda_0)^N + (-1)^{2N} c_N c_1 c_2 \dots c_{N-1}$$
$$= (-1)^{2N+1} \lambda_0^N + \lambda = -\lambda + \lambda = 0,$$

because $\lambda_0^N = \lambda$.

Since the first subdeterminant of order $N-1$ is (by assumption) different from zero, we solve the system (2.8) by cancelling the last equation and by putting

$$a(\varepsilon) = a,$$

where a is an arbitrary complex number. We obtain

$$a(\varepsilon^2) = \frac{c_1}{\lambda_0} a,$$

$$a(\varepsilon^{m+1}) = \frac{c_m}{\lambda_0} a(\varepsilon^m) \quad \text{for } m = 2, 3, \dots, N-1.$$

Hence

$$a(\varepsilon^{m+1}) = \frac{c_m c_{m-1} \dots c_1}{\lambda_0^m} a,$$

a is an arbitrary complex number ($m = 1, 2, \dots, N-1$).

We have determined

$$a(\varepsilon^{m+1}) = \sum_{k=0}^{N-1} a_k \varepsilon^{(m+1)k}.$$

Now we shall determine the constants a_k. We obtain the following system of equations:

$$\sum_{k=0}^{N-1} a_k \varepsilon^{(m+1)k} = \frac{c_m c_{m-1} \dots c_1}{\lambda_0^m} \quad (m = 1, 2, \dots, N).$$

If we remark that by definition

$$a = a(\varepsilon) = \sum_{k=1}^{N-1} a_k \varepsilon^k = \sum_{k=0}^{N-1} a_k \varepsilon^{(N+1)k}$$

and $\lambda_0^N = \lambda$, $c_N c_{N-1} \dots c_1 = \lambda$, we can write this system as follows:

$$\sum_{k=0}^{N-1} a_k \varepsilon^{(m+1)k} = \frac{c_m c_{m-1} \dots c_1}{\lambda_0^m} \quad (m = 1, 2, \dots, N). \tag{2.9}$$

We have obtained the system of N linear non-homogeneous equations with N unknowns $a_1, \dots, a_N$. The determinant V of the system (2.9) is the Van der Monde determinant of numbers $\varepsilon^2, \varepsilon^3, \dots, \varepsilon^{N+1} = \varepsilon$ different one from the others. Hence

$$V = \prod_{1 \leq k, m \leq N, k \neq m} (\varepsilon^k - \varepsilon^m) \neq 0.$$

Let us denote by $V_{k,m}$ the subdeterminant of V obtained by cancelling the $(k+1)$-th column and the m-th row. The unique solution of (2.9) is then of the form

$$a_k = \frac{a}{V} \sum_{m=1}^{N} \frac{c_1 \dots c_m}{\lambda_0^m} V_{k,m} \quad (k = 0, 1, \dots, N-1).$$

Since a is an arbitrary complex number, write $d = a/V$ and we obtain the thesis of the theorem if we put

$$d_k = \sum_{m=1}^{N} \lambda_0^{-m} c_1 \dots c_m V_{k,m}.$$

Now we prove the general theorem without any assumption concerning $\dim Z_{D-\lambda_0 I}$.

THEOREM 2.3.

$$Z_A = \{z \in X : z = d \sum_{k=0}^{N-1} d_k S^k z_0;\ z_0 \in Z_{D-\lambda_0 I} \text{ and the scalar } d \text{ is arbitrary},$$

$$d_k = \sum_{m=1}^{N} \lambda_0^{-m} c_1 \dots c_m V_{k,m}\},$$

where we denote by $V_{k,m}$ the subdeterminant obtained by cancelling the $(k+1)$-th column and the m-th row of the Van der Monde determinant V of numbers $\varepsilon^2, \dots, \varepsilon^N, \varepsilon$ and $c_m = a(\varepsilon^m)/b(\varepsilon^m)$.

Proof. Since $Z_A \subset Z_{D^N - \lambda I}$ (Lemma 2.2), we infer that $z \in Z_A$ is of the form

$$z = \sum_{k=0}^{N-1} S^k z_k, \quad z_k \in Z_{D-\lambda_0 I}$$

(Theorem 2.1). If we write $d(S) = -a(S) b^{-1}(S)$, we infer that the equation

$$Az = [a(S) - b(S) D] z = 0$$

is equivalent to the equation

$$[D - d(S)] z = 0. \tag{2.10}$$

We write

$$d(S) = \sum_{m=0}^{N-1} d_m S^m.$$

Since $z \in Z_{D^N - \lambda I}$, we have

$$Dz = D \sum_{k=0}^{N-1} S^k z_k = \sum_{k=0}^{N-1} D S^k z_k = \sum_{k=0}^{N-1} \varepsilon^k S^k D z_k = \sum_{k=0}^{N-1} \varepsilon^k \lambda_0 S^k z_k.$$

Hence, if $z \in Z_A$, then

$$0 = [D - d(S)] z = \sum_{k=0}^{N-1} \varepsilon^k \lambda_0 S^k z_k - \sum_{k=0}^{N-1} d(S) S^k z_k = \sum_{k=0}^{N-1} \varepsilon^k \lambda_0 S^k z_k$$
$$= \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} d_m S^{m+k} z_k.$$

Since $S^j z_k \in Z_{D-\lambda_j I}$ (Theorem 2.1), all elements $S^j z_k$ are linearly independent. This implies that equation (2.9) is equivalent to the following system equations:

$$\begin{aligned}
(d_0 - \lambda_0) z_0 + d_{N-1} z_1 + d_{N-2} z_2 + \dots + d_1 z_{N-1} &= 0, \\
d_1 S z_0 + (d_0 - \varepsilon \lambda_0) S z_1 + d_{N-1} S z_2 + \dots + d_2 S z_{N-1} &= 0, \\
\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\
d_{N-1} S^{N-1} z_0 + d_{N-2} S S^{N-1} z_1 + d_{N-3} S^{N-1} z_2 + \dots + (d_0 - \varepsilon^{N-1} \lambda_0) S^{N-1} z_{N-1} &= 0.
\end{aligned}$$

Acting on both sides of the k-th equation with the operator S^{N-k} $(k = 0, 1, \dots, N-1)$ and using the identity $S^N = I$, we obtain the following system of equations:

$$\begin{aligned}
(d_0 - \lambda_0) z_0 + d_{N-1} z_1 + \dots + d_1 z_{N-1} &= 0, \\
d_1 z_0 + (d_0 - \varepsilon \lambda_0) z_1 + \dots + d_2 z_{N-1} &= 0, \\
\dots \dots \dots \dots \dots \dots \\
d_{N-1} z_0 + d_{N-2} z_1 + \dots + (d_0 - \varepsilon^{N-1} \lambda_0) z_{N-1} &= 0.
\end{aligned} \tag{2.11}$$

Let us consider the matrix M of the system (2.11). First we remark that this matrix does not depend on the dimension of the space $Z_{D-\lambda_0 I}$. This implies also that the rank of the matrix M does not depend on the dimension of the space $Z_{D-\lambda_0 I}$. An immediate consequence of Proposition

2.2 is that rank $M = N-1$ in the case $\dim Z_{D-\lambda_0 I} = 1$. Hence we must have rank $M = N-1$ also in the general case. Further considerations follow the same way as Proposition 2.2.

3. The notation and assumptions of the preceding section remain unchanged. We now determine the general form of solutions of the non-homogeneous equation $Ax = y$.

LEMMA 3.1. *Let $d(S) = a(S)b^{-1}(S)$. Then*

$$\prod_{m=0}^{N-1} d(\varepsilon^m S) = \lambda I.$$

Proof. Property 1.7 implies

$$d(\varepsilon^m S) = \sum_{\nu=1}^{N} d(\varepsilon^{m+\nu})P_\nu = \sum_{\nu=1}^{N} a(\varepsilon^{m+\nu})b^{-1}(\varepsilon^{m+\nu})P_\nu \quad (m = 1, 2, \ldots, N-1).$$

Property 1.6 implies

$$\prod_{m=0}^{N-1} d(\varepsilon^m S) = \sum_{\nu=1}^{N}\Big[\prod_{m=0}^{N-1} d(\varepsilon^{m+\nu})\Big]P_\nu = \sum_{\nu=1}^{N}\Big[\prod_{m=0}^{N-1}\frac{a(\varepsilon^{m+\nu})}{b(\varepsilon^{m+\nu})}\Big]P_\nu.$$

We consider the coefficients in the last sum. For $\nu = 1$ we have

$$\prod_{m=0}^{N-1}\frac{a(\varepsilon^{m+1})}{b(\varepsilon^{m+1})} = \frac{a(\varepsilon)a(\varepsilon^2)\ldots a(\varepsilon^N)}{b(\varepsilon)b(\varepsilon^2)\ldots b(\varepsilon^N)} = c_1c_2\ldots c_N = \lambda.$$

For arbitrary $1 < \nu \leqslant N$ we obtain a product of N numbers $a(\varepsilon^{m+\nu})$ for N different values $m+\nu$. Using the equality $\varepsilon^{N+k} = \varepsilon^k$ for $k = 1, 2, \ldots, N$, we obtain the product of the same numbers $c_1, \ldots, c_N$ but in different order for each ν. Hence

$$\prod_{m=0}^{N-1}\frac{a(\varepsilon^{m+\nu})}{b(\varepsilon^{m+\nu})} = \lambda \quad \text{for } \nu = 1, 2, \ldots, N.$$

Therefore Corollary 1.1 implies

$$\prod_{m=0}^{N-1} d(\varepsilon^m S) = \lambda I.$$

LEMMA 3.2. *Let $\tilde{A} = D^{N-1}+T$, where*

$$T = d(\varepsilon^{N-1}S)D^{N-2}+d(\varepsilon^{N-1}S)d(\varepsilon^{N-2}S)D^{N-3}+\ldots+d(\varepsilon^{N-1}S)\ldots d(\varepsilon^2S)D+ +d(\varepsilon^{N-1}S)\ldots d(\varepsilon S).$$

Then

$$(3.1)\qquad \big(D-d(S)\big)\tilde{A} = \tilde{A}\big(D-d(S)\big) = D^N-\lambda I.$$

Proof. Let $x\in X$ be arbitrarily fixed and let $u = [d(S)-D]x$. Then $Dx = d(S)x-u$.

Acting on both sides of this equation with powers of D and applying Corollary 1.3, we obtain successively:

$$D^2x = D[d(S)x-u] = d(\varepsilon S)Dx-Du = d(\varepsilon S)[d(S)x-u-Du] = d(\varepsilon S)d(S)x-d(\varepsilon S)u-Du,$$

$$D^3x = D[d(\varepsilon S)d(S)]x-d(\varepsilon S)u-Du = d(\varepsilon^2S)d(\varepsilon S)Dx-d(\varepsilon^2S)Du-D^2u = d(\varepsilon^2S)d(\varepsilon S)[d(S)x-u]-d(\varepsilon^2S)Du-Du^2 = d(\varepsilon^2S)d(\varepsilon S)d(S)x-d(\varepsilon^2S)d(\varepsilon S)u-d(\varepsilon^2S)Du-Du^2,$$

. . . .

$$D^Nx = d(\varepsilon^{N-1}S)\ldots d(S)x-[d(\varepsilon^{N-1}S)\ldots d(\varepsilon S)+d(\varepsilon^{N-1}S)\ldots d(\varepsilon^2S)D+ +\ldots+d(\varepsilon^{N-1}S)D^{N-2}+D^{N-1}]u.$$

Lemma 3.1 implies $D^Nx = \lambda x-\tilde{A}u$. But $u = [d(S)-D]x$. Hence $(D^N-\lambda I)x = -\tilde{A}[d(S)-D]x = \tilde{A}[D-d(S)]x$. Since x was arbitrarily chosen, we find $D^N-\lambda I = \tilde{A}[D-d(S)]$.

To prove the first part of formula (3.1), we show that

$$(3.2)\qquad d(S)\tilde{A} = DT+\lambda I.$$

Indeed, by Lemma 3.1 and Property 1.8,

$$\begin{aligned} d(S)\tilde{A} &= d(S)D^{N-1}+d(S)T \\ &= d(\varepsilon^N S)D^{N-1}+d(\varepsilon^N S)T \\ &= Dd(\varepsilon^{N-1}S)D^{N-2}+d(\varepsilon^N S)D\times \\ &\quad\times[d(\varepsilon^{N-2}S)D^{N-3}+d(\varepsilon^{N-2}S)d(\varepsilon^{N-3})D^{N-4}+\ldots+ \\ &\quad+d(\varepsilon^{N-2}S)\ldots d(\varepsilon S)]+d(\varepsilon^N S)d(\varepsilon^{N-1}S)\ldots d(\varepsilon S) \\ &= D[d(\varepsilon^{N-1}S)D^{N-2}+d(\varepsilon^{N-1}S)d(\varepsilon^{N-2}S)D^{N-3}+ \\ &\quad+\ldots+d(\varepsilon^{N-1}S)\ldots d(\varepsilon S)]+d(\varepsilon^{N-1}S)\ldots d(\varepsilon S)d(S) = DT+\lambda I. \end{aligned}$$

On the other hand, by definition

$$(3.3)\qquad D\tilde{A} = D^N+DT.$$

It follows from (3.2) and (3.3) that

$$[D-d(S)]\tilde{A} = D^N+DT-(DT+\lambda I) = D^N-\lambda I.$$

LEMMA 3.3. *Let $R_A = -\tilde{A}b^{-1}(S)$ (where $\tilde{A}$ is determined in Lemma 3.2). Then $R_A\tilde{A} = \tilde{A}R_A = D^N-\lambda I$.*

Proof. Since

$$D-d(S) = D-a(S)b^{-1}(S) = -b^{-1}(S)[a(S)-b(S)D] = -b^{-1}(S)A,$$

we find

$$D^N-\lambda I = \tilde{A}[D-d(S)] = \tilde{A}[-b^{-1}(S)A] = [-\tilde{A}b^{-1}(S)]A = R_A A.$$

On the other hand,

$$D^N-\lambda I = [D-d(S)]\tilde{A} = -b^{-1}(S)A\tilde{A}.$$

Hence $-A\tilde{A} = b(S)(D^N-\lambda I)$. But Property 1.4 implies that $(D^N-\lambda I)$ is commuting with S. Therefore $b(S)$ is commuting with $D^N-\lambda I$ and $-A\tilde{A} = (D^N-\lambda I)b(S)$. Hence

$$D^N-\lambda I = -A\tilde{A}b^{-1}(S) = A[-\tilde{A}b^{-1}(S)] = AR_A.$$

From Lemma 3.3 it immediately follows

PROPOSITION 3.1. *The operators A and R_A are commuting.*

LEMMA 3.4. *If $\tilde{x}$ is a solution of the equation $(D^N-\lambda I)\tilde{x} = y$, then $x = R_A\tilde{x}$ is a solution of the equation $Ax = y$.*

Indeed, by Lemma 3.3,

$$Ax = AR_A\tilde{x} = (D^N-\lambda I)\tilde{x} = y.$$

Now we can formulate the main theorem:

THEOREM 3.1. *Let S be an involution of order N acting in the linear space X (over complex scalars) and let D be permuting S. Let $A = a(S)-b(S)D$, where*

$$a(S) = \sum_{k=0}^{N-1} a_k S^k, \qquad b(S) = \sum_{k=0}^{N-1} b_k S^k$$

are polynomials with constant complex coefficients, such that $a(\varepsilon^\nu) \neq 0 \neq b(\varepsilon^\nu)$ for $\nu = 1, 2, \ldots, N$ and $\varepsilon = e^{2\pi i/N}$. Then every solution of the equation $Ax = y$ is of the form

$$x = R_A\tilde{x} + d\sum_{k=0}^{N-1} d_k S^k z_0,$$

where:

$$R_A = -[D^{N-1}+d(\varepsilon^{N-1}S)D^{N-2}+d(\varepsilon^{N-1}S)d(\varepsilon^{N-2}S)D^{N-3}+ +\ldots+d(\varepsilon^{N-1}S)\ldots d(\varepsilon S)]b^{-1}(S);$$

$d(S) = b^{-1}(S)a(S)$;

$\tilde{x}$ is a solution of the equation $(D^N-\lambda I)\tilde{x} = y$;

$\lambda = \prod_{m=1}^{N} c_m$;

$c_m = a(\varepsilon^m)/b(\varepsilon^m)$;

d is an arbitrary complex number;

$$d_k = \sum_{m=1}^{N} \lambda_0^{-m} c_1\ldots c_m V_{k,m} \quad (k = 0, 1, \ldots, N-1), \qquad \lambda_0 = \sqrt[N]{|\lambda|}\, e^{\frac{\varphi}{N}i},$$

$$\varphi = \operatorname{Arg}\lambda \quad (0 \leqslant \varphi < 2\pi);$$

$V_{k,m}$ is the subdeterminant obtained by cancelling the $(k+1)$-th column and the m-th row of the Van der Monde determinant V of numbers $\varepsilon^2, \varepsilon^3, \ldots, \varepsilon^N, \varepsilon^1$ $(m = 1, 2, \ldots, N;\ k = 0, 1, \ldots, N-1)$;

z_0 is an arbitrary solution of the equation $(D-\lambda_0 I)z = 0$.

The proof immediately follows from Theorem 2.3, Lemmas 3.2, 3.3, 3.4 and from the linearity of the operator A.

4. Let S be an involution of order N acting in a linear space X and let D be permuting S. Let us consider the operator $A = a(S)-b(S)D$, where $a(S)$ and $b(S)$ are polynomials with constant complex coefficients. In the two last sections we have assumed that $a(\varepsilon^\nu) \neq 0 \neq b(\varepsilon^\nu)$ for $\nu = 1, 2, \ldots, N$. Now we will drop this assumption. We shall consider some most typical cases.

Similarly as in Lemma 2.1, formula (2.3), the equation

$$Ax = y \tag{4.1}$$

can be written as an equivalent system of equations

$$\begin{aligned} &a(\varepsilon^m)x_{(m)}-b(\varepsilon^m)Dx_{(m+1)} = y_{(m)} \quad \text{for } m = 1, 2, \ldots, N-1, \\ &a(\varepsilon^N)x_{(N)}-b(\varepsilon^N)Dx_{(1)} = y_{(N)}, \end{aligned} \tag{4.2}$$

where $x_{(m)} = P_m x$, $y_{(m)} = P_m y$. Of course, if $a(\varepsilon^m) = b(\varepsilon^m) = 0$ for $m = 1, 2, \ldots, N$, then $A = 0$ (Corollary 1.1).

1° If $b(\varepsilon^m) = 0$ for $m = 1, 2, \ldots, N$, then the solution of (4.1) was given in [1] (see also [3], p. 89), and it is of the form

$$x = \sum_{m:a(\varepsilon^m)\neq 0} \frac{1}{a(\varepsilon^m)} P_m y + \sum_{m:a(\varepsilon^m)=0} z_{(m)}$$

(the first sum runs over all m such that $1 \leqslant m \leqslant N$ and $a(\varepsilon^m) \neq 0$, the second one over all m such that $1 \leqslant m \leqslant N$ and $a(\varepsilon^m) = 0$) under the necessary and sufficient condition

$$P_m y = 0 \quad \text{for all } m \text{ such that } a(\varepsilon^m) = 0,$$

where $z_{(m)}$ is an arbitrary element of the space $X_{(m)} = P_m X$.

2° If $a(\varepsilon^m) = 0$ for $m = 1, 2, \ldots, N$, then we solve equation (4.1) with respect to the unknown Dx. We reduce our problem (similarly as in 1°) to the equation

$$Dx = y_0,$$

where

$$y_0 = -\sum_{m:b(\varepsilon^m)\neq 0} \frac{1}{b(\varepsilon^m)} P_m y - \sum_{m:b(\varepsilon^m)=0} z_{(m)} \quad (m = 1, 2, \ldots, N),$$

under the necessary and sufficient condition $P_m y = 0$ for all m such that $b(\varepsilon^m) = 0$, and $z_{(m)} \in X_{(m)}$ are arbitrary.

3° Let us suppose that $a(\varepsilon^m) \neq 0$ for all m and $b(\varepsilon^m) = 0$ for at least one m. Without loss of generality we can consider the case $b(\varepsilon^N) = 0$. From the last equation of (4.2) we obtain $x_{(N)} = a^{-1}(\varepsilon^N) y_{(N)}$, and solving the system (4.2) successively, we have

$$x_{(N)} = \frac{1}{a(\varepsilon^N)} y_{(N)},$$

$$x_{(m)} = \frac{1}{a(\varepsilon^m)} y_{(m)} + \frac{b(\varepsilon^m)}{a(\varepsilon^m)} D x_{(m+1)} \quad (m = 1, 2, \ldots, N-1).$$

Hence, by Properties 1.2 and 1.3,

$$\begin{aligned} x_{(m)} &= \frac{1}{a(\varepsilon^m)} \left[y_{(m)} + \frac{b(\varepsilon^m)}{a(\varepsilon^{m+1})} D y_{(m+1)} + \ldots + \frac{b(\varepsilon^m) \ldots b(\varepsilon^{N-1})}{a(\varepsilon^{m+1}) \ldots a(\varepsilon^N)} D^{N-m} y_{(N)} \right] \\ &= \frac{1}{a(\varepsilon^m)} \left[P_m + \frac{b(\varepsilon^m)}{a(\varepsilon^{m+1})} D P_{m+1} + \ldots + \frac{b(\varepsilon^m) \ldots b(\varepsilon^{N-1})}{a(\varepsilon^{m+1}) \ldots a(\varepsilon^N)} D^{N-m} P_N \right] y \\ &= \frac{1}{a(\varepsilon^m)} P_m \left[I + \frac{b(\varepsilon^m)}{a(\varepsilon^{m+1})} D + \ldots + \frac{b(\varepsilon^m) \ldots b(\varepsilon^{N-1})}{a(\varepsilon^{m+1}) \ldots a(\varepsilon^N)} D^{N-m} \right] y \end{aligned}$$

and

$$\begin{aligned} x &= \sum_{m=1}^{N} x_{(m)} \\ &= \sum_{m=1}^{N} \frac{1}{a(\varepsilon^m)} P_m \left[I + \frac{b(\varepsilon^m)}{a(\varepsilon^{m+1})} D + \ldots + \frac{b(\varepsilon^m) \ldots b(\varepsilon^{N-1})}{a(\varepsilon^{m+1}) \ldots a(\varepsilon^N)} D^{N-m} \right] y. \end{aligned}$$

In a similar way we determine the solution of (4.2) if $b(\varepsilon^m) = 0$ for an $m \neq N$.

5° Let us suppose that $b(\varepsilon^m) \neq 0$ for all m and $a(\varepsilon^m) = 0$ for at least one m. As previously, we consider the case $a(\varepsilon^N) = 0$. Then we determine $x_{(1)}$ from the equation

$$D x_{(1)} = -\frac{1}{b(\varepsilon^N)} y_{(N)}$$

obtained from the last equation (4.2). Having $x_{(1)}$, we successively solve the equations

$$D x_{(m+1)} = \frac{-1}{b(\varepsilon^m)} y_{(m)} + \frac{a(\varepsilon^m)}{b(\varepsilon^m)} x_{(m)} \quad (m = 1, 2, \ldots, N-1)$$

obtained from the first $N-1$ equations (4.2). Similarly, we solve equation (4.1) if $a(\varepsilon^m) = 0$ for an $m \neq N$.

5. Example. Let us consider on the complex plane the differential equation

$$(5.1) \qquad \sum_{k=0}^{N-1} a_k x(\varepsilon^k t + \beta_k) + \sum_{k=0}^{N-1} b_k x'(\varepsilon^k t + \beta_k) = y(t),$$

where a_k, b_k, β_k are constant complex numbers and $\varepsilon = e^{2\pi i/N}$, $N \geqslant 2$. Let us consider the following operator:

$$(Sx)(t) = x(\varepsilon t + \beta_0).$$

It is an involution of order N in the space of all functions of one complex variable. Indeed, it is easy to check that

$$(5.2) \qquad (S^m x)(t) = x\big(\varepsilon^m t + \beta_0(\varepsilon^{m-1} + \varepsilon^{m-2} + \ldots + \varepsilon + 1)\big).$$

Hence

$$(S^N x)(t) = x\big(\varepsilon^N t + \beta_0(\varepsilon^{N-1} + \ldots + \varepsilon + 1)\big).$$

But $\varepsilon^N = 1$ and, by formula (1.3), $\varepsilon^{N-1} + \ldots + \varepsilon + 1 = 0$. Then $(S^N x)(t) = x(t)$.

The differentiation operator is permuting S. Indeed,

$$(DSx)(t) = [x(\varepsilon t + \beta_0)]' = \varepsilon x'(\varepsilon t + \beta_0) = \varepsilon (SDx)(t).$$

Hence all previous considerations can be applied to equation (5.1) if we assume additionally, according to (5.2), that

$$(5.3) \qquad \beta_k = \beta_0(\varepsilon^{k-1} + \ldots + \varepsilon + 1) \quad \text{for } k = 1, 2, \ldots, N-1.$$

For example, if

$$a(\varepsilon^m) = \sum_{k=0}^{N-1} a_k \varepsilon^{mk} \neq 0 \neq b(\varepsilon^m) = \sum_{k=0}^{N-1} b_k \varepsilon^{mk} \quad (m = 1, 2, \ldots, N),$$

according to Theorem 3.1, to solve equation (5.1) it is sufficient to know all solutions of the equation $z' - \lambda_0 z = 0$ and a solution of the equation $\tilde{x}^{(N)} - \lambda \tilde{x} = y$, where

$$\lambda = \prod_{m=1}^{N} a(\varepsilon^m)/b(\varepsilon^m)$$

and

$$\lambda_0 = \sqrt[n]{|\lambda|}\, e^{\varphi i/N}, \qquad \varphi = \operatorname{Arg} \lambda.$$

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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On an equation with reflection of order n

by

BARBARA MAŻBIC-KULMA (Warszawa)

If a differential equation contains together with the unknown function $x(t)$ the function $x(-t)$, then it is called a *differential equation with reflection*.

D. Przeworska-Rolewicz gives in [1] the general solution of an equation with reflection of order 1, i.e. of the equation

$$a_0x(t)+b_0x(-t)+a_1x'(t)+b_1x'(-t) = y(t),$$

where a_0, a_1, b_0 and b_1 are scalars.

In the present paper we consider the differential equation with reflection of order n,

$$a_0x(t)+b_0x(-t)+\ldots+a_nx^{(n)}(t)+b_nx^{(n)}(-t) = y(t), \tag{1}$$

where the coefficients $a_0, \ldots, a_n, b_0, \ldots, b_n$ are constants. We give a general form of the solution of (1) under the following assumptions:

1° $a_n^2-b_n^2 \neq 0$;

2° $a_{j-k}a_k-b_{j-k}b_k \neq 0$ $(k = 0, 1, \ldots, n$ and $j = k+1, \ldots, k+n)$;

3° the polynomial $\sum_{j=0}^{n} \lambda_{2j}t^j$ has single roots only for $k = 0, 1, \ldots, n$,

where

$$\lambda_j = \begin{cases} \sum_{k=0}^{j} c_{jk} & \text{for } 0 \leqslant j \leqslant n, \\ \sum_{k=j-n}^{n} c_{jk} & \text{for } n < j \leqslant 2n, \end{cases} \tag{i}$$

$$c_{jk} = (-1)^{j-k}(a_{j-k}a_k-b_{j-k}b_k)(a_n^2-b_n^2)^{-1}. \tag{ii}$$

1. Let S be a reflection: $Sx(t) = x(-t)$. Since $S^2 = I$, where I is the identity operator, S is an involution. We write

$$Dx(t) = x'(t). \tag{2}$$