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Approximating unbounded functions with linear operators generated by moment sequences

by

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1. Introduction. Recently (see, e.g. [3], [4] and [7]) attention has been given to the problem of uniform approximation on the intervals $(-\infty, \infty)$ and $[0, \infty)$ of functions $f(x)$ having certain growth rate as $x \to \pm\infty$ by means of linear operators which are positive on some finite interval. The technique employed to solve this problem is known as *multiplier enlargement*. In this paper we apply multiplier enlargement to the approximation of continuous functions by means of generalized Bernstein polynomials and Bernstein power series which are generated by moment sequences. In particular, we obtain as a corollary an extension of a result for the Bernstein polynomials due to Chlodovsky ([5], p. 36).

2. Definitions and preliminaries. The operators we shall consider are defined below.

Let $\{\mu_n(x)\}$ be a sequence of real-valued functions defined on $[0, 1]$. Denote by $(h_{nk}(x))$ and $(p_{nk}(x))$ respectively the matrices generated by $\{\mu_n(x)\}$ as follows:

$$h_{nk}(x) = \begin{cases} \binom{n}{k} \Delta^{n-k} \mu_k(x), & 0 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases} \tag{2.1}$$

and

$$p_{nk}(x) = \begin{cases} 0, & k < n, \\ \binom{k}{n} \Delta^{k-n} \mu_{n+1}(x), & k \geqslant n, \end{cases} \tag{2.2}$$

where, for any non-negative integers n and p,

$$\Delta^p \mu_n(x) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \mu_{n+j}(x). \tag{2.3}$$

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$\{\mu_n(x)\}$ is called a *generalized moment sequence* if there exists a function $\beta(x, t)$ of bounded variation in t for each $x \in [0, 1]$ such that for all $x \in [0, 1]$

$$\mu_n(x) = \int_0^1 t^n \, d\beta(x, t), \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

The sequence $\{\mu_n(x)\}$ is called *totally monotone* if $\Delta^p \mu_n(x) \geqslant 0$ for all $x \in [0, 1]$ and all integers $n, p \geqslant 0$.

Let $\{\mu_n(x)\}$ be a generalized moment sequence. For all functions f defined on the interval $[0, 1]$ associate the linear operator

$$H_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) h_{nk}(x) \tag{2.5}$$

with the matrix (2.1) and associate the linear operator

$$P_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k-n}{k}\right) p_{nk}(x) \tag{2.6}$$

with the matrix (2.2).

If $\mu_n(x) \equiv x^n$ $(n = 0, 1, 2, \ldots)$, then H_n becomes the n-th Bernstein polynomial. When $\mu_n(x) \equiv (1-x)^n$ for all n, P_n becomes a modified Bernstein power series [1].

We shall have need of the following extension of Theorem 1 of [4] (see also [7], Theorem 1):

THEOREM 2.1. *Let $f(x)$ be defined and continuous on $[0, \infty)$ and let $L_n(f(t); x)$ $(n = 1, 2, \ldots)$ be a sequence of linear operators which are positive on $[0, 1]$. Let the "bounding function" $\Omega(|x|)$ satisfy*

$$\Omega(|x|) \geqslant 1, \qquad \Omega(|x|) \uparrow \infty \qquad (|x| \uparrow \infty),$$

and suppose $f(x) = O(\Omega(|x|))$ $(|x| \to \infty)$. Let α_n be increasing to $+\infty$ with n and let $\{L_n(1; \alpha_n^{-1}x)\}$ be almost convergent to 1 uniformly on every finite interval of $[0, \infty)$, and $\{L_n((\alpha_n t - x)^2 \Omega((\alpha_n t)); \alpha_n^{-1}x)\}$ be almost convergent to 0 uniformly on every finite interval of $[0, \infty)$.

Then $\{L_n(f(\alpha_n t); \alpha_n^{-1}x)\}$ is almost convergent to $f(x)$ uniformly on every finite interval of $[0, \infty)$.

3. The generalized Bernstein polynomial H_n. In the sequel let $e_k(x) = x^k$ $(k = 0, 1, 2, \ldots)$.

THEOREM 3.1. *Let $\{\mu_n(x)\}$ be a totally monotone generalized moment sequence. Let α_n be increasing to $+\infty$ with n and let $\alpha_n = o(n)$. Let $f(x)$ be defined and continuous on $[0, \infty)$ and suppose $f(x) = O(e^{ax})$ $(x > 0)$ for some $a > 0$. Assume that, uniformly in j $(j = 0, 1, 2, \ldots)$, $\{\alpha_n^j \mu_j(x/\alpha_n)\}$ is convergent (almost convergent) to x^j, uniformly on any finite interval of $[0, \infty)$.*

Then $\{H_n(f(\alpha_n t); \alpha_n^{-1}x)\}$ is convergent (almost convergent) to $f(x)$, uniformly on any finite interval of $[0, \infty)$.

Proof. Since $\{\mu_n(x)\}$ is totally monotone, H_n is positive on $[0, 1]$. Thus $H_n(f(\alpha_n t); \alpha_n^{-1}x)$ is positive on $[a, b] \subset [0, \infty)$ for n large. Let $0 < x < \infty$ and $g(\zeta) = (\zeta - x)^2 e^{a\zeta}$. To prove the theorem it suffices, by Theorem 1 of [4] and Theorem 2.1, to show that $\{H_n(g(\alpha_n\zeta); \alpha_n^{-1}x)\}$ is convergent (almost convergent) to 0. We have

$$\begin{aligned}
&H_n(g(\alpha_n\zeta); \alpha_n^{-1}x) \\
&= H_n(e^{a\alpha_n\zeta} e_2(\alpha_n\zeta); \alpha_n^{-1}x) - 2xH_n(e^{a\alpha_n\zeta} e_1(\alpha_n\zeta); \alpha_n^{-1}x) + x^2 H_n(e^{a\alpha_n\zeta}; \alpha_n^{-1}x) \\
&= \sum_{k=0}^{n} e^{\alpha_n ak/n} \binom{n}{k} \int_0^1 (1-t)^{n-k} t^k \, d\beta(\alpha_n^{-1}x, t) \left\{\left(\frac{\alpha_n k}{n}\right)^2 - 2x\left(\frac{\alpha_n k}{n}\right) + x^2\right\} \\
&= \sum_{j=0}^{\infty} \frac{a^j}{j!} \int_0^1 \sum_{k=0}^{n} \left\{\left(\frac{\alpha_n k}{n}\right)^{j+2} - 2x\left(\frac{\alpha_n k}{n}\right)^{j+1} + x^2\left(\frac{\alpha_n k}{n}\right)^{j}\right\} \binom{n}{k} (1-t)^{n-k} t^k \, d\beta(\alpha_n^{-1}x, t) \\
&= \sum_{j=0}^{\infty} \frac{a^j}{j!} \int_0^1 \{B_n(e_{j+2}(\alpha_n\zeta); t) - 2xB_n(e_{j+1}(\alpha_n\zeta); t) + x^2 B_n(e_j(\alpha_n\zeta); t)\} \, d\beta(\alpha_n^{-1}x, t),
\end{aligned}$$

where B_n is the n-th order Bernstein polynomial. It follows from the proof of Theorem 3 of [7] that

$$B_n(e_r(\alpha_n\zeta); t) = \underbrace{(\alpha_n t)^r \frac{n(n-1)\ldots(n-r+1)}{n^r} + \ldots + (\alpha_n t)\left(\frac{\alpha_n}{n}\right)^{r-1}}_{r \text{ terms}}$$

for $r = 1, 2, 3, \ldots$ Therefore,

$$\begin{aligned}
&H_n(g(\alpha_n\zeta); \alpha_n^{-1}x) \\
&= \sum_{j=0}^{\infty} \frac{a^j}{j!} \Bigg\{\Bigg(\alpha_n^{j+2} \frac{n\ldots(n-j-1)}{n^{j+2}} \int_0^1 t^{j+2} \, d\beta(\alpha_n^{-1}x, t) + \ldots \\
&\qquad\qquad + \alpha_n \left(\frac{\alpha_n}{n}\right)^{j+1} \int_0^1 t \, d\beta(\alpha_n^{-1}x, t)\Bigg) - \\
&\quad - 2x\Bigg(\alpha_n^{j+1} \frac{n\ldots(n-j)}{n^{j+1}} \int_0^1 t^{j+1} \, d\beta(\alpha_n^{-1}x, t) + \ldots + \alpha_n \left(\frac{\alpha_n}{n}\right)^{j} \int_0^1 t \, d\beta(\alpha_n^{-1}x, t)\Bigg) + \\
&\quad + x^2\Bigg(\alpha_n^{j} \frac{n\ldots(n-j+1)}{n^{j}} \int_0^1 t^{j} \, d\beta(\alpha_n^{-1}x, t) + \ldots + \alpha_n \left(\frac{\alpha_n}{n}\right)^{j-1} \int_0^1 t \, d\beta(\alpha_n^{-1}x, t)\Bigg)\Bigg\}.
\end{aligned}$$

By hypothesis

$$\{a_n^j \int_0^1 t^j \, d\beta(a_n^{-1}x, t)\} = \{a_n^j \mu_j(x/a_n)\}$$

is convergent (almost convergent) to x^j uniformly in j ($j = 0, 1, 2, \ldots$). The theorem follows immediately from the above.

Corollary 3.2. *Let $f(x)$ be defined and continuous on $[0, \infty)$. Let a_n be increasing to $+\infty$ with n and let $a_n = o(n)$. Let $B_n(f(t); x)$ denote the n-th Bernstein polynomial. If*

$$\max\{|f(x)|: 0 \leqslant x \leqslant a_n\} = o(e^{na/a_n}) \tag{3.1}$$

for each $a > 0$, then $\{B_n(f(a_n t); a_n^{-1}x)\}$ converges to $f(x)$ uniformly on any finite interval of $[0, \infty)$. If

$$f(x) = O(e^{ax}) \quad (x > 0) \tag{3.2}$$

for some $a > 0$, then $\{B_n(f(a_n t); a_n^{-1}x)\}$ converges to $f(x)$ uniformly on any finite interval of $[0, \infty)$.

Remarks. Result (3.1) is due to Choldovsky [5], p. 36. Result (3.2) follows from Theorem 3.1 by choosing $\mu_j(x) = x^j$ ($j = 0, 1, 2, \ldots$). The example $a_n = n^{1/3}$ and $f(x) = e^{x^3/2}$ shows that (3.1) does not imply (3.2) and the example $a_n = n^{2/3}$ and $f(x) = e^x$ shows that (3.2) does not imply (3.1).

It is interesting to note the following characterization of the Bernstein polynomials, the proof of which was conveyed to the authors by Professor Dany Leviatan.

Theorem 3.3. *Let $\{\mu_j(x)\}$ be a generalized moment sequence and $\{H_n\}$ the sequence of operators defined in (2.5). Then a necessary and sufficient condition that*

$$\lim_{n\to\infty} H_n(f; x) = f(x) \text{ uniformly on } [0, 1],$$

for each $f \in C[0, 1]$, is $\mu_j(x) \equiv x^j$ for $j = 0, 1, \ldots$

Proof. If $\mu_j(x) \equiv x^j$ ($j = 0, 1, 2, \ldots$), then H_n is the n-th Bernstein polynomial. Hence the sufficiency follows from [5], p. 5.

On the other hand, if $\lim_{n\to\infty} H_n(f; x) = f(x)$ for all $f \in C[0, 1]$, then

$$\lim_{n\to\infty} H_n(e_k; x) = x^k. \tag{3.3}$$

Also

$$\lim_{n\to\infty} H_n(e_k; x) = \lim_{n\to\infty} \sum_{m=0}^{n} \left(\frac{m}{n}\right)^k \binom{n}{m} \int_0^1 (1-t)^{n-m} t^m \, d\beta(x,t)$$

$$= \int_0^1 \lim_{n\to\infty} \sum_{m=0}^{n} \left(\frac{m}{n}\right)^k \binom{n}{m} (1-t)^{n-m} t^m \, d\beta(x,t) = \int_0^1 t^k \, d\beta(x,t).$$

Hence

$$\lim_{n\to\infty} H_n(e_k; x) = \mu_k(x) \tag{3.4}$$

and the necessity follows from (3.3) and (3.4).

4. The generalized Bernstein power series P_n. The main results of this section (Theorem 4.2 and Theorem 4.4) depend on the following lemma:

Lemma 4.1. *Let $\{a_n\}$ be a sequence of non-zero real numbers and $\{P_n\}$ the sequence of linear operators defined in (2.6). Then*

$$P_n(1; a_n^{-1}x) = \mu_0(a_n^{-1}x),$$

$$P_n(a_n t; a_n^{-1}x) = a_n [\mu_0(a_n^{-1}x) - \mu_1(a_n^{-1}x)],$$

and

$$a_n^2[\mu_0(a_n^{-1}x) - 2\mu_1(a_n^{-1}x) + \mu_2(a_n^{-1}x)] \leqslant P_n((a_n t)^2; a_n^{-1}x)$$

$$\leqslant a_n^2[\mu_0(a_n^{-1}x) - 2\mu_1(a_n^{-1}x) + \mu_2(a_n^{-1}x)] + \frac{a_n^2}{n}[\mu_0(a_n^{-1}x) - \mu_1(a_n^{-1}x)].$$

Proof. The result follows from a slight modification of the proof of [2], Theorem 3.1.

Theorem 4.2. *Let a_n be positive and increasing to $+\infty$ with n, and $a_n = o(n)$. Let $f(x)$ be defined, bounded and continuous on $[0, \infty)$. Assume that $\{\mu_0(a_n^{-1}x)\}$ is convergent (almost convergent) to 1, $\{a_n[\mu_0(a_n^{-1}x) - \mu_1(a_n^{-1}x)]\}$ is convergent (almost convergent) to x, and $\{a_n^2(\mu_0(a_n^{-1}x) - 2\mu_1(a_n^{-1}x) + \mu_2(a_n^{-1}x))\}$ is convergent (almost convergent) to x^2, uniformly on any finite interval of $[0, \infty)$. Then $\{P_n(f(a_n t); a_n^{-1}x)\}$ is convergent (almost convergent) to $f(x)$ uniformly on any finite interval of $[0, \infty)$.*

Proof. The conclusion follows from Lemma 4.1 and [3], Theorem 1, with $m = 1$ for convergence, and from Lemma 4.1 and [7], Theorem 1, for almost convergence.

Corollary 4.3. *Let a_n be increasing to $+\infty$ with n and $a_n = o(n)$. If $\mu_n(x) = (1-x)^n$ for $n = 0, 1, 2, \ldots$, then $\{P_n(f(a_n t); a_n^{-1}x)\}$ converges to $f(x)$ uniformly on any finite interval of $[0, \infty)$ for all functions $f(x)$ which are defined, bounded, and continuous on $[0, \infty)$.*

Thus we have a convergence theorem for the linear operator

$$P_n(f; x) = \sum_{k=n}^{\infty} f\left(\frac{k-n}{k}\right) \binom{k}{n} x^{k-n} (1-x)^{n+1} \tag{4.1}$$

which differs slightly from the Bernstein power series

$$M_n(f; x) = \sum_{k=n}^{\infty} f\left(\frac{k-n}{k}\right) \binom{k-1}{n-1} x^{k-n} (1-x)^n. \tag{4.2}$$

However, by comparing Theorem 4.4 below and [6], Theorem 1, it is easy to see that (4.1) and (4.2) have the same approximation properties and are essentially the same. Hence Theorem 4.4 may be considered as a characterization of the Bernstein power series.

THEOREM 4.4. *Let $0 < a < 1$. Then a necessary and sufficient condition that $\{P_n(f; x)\}$ converge to $f(x)$ uniformly on $[0, a]$, for each $f \in C[0, 1]$, is* $\mu_j(x) = (1-x)^j$ *for* $j = 0, 1, 2, \ldots$

Proof. By applying the Korovkin theorem and Lemma 4.1 with $a_n = 1$ for all n, we see that the condition is sufficient. The proof of necessity is similar to the proof given in Theorem 3.3.

References

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Пример гладкого пространства, сопряженное к которому не является строго нормированным

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Пространство Банаха X называется *строго нормированным*, если из условий

$$\|x\| = \|y\| = 1, \quad \|x+y\| = 2 \quad (x, y \in X)$$

следует, что x и y совпадают.

Пространство Банаха X называется *гладким*, если его норма дифференцируема по Гато, т.е. если для любых $x, y \in X$ ($\|x\| > 0$) выполнено следующее условие:

$$\lim_{\tau\to 0} \frac{1}{\tau}(\|x+\tau y\| + \|x-\tau y\| - 2\|x\|) = 0.$$

Алаоглу и Биркхоф [1] показали, что

(а) пространство X строго нормированно, если его сопряженное X^* гладко,

(б) пространство X гладко, если X^* строго нормированно.

Известны примеры (см. напр. Дэй [2], стр. 191) строго нормированного пространства, сопряженное к которому не является гладким.

Цель настоящей заметки построить пример гладкого пространства, сопряженное к которым не является строго нормированным.

Через l обозначим банахово пространство, состоящее из действительных числовых последовательностей $\{a_i\}_{i=1}^{\infty}$, ряд из которых абсолютно сходится:

$$\|\{a_i\}_{i=1}^{\infty}\| = \sum_{i=1}^{\infty} |a_i| \quad (\{a_i\}_{i=1}^{\infty} \in l).$$

Общий вид линейного функционала в l записывается в виде:

$$\sum_{i=1}^{\infty} a_i \xi_i \quad (\{a_i\}_{i=1}^{\infty} \in l),$$

где $\{\xi_i\}_{i=1}^{\infty}$ есть ограниченная последовательность действительных чисел. Сопряженным пространством к l является пространство m,