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The differential equation of an inner function*

by

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1. Introduction. A *unitary function* in a Hilbert space $\mathscr{H}$ (always assumed separable, and allowed to be finite-dimensional) is a function $U(x)$ defined on the line, taking values in the set of unitary operators in $\mathscr{H}$, and measurable. (The common definitions of measurability coincide when $\mathscr{H}$ is separable.) $L^2_{\mathscr{H}}$ is the Lebesgue space of measurable functions on the line with values in $\mathscr{H}$, under the norm

$$\|F\| = \Big[\int_{-\infty}^{\infty} \|F(x)\|^2\,dx\Big]^{1/2}.$$

A unitary function defines a unitary operator in $L^2_{\mathscr{H}}$ by multiplication: UF is the function with values $U(x)F(x)$. $H^2_{\mathscr{H}}$, the Hardy space of the upper half-plane, is the subspace of $L^2_{\mathscr{H}}$ consisting of those F having an analytic extension to the upper half-plane with

$$\int_{-\infty}^{\infty} \|F(x+iu)\|^2\,dx \leqslant K < \infty, \qquad \text{all } u > 0.$$

A unitary function U is *inner* if U is non-constant, and UF is in $H^2_{\mathscr{H}}$ for each F in $H^2_{\mathscr{H}}$. This is the case just if $U(x)$ is the boundary function of an analytic operator function $U(z)$, defined in the upper half-plane and satisfying $\|U(z)\| \leqslant 1$ there.

The *invariant subspace problem* is to decide whether each bounded operator T in $\mathscr{H}$ has a non-trivial closed invariant subspace. We exclude the case where $\mathscr{H}$ is 1-dimensional, and the answer trivially negative. This general problem about operators has been transformed [2] into a problem about inner functions: can every inner function U (with trivial exceptions) be written in the form VW, where V and W are inner functions?

More generally, spectral properties of operators are related to the structure of associated inner functions. Therefore it is not surprising that information about inner functions in general is difficult to obtain.

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A differentiable unitary function U satisfies a differential equation

$$U'(x) = iM(x)\,U(x), \tag{1}$$

where the operator function M is determined by the equation. U being unitary means that $M(x)$ is self-adjoint for each x. Which self-adjoint functions M are obtained from unitary functions U that are inner? This question is a stability problem for ordinary differential equations, but no information seems to be available about it. A good answer would undoubtedly contain a theorem about invariant subspaces.

The invariant subspace problem for quasi-nilpotent T leads to a function $M(x)$ having an analytic extension $M(z)$ everywhere in the plane except to $\pm i$ (and indeed $z^2M(z)$ is analytic at infinity). U is inner if and only if $z = i$ is an *apparent singularity* of the differential equation. Even when $\mathscr{H}$ is finite-dimensional no simple criterion is known on which to decide whether a singularity of $M(z)$ is apparent.

At the other extreme, if $M(z)$ is entire, the solution $U(z)$ will be entire also; we seek conditions on M implying that $U(z)$ is bounded in the upper half-plane.

The trivial non-factorable inner functions (which exist in every dimension) lead to M essentially of the form $(1+x^2)^{-1}\,M_0$, where M_0 is a constant projection. This paper arose out of the attempt to find a non-factorable inner function by solving (1) with $M(x)$ a non-constant projection function. This hope was futile, for I proved in 1966 that *an inner function U is surely factorable if M is non-constant and projection-valued.* The main theorem of this paper asserts that this result is empty: *if U is inner and M projection-valued, then M is constant.*

The theorem is a negative one as far as invariant subspaces go, but it seems to be of a new type in differential equations. At least the proof given here seems unlike those current in the field.

2. The spectral resolution. To a unitary function U is associated a closed subspace $\mathscr{M} = U\cdot H^2_{\mathscr{H}}$ of $L^2_{\mathscr{H}}$ (the set of all UF, F in $H^2_{\mathscr{H}}$). The subspace $\mathscr{M}$ determines U up to multiplication on the right by a constant unitary operator. For two unitary operators U and V

$$U\cdot H^2_{\mathscr{H}} \subset V\cdot H^2_{\mathscr{H}} \text{ if and only if } V^*U \text{ is inner.}$$

For each real a let S_a denote the operation of multiplication by e^{iax} in $L^2_{\mathscr{H}}$. Then for $\mathscr{M} = U\cdot H^2_{\mathscr{H}}$, U any unitary function, we have [3]

$$S_a\mathscr{M} \subset S_\beta\mathscr{M} \quad \text{if } a > \beta,$$

$$\bigcap_{a<\infty} S_a\mathscr{M} = \{0\}, \qquad \bigcup_{a>-\infty} S_a\mathscr{M} \text{ dense in } L^2_{\mathscr{H}}.$$

Any closed subspace $\mathscr{M}$ of $L^2_{\mathscr{H}}$ having these properties will be called a *normal invariant subspace*. A simple but fundamental result, generalizing a theorem of Beurling and first completely proved by Halmos [1], states that every normal invariant subspace of $L^2_{\mathscr{H}}$ has the form $U\cdot H^2_{\mathscr{H}}$ for some unitary function U; and U is determined by the subspace up to multiplication on the right by a constant unitary operator.

Let U be a unitary function, $\mathscr{M} = U\cdot H^2_{\mathscr{H}'}$, and P_a the self-adjoint projection of $L^2_{\mathscr{H}}$ on $S_a\mathscr{M}$ for each real a. Then $\{I-P_a\}$ is a decomposition of the identity in $L^2_{\mathscr{H}}$. We construct the unitary group

$$V_t = -\int_{-\infty}^{\infty} e^{i\lambda t}\,dP_\lambda$$

with self-adjoint generator

$$B = -\int_{-\infty}^{\infty} \lambda\,dP_\lambda.$$

From definition,

$$P_{\lambda+\tau} = S_\lambda P_\tau S_{-\lambda} \quad (\text{all real } \lambda, \tau).$$

In consequence, we have the two *commutation relations*

$$V_tS_\lambda = e^{i\lambda t}\,S_\lambda V_t, \qquad S_{-\lambda}BS_\lambda = \lambda I + B \quad (\text{all real } \lambda, t). \tag{2}$$

The second relation means, in particular, that the domain of B is invariant under all S_λ.

Each continuous unitary group $\{V_t\}$ or self-adjoint operator B, acting in $L^2_{\mathscr{H}}$ and satisfying (2), leads backwards to a unique subspace $\mathscr{M}$ and almost unique unitary function U.

If $\mathscr{M} = H^2_{\mathscr{H}'}$, or U is constant, it is easy to verify that

$$V_tF(x) = F(x+t), \qquad BF(x) = -i\frac{d}{dx}F(x).$$

The domain of B is the set of F in $L^2_{\mathscr{H}}$ whose Fourier transforms $\hat{F}$ satisfy

$$\int_{-\infty}^{\infty} u^2\,\|\hat{F}(u)\|^2\,du < \infty.$$

Such functions are continuous, and have this property (which will be required later): if $BF = 0$ almost everywhere on an interval, then F is constant on the interval.

These operators V_t, B associated with $H^2_{\mathscr{H}}$ will be called T_t and B_0. We want to express V_t and B for an arbitrary subspace $\mathscr{M} = U\cdot H^2_{\mathscr{H}}$ in terms of T_t, B_0, and U. Let Q_a be the projection of $L^2_{\mathscr{H}}$ on $S_aH^2_{\mathscr{H}}$; then we have

$$P_a = UQ_aU^*, \qquad V_t = UT_tU^*, \qquad B = UB_0U^*. \tag{3}$$

The first relation is immediate, because the right side is an orthogonal projection whose range in $US_aH^2_{\mathscr{H}} = S_a\mathscr{M}$. The other two relations are obtained by setting the first one into the integrals defining V_t and B. Now define

$$A_t(x) = U(x)U^*(x+t),$$

and denote the operation of mutliplication by A_t in $L^2_{\mathscr{H}}$ again by A_t. The second relation in (3) takes the form

$$V_tF = A_tT_tF.$$

In a scalar context, where these calculations have been carried out before [3], the function $A_t(x)$ of two variables was called a *cocycle*.

The description of B in terms of B_0 is obtained in the next section.

3. The differential equation. The last relation in (3) gives formally

$$BF = -iU(U^{*\prime}F+U^*F') = B_0F-MF, \qquad M = iUU^{*\prime}. \tag{4}$$

The calculation is meaningful if F belongs at the same time to the domains of B and of B_0; but such functions may be rare. A condition of smoothness on U is needed to ensure the existence of many such F. In applications (for example to the invariant subspace problem) U may be assumed analytic on the real axis. A weaker hypothesis, convenient for present purposes, is this:

$t^{-1}[U(x+t)-U(x)]$ should converge in norm to a bounded operator $U'(x)$ for each x, and uniformly on every finite interval of x.

We shall call U a *smooth unitary function* if it has this property.

The derivative of a smooth unitary function is continuous in norm. Also

$$\frac{U^*(x+t)-U^*(x)}{t} = U^*(x+t)\,\frac{U(x)-U(x+t)}{t}\,U^*(x),$$

so that U^* is also smooth, and $U^{*\prime}(x) = -U^*(x)U'(x)U^*(x)$.

Let $\mathscr{D}$ denote the set of F in the domain of B_0 that are compactly supported.

LEMMA. *$U\mathscr{D} = \mathscr{D}$ if U is a smooth unitary function.*

Let F belong to $\mathscr{D}$, and write

$$\begin{aligned} &t^{-1}[U(x+t)F(x+t)-U(x)F(x)] \\ &\qquad = t^{-1}U(x+t)[F(x+t)-F(x)]+t^{-1}[U(x+t)-U(x)]F(x). \end{aligned}$$

As t tends to 0, the first term on the right converges in $L^2_{\mathscr{H}}$ because U is continuous in norm and F is in the domain of B_0. The second term converges uniformly and in $L^2_{\mathscr{H}}$ because F is compactly supported. Thus

the left side converges in $L^2_{\mathscr{H}}$, which means that UF is in the domain of B_0. Since UF is compactly supported, it belongs to $\mathscr{D}$.

We have shown that $U\mathscr{D}$ is contained in $\mathscr{D}$. The same result holds for U^*, and so $U\mathscr{D} = \mathscr{D}$.

The lemma in connection with (3) implies that $\mathscr{D}$ is contained in the domain of B, and obviously (4) holds for F in $\mathscr{D}$.

The expression for M in (4) can be written

$$M = iUU^{*\prime} = -iU[U^*U'U^*] = -iU'U^*, \tag{5}$$

showing that U satisfies the differential equation (1). $M(x)$ is self-adjoint for each x, because the adjoint of $iUU^{*\prime}$ is $-iU'U^*$, both equal to M. Finally, M is continuous in norm, because it is the product of norm-continuous functions.

The differential equation (1) holds in the complex plane wherever $M(z) = -iU'(z)U(z)^{-1}$ is analytic. This region is symmetric about the real axis.

We have associated to a smooth unitary function the invariant subspace $\mathscr{M} = U\cdot H^2_{\mathscr{H}}$, self-adjoint operator B acting in $L^2_{\mathscr{H}}$, and finally the self-adjoint operator function $M(x)$ acting in $\mathscr{H}$ for each x. This path can be retraced. Some points require care, and so the result is stated formally.

THEOREM 1. *Let M be a norm-continuous self-adjoint operator function defined on the line. Then B_0-M is a symmetric operator on $\mathscr{D}$. It possesses just one self-adjoint extension B that satisfies the commutation relation (2). B is derived from a unique normal invariant subspace $U\cdot H^2_{\mathscr{H}}$. U is a smooth unitary function, and the solutions of the vectorial differential equation $F'(x) = iM(x)F(x)$ are exactly the functions $U\varphi$, φ in $\mathscr{H}$.*

For each φ in $\mathscr{H}$ construct a solution F_φ of this differential equation such that $F_\varphi(0) = \varphi$. Any solution function F has constant norm:

$$\frac{d}{dx}\|F(x)\|^2 = 2\,\mathrm{Re}\big(F'(x), F(x)\big) = 2\,\mathrm{Re}\big(iM(x)F(x), F(x)\big) = 0$$

because $M(x)$ is self-adjoint. Now define $U(x)\varphi = F_\varphi(x)$ for each x and φ. Standard facts about the differential equation imply that $U(x)$ is unitary.

In order to solve (1), in the classical theory of differential equations, one actually solves the integral equation

$$U(x) = I+i\int_0^x M(y)U(y)\,dy. \tag{6}$$

It follows (since M is locally bounded) that U is norm-continuous. A second appeal to (6), using the continuity of U and of M, shows that U is smooth.

Let B be the self-adjoint operator associated with U. Then $B = B_0 - M$ on $\mathcal{D}$, because U is smooth and satisfies the differential equation. The unicity of B requires proof. Suppose that $\tilde{B}$ is another self-adjoint operator satisfying the commutation relation and equal to $B_0 - M$ on $\mathcal{D}$. Let $\tilde{U}$ be the corresponding unitary function, determined up to multiplication on the right by a constant unitary operator. If $\tilde{U}$ is smooth, normalize it so that $\tilde{U}(0) = I$. Then $U\varphi$ and $\tilde{U}\varphi$ are two solutions of the differential equation equal to φ at $x = 0$; the uniqueness theorem for solutions of the differential equation implies that these solutions are identical, and we are finished. But we want this conclusion without supposing that $\tilde{U}$ is smooth. For that we must in effect prove a stronger result about uniqueness of solutions of the differential equation.

We are given that

$$\tilde{U}B_0\tilde{U}^*F = UB_0U^*F \qquad (\text{all } F \text{ in } \mathcal{D}).$$

Writing $F = UG$ and taking account of the lemma we have

$$U^*\tilde{U}B_0\tilde{U}^*UG = B_0G \qquad (\text{all } G \text{ in } \mathcal{D}).$$

Take $G(x) = f(x)\varphi$, where f is a scalar function that is suitably smooth, compactly supported, and equal to 1 on an interval $(-A, A)$. Then G is in $\mathcal{D}$ and B_0G vanishes on the interval. Hence $B_0\tilde{U}^*UG$ vanishes on the interval too. As we have observed, this implies that $\tilde{U}^*UG = \tilde{U}^*U\varphi$ is continuous and constant on the interval. Now A can be as large as we please, and φ is arbitrary in $\mathcal{H}$. Hence $\tilde{U}^*U$ is a constant unitary operator, so $\tilde{U}$ determines the same invariant subspace as U, and finally $\tilde{B} = B$. This completes the proof of the theorem.

4. The order relation. Let U and V be smooth unitary functions with associated self-adjoint functions M and N respectively. Write $M \succ N$ to mean that $U\cdot H^2_{\mathcal{H}}$ is contained in $V\cdot H^2_{\mathcal{H}}$, or, in other words, that V^*U is inner. We should like to describe this order relation by analytic properties of M and N. In particular, it would be interesting to characterize those M such that $M \succ 0$, corresponding to inner functions U. Concretely, we ask *for which functions M does every solution of the differential equation $F' = iMF$ have a bounded analytic extension to the upper half-plane?*

THEOREM 2. *Let B and C be the self-adjoint operators associated with M and N respectively. $M \succ N$ if and only if $f(B) \leqslant f(C)$ for every function f that is real, increasing and bounded on the line.*

The operators $f(B)$ and $f(C)$ are defined by the spectral resolutions of B and C, and the inequality refers to the ordinary order relation for bounded self-adjoint operators.

Let B and C have spectral resolutions $\{I - P_\lambda\}$, $\{I - Q_\lambda\}$ respectively. Then

$$f(B) = -\int_{-\infty}^{\infty} f(\lambda)\,dP_\lambda, \qquad f(C) = -\int_{-\infty}^{\infty} f(\lambda)\,dQ_\lambda.$$

Set $q(x) = 0$ for $x < 0$, $= 1$ for $x \geqslant 0$. This is a function of the type mentioned in the theorem, and $q(B) = P_0$, $q(C) = Q_0$. (The spectral resolutions are continuous, so there is no question of left or right limits.) By definition, $M \succ N$ if and only if $P_0 \leqslant Q_0$, which is to say $q(B) \leqslant q(C)$.

If this is the case, it follows (by considering linear combinations of translates of q) that $f(B) \leqslant f(C)$ for all increasing bounded functions f, and the theorem is proved.

COROLLARY. *If $M \succ N$, then $M(x) \geqslant N(x)$ for all x. In particular if $M \succ 0$, then $M(x) \geqslant 0$ for all x.*

Let f_n be bounded monotonic functions tending to the unbounded function $f(x) = x$. For each n, $f_n(B) \leqslant f_n(C)$. Passing to the limit and using the fact that $B = B_0 - M$ and $C = B_0 - N$ on $\mathcal{D}$, we have for F in $\mathcal{D}$

$$\int_{-\infty}^{\infty} (MF, F)\,dx \geqslant \int_{-\infty}^{\infty} (NF, F)\,dx.$$

This implies that $M(x) \geqslant N(x)$ for each x.

There is another way to prove the corollary directly from the differential equation. This proof is less general but throws light on the result from another direction. Suppose that $M \succ 0$ and that $U(z)$ is analytic on the real axis; let us show that $M(x) \geqslant 0$ for each x. For any φ and $\operatorname{Im} z \geqslant 0$ we have $\|U(z)\varphi\| \leqslant \|\varphi\|$. Therefore, setting $z = x + iy$ and differentiating at $y = 0$,

$$0 \geqslant \frac{\partial}{\partial y}\big(U(z)\varphi, U(z)\varphi\big) = 2\operatorname{Re}\big(iU'(x)\varphi, U(x)\varphi\big)$$

$$= 2\operatorname{Re}\big(-M(x)U(x)\varphi, U(x)\varphi\big).$$

Since $M(x)$ is self-adjoint, this means $M(x) \geqslant 0$.

THEOREM 3. *If U, V, and W are smooth unitary functions, and $U = VW$, then the corresponding self-adjoint functions satisfy*

$$M(U) = M(V) + VM(W)V^*.$$

The proof is a straightforward verification.

5. The main theorem.

THEOREM 4. *Suppose that every vector function F satisfying the differential equation*

$$F'(x) = i\varrho(x)\, M(x) F(x) \tag{7}$$

has a bounded analytic extension to the upper half-plane. Assume that $M(x)$ is norm-continuous and projection-valued. Assume, furthermore, that $\varrho(x)$ is a continuous bounded real function on the line such that $1/\varrho(z)$ is entire, and such that $-\log|\varrho(z)|$ possesses a positive harmonic majorant in the upper half-plane. Then $M(x)$ is constant.

For the sake of clarity the proof will be given for the case that $\varrho(x) \equiv 1$; the modifications necessary for the general result will be sketched afterwards. First we prove two lemmas that are well known for scalar inner functions.

LEMMA. *An inner function that is norm-continuous on the real axis is necessarily analytic on the real axis.*

Any inner function U has a representation in the upper half-plane by a Poisson integral along the axis. The continuity of U enables one to show, as for scalar functions, that $U(z)$ is continuous in norm on the closed upper half-plane. Therefore $U(z)$ is continuously, and therefore analytically, invertible near the real axis. Define $Q(\bar{z})^* = U(z)^{-1}$ in the upper half-plane, near the real axis where $U(z)$ is invertible. Then $Q(z)$ is bounded and analytic in the lower half-plane near the real axis, with boundary values equal to those of $U(z)$. Hence Q and U continue each other across the axis.

LEMMA. *Let U, V, and W be inner functions, $U = VW$, and U analytic on the real axis. Then V and W are analytic on the real axis.*

(This fact is contained in the lemmas of p. 76 and p. 79 of [2]; however, a short independent proof is desirable. Mr. C. Jacewitz has suggested this one.)

Define $Q(\bar{z})^* = W(z)\,U(z)^{-1}$, analytic and bounded in the upper half-plane near the real axis. Then $Q(z)$ is analytic and bounded in the lower half-plane near the axis, and its boundary values are $V(x)$ almost everywhere. Hence V and Q continue each other across the axis. Now V, being analytic and unitary on the axis, is analytically invertible there, so W is analytic on the axis also.

There is one more fact to prove before beginning the main argument. If U is a smooth inner function satisfying the differential equation (1), then obviously $U'(x)$ has norm at most 1 (assuming that M is projection-valued). We want to show that $\|U'(z)\| \leqslant 1$ in the upper half-plane. From

the integral equation (6) we have for all real $t \neq 0$

$$\|t^{-1}[U(x+t) - U(x)]\| = \Big\|t^{-1}\int_x^{x+t} M(y)\,U(y)\,dy\Big\| \leqslant 1. \tag{8}$$

For fixed t the quantity

$$\|t^{-1}[U(z+t) - U(z)]\| \tag{9}$$

is bounded in the upper half-plane, because $U(z)$ is bounded; and for z real the bound is 1. Hence (9) is bounded by 1 in the upper half-plane, uniformly in t. Letting t tend to 0 gives the same bound for $\|U'(z)\|$.

The proof of the theorem itself makes use of the *analytic range functions* studied in [2]. For each x, let $J(x)$ be the range of $M(x)$. Since $J(x)$ is also the range of the bounded analytic operator function $U'(x)$, J is an analytic range function. The complement $J(x)^{\perp}$ of $J(x)$ is also analytic, because it is the range of $iU - U' = i(I-M)U$. (The hypothesis that M is projection-valued is crucial at this point.)

For any range function K, $\mathscr{A}_K$ denotes the set of all F in $H^2_{\mathscr{H}}$ with values in K almost everywhere. Set $\mathscr{N} = \mathscr{A}_J \oplus \mathscr{A}_J^{\perp}$. Clearly, $\mathscr{N}$ is an invariant subspace of $H^2_{\mathscr{H}}$; it is a normal invariant subspace because J and $J^{\perp}$ are analytic range functions. Call its inner function V. A function F in $H^2_{\mathscr{H}}$ belongs to $\mathscr{N} = V\cdot H^2_{\mathscr{H}}$ if and only if MF, the projection of F in J at each point, is a function in $H^2_{\mathscr{H}}$.

Now $U\cdot H^2_{\mathscr{H}}$ is contained in $V\cdot H^2_{\mathscr{H}}$. For if $G = UF$, F in $H^2_{\mathscr{H}}$, then $iMG = iMUF = U'F$ is in $H^2_{\mathscr{H}}$, so G is in $V\cdot H^2_{\mathscr{H}}$. Hence $V^*U = W$ is inner, so U has the inner factoring VW.

U is smooth and therefore analytic on the real axis (by the first lemma of this section). Therefore V and W are analytic on the axis (by the second lemma). Let the self-adjoint functions corresponding to V and W be M_1, M_2 respectively. Then $M_1(x)$ and $M_2(x)$ are positive for all x, and $M = M_1 + VM_2V^*$. Therefore

$$0 \leqslant VM_2V^* \leqslant M \quad \text{or} \quad 0 \leqslant M_2 \leqslant V^*MV. \tag{10}$$

The structure of V can be described to this extent: there exist complementary subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ of $\mathscr{H}$ such that for all x

$$V(x)\,\mathscr{H}_1 = J(x), \qquad V(x)\,\mathscr{H}_2 = J(x)^{\perp}. \tag{11}$$

The fact that $\mathscr{H}_1$ and $\mathscr{H}_2$ are independent of x, which is the important point, is seen by constructing ([2], p. 66) the outer isometry functions mapping $\mathscr{H}_1$ onto J and $\mathscr{H}_2$ onto $J^{\perp}$ respectively, and observing that V must be the sum of these isometry functions.

From (10) and (11) we conclude that V^*MV is the constant projection of $\mathscr{H}$ upon $\mathscr{H}_1$. Then (10) implies

$$M_2\mathscr{H}_1 \subset \mathscr{H}_1, \qquad M_2\mathscr{H}_2 = \{0\}.$$

The differential equation satisfied by W^*

$$W^{*\prime} = -iW^*M_2$$

gives $W^{*\prime}\varphi = 0$ for φ in $\mathscr{H}_2$, or $W^*\varphi =$ constant.

If we assume, as we may, that $U(0) = V(0) = W(0) = I$, then W^* is the identity on $\mathscr{H}_2$. Therefore $W^*\mathscr{H}_1 = \mathscr{H}_1$, and W also leaves $\mathscr{H}_1$, $\mathscr{H}_2$ invariant. Let φ be in $\mathscr{H}_2$. Then $U\varphi = VW\varphi$ is in $V\mathscr{H}_2 = J^\perp$, so that

$$U'\varphi = iMU\varphi = 0.$$

Hence $U\varphi = \varphi$ for all φ in $\mathscr{H}_2$, $U\mathscr{H}_2 = J^\perp$ is constant, J is constant and finally M (the projection on J) is constant. This completes the proof under the assumption that $\varrho(x) \equiv 1$.

In the general case, we have to prove first that $U'(z)$ is bounded in the upper half-plane. $U'(x)$ is still bounded so this is not surprising. An analogue of (8) holds, with $\varrho(y)$ supplied in the integral and the constant on the right changed to $\|\varrho\|_\infty$. The conclusion follows as before.

It follows that J, the range of U', is analytic. Now $J^\perp$ is the range of

$$i(I-M)U = iU - \varrho^{-1}U',$$

and we have to show that $U'(z)/\varrho(z)$ is analytic and bounded in the upper half-plane. This is a conclusion of Phragmén-Lindelöf type. Let $u(z)$ be a positive harmonic majorant for $\log|\varrho(z)^{-1}|$ in the upper half-plane. Since $U'(z)$ is bounded, we have

$$\|\varrho(z)^{-1}U'(z)\| \leqslant Ke^{u(z)} \qquad (\operatorname{Im} z > 0).$$

The left side is bounded on the real axis by the differential equation, and we conclude that it is bounded in the half-plane.

(The growth condition on $\varrho(z)^{-1}$ is only needed for this application of the Phragmén-Lindelöf principle, and the condition can take various forms. It is enough to require that $\varrho(z)^{-1} = O(\exp|z|^a)$ for some $a < 1$, uniformly in the upper half-plane. This hypothesis covers the case $\varrho(x) = (1+x^2)^{-1}$ mentioned in the Introduction.)

Having established that J and $J^\perp$ are analytic range functions, and that U'/ϱ is bounded in the upper half-plane, we can follow the original proof with only obvious modifications.

References

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