References

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Moscow 1967 (Russian).

[2] R. G. Bartle and L. M. Graves, *Mappings between function space*, Trans. Amer. Math. Soc. 72 (1952), p. 400-413.

[3] C. Bessaga and A. Pełczyński, *A topological proof that every separable Banach space is homeomorphic to a countable product of lines*, Bull. Acad. Pol. Sc., Sér. Math., Astr. et Phys., 1969.

[4] B. F. Bylov, R. E. Vinogradov, D. M. Grobman and V. V. Niemytzki, *Theory of Liapunov's characteristic exponents*, Moscow 1966 (Russian).

[5] A. Douady, *Un espace de Banach dont le groupe linéaire n'est pas connexe*, Ind. Math. 27.5 (1965).

[6] N. Dunford and J. T. Schwartz, *Linear operators I*, New York 1958.

[7] — *Linear operators II*, New York 1963.

[8] P. Hartman, *Ordinary differential equations*, New York 1964.

[9] P. J. Hilton and S. Wylie, *Homology theory*, Cambridge 1960.

[10] Hu Sze-Tsen, *Homotopy theory*, New York 1959.

[11] M. I. Kadec, *Proof of topological equivalence of all separable, infinite-dimensional Banach spaces*, Funkc. Analiz 4 (1967), p. 61-70 (Russian).

[12] V. Klee, *A note on topological properties of normed-linear spaces*, Proc. Amer. Math. Soc. 7 (1959), p. 673-674.

[13] N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology 3.1 (1965), p. 19-30.

[14] F. Riesz et B. Nagy, *Leçons d'analyse fonctionnelle*, Budapest 1965.

[15] S. Rolewicz, *On orbits of elements*, Studia Math. 32 (1969), p. 17-22.

[16] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), p. 97-116.

[17] — *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), p. 747-817.

[18] E. M. Vaisbord, *On equivalence of systems of differential equations near singular point*, Nauc. Dokl. Vizh. Shk. (fiz.-mat.) 1 (1958), p. 37-42 (Russian).

[19] R. Y. T. Wong, *On homeomorphism of certain infinite dimensional spaces*, Trans. Amer. Math. Soc. 128 (1967).

Reçu par la Rédaction le 18. 5. 1969

STUDIA MATHEMATICA, T. XXXV. (1970)

Analytic functions in Banach spaces

by

JACEK BOCHNAK (Kraków)

There are not many papers devoted to the theory of analytic mappings in Banach spaces, especially in real Banach spaces; recently, however, a number of papers concerning that branch have been published (Cartan, Douady, Lelong, Ramis and others). N. Bourbaki in his forthcoming books [5] ("Fascicule de résultats" is already available) will give the systematic theory of such mappings.

This paper gives some results on analytic mappings, with values in a Banach space, defined on open subsets of Banach spaces (real or complex). In particular, we prove a natural criterion of the analyticity of mappings (Theorem 6). Some versions of that criterion were given by Alexiewicz and Orlicz [1] and Siciak [17]; the author of the present paper has been inspired by some ideas of [1], and [9].

The plan of the article is as follows. We start in Section I by proving some results on formal series in Banach spaces. In Section II we consider Gateaux-differentiable mappings. In Section III we state and prove some criteria of the holomorphicity (Theorem 4, complex case) and analyticity (Theorem 6, real case) of the mappings. Finally, in Section IV, we give the applications of the preceding results, namely: a proof of the Weierstrass preparation theorem for analytic functions in Banach spaces (another proof of that theorem was given by Ramis [14]), the generalization of a theorem of Malgrange, and some other theorems.

Theorem 4 is stated in [5] without proof (see also [9], [21]). The proof of case I of Theorem C has been communicated to me by Professor S. Łojasiewicz.

I would like to express my gratitude to Professor S. Łojasiewicz for his guidance and valuable remarks. I also want to thank Professor J. Siciak for helpful conversations.

I. FORMAL SERIES IN BANACH SPACES

Let E and F be real or complex Banach spaces. Denote by $\mathrm{Hom}^k(E, F)$ (resp. $L^k(E, F)$) the space of k-linear, symmetric (resp. k-linear, symmetric and continuous) mappings from E^k to F. (Thus $\mathrm{Hom}^0(E,F)$ is the set of constant mappings from E to F.) Let $\pi_k\colon E\ni h\to(h,\ldots,h)\in E^k$ and

$$Q^k(E,F)=\{f\colon f=\bar f\circ\pi_k \text{ for some } \bar f\in\mathrm{Hom}^k(E,F)\},$$
$$P^k(E,F)=\{f\colon f=\bar f\circ\pi_k \text{ for some } \bar f\in L^k(E,F)\}.$$

A mapping $f\in Q^k(E,F)$ (resp. $f\in P^k(E,F)$) is called a *homogeneous* (resp. *continuous homogeneous*) *polynomial of degree* k. More generally, a finite sum of homogeneous polynomials is called a *polynomial*. The composition of two polynomials is a polynomial.

The linear mapping given by the formula

$$\mathrm{Hom}^k(E,F)\ni\bar f\to\bar f\circ\pi_k\in Q^k(E,F)$$

is an isomorphism and it sends $L^k(E,F)$ onto $P^k(E,F)$. Its inverse can be expressed by ([13], [19])

$$Q^k(E,F)\ni f\to\frac{1}{k!}\sum_{\substack{\varepsilon_1=0,1\\ \cdots\cdots\\ \varepsilon_k=0,1}}(-1)^{k-(\varepsilon_1+\ldots+\varepsilon_k)}f\circ\sigma_\varepsilon\in\mathrm{Hom}^k(E,F),$$

where $\sigma_\varepsilon\colon E^k\ni(h_1,\ldots,h_k)\to\sum_{i=1}^k\varepsilon_i h_i\in E$.

Thus to every homogeneous polynomial f of degree k we have a uniquely determined k-linear symmetric mapping, which will be denoted by $\bar f$ $(f=\bar f\circ\pi_k)$.

The elements of the cartesian product $Q=\prod_{k=0}^\infty Q^k(E,F)$ (resp. $P=\prod_{k=0}^\infty P^k(E,F)$) are called *formal series* (resp. *continuous formal series*) and denoted by $\sum_{k=0}^\infty f_k$.

We are now going to distinguish some subsets $\mathscr{K}$ and $\mathscr{H}$ of the space of formal series. Put

$$\mathscr{K}=\Big\{\sum_{n=0}^\infty f_n\in Q\colon \sum_{n=0}^\infty f_n(x) \text{ is convergent for every point } x \text{ of a neighbourhood of zero in } E\Big\}.$$

We say that a subset $A\subset E$ is a *λ-neighbourhood of zero in E* if, for every 1-dimensional subspace $\mu\subset E$, the intersection $A\cap\mu$ is a neighbourhood of zero on that μ. Then we put

$$\mathscr{H}=\Big\{\sum_{n=0}^\infty f_n\in Q\colon \sum_{n=0}^\infty f_n(x) \text{ is convergent for every point } x \text{ of a } \lambda\text{-neighbourhood of zero in } E\Big\}.$$

The elements of the set $\mathscr{K}$ (resp. $\mathscr{H}$) are called *convergent formal series* (resp. *quasi-convergent formal series*).

We say that $\sum_{n=0}^\infty f_n$ is *normally convergent* if, for a neighbourhood U of zero in E,

$$\sum_{n=0}^\infty\sup_{x\in U}\|f_n(x)\|<\infty.$$

These three notions are equivalent for continuous formal series. Namely, we will prove the following

THEOREM 1. *Any quasi-convergent continuous formal series* $\sum_{n=0}^\infty f_n$ *is normally convergent.*

Before proving Theorem 1 we give some lemmas.

Let E be a real Banach space, $\hat E=E\otimes_R C$. For $y\in\hat E$ put ([8], [15])

$$\|y\|^\wedge=\inf\{\textstyle\sum|z_i|\,\|x_i\|\colon y=\sum x_i\otimes z_i;\ x_i\in E,\ z_i\in C\}\text{(1)}.$$

LEMMA 1. *The space* $(\hat E,\|\ \|^\wedge)$ *is a complex Banach space, and the mapping*

$$E\ni a\to a\otimes 1\in\hat E$$

is an $\boldsymbol{R}$*-linear isometry.*

Proof. For any points $y_1,y_2\in\hat E$ the trivial inclusion

$$\{\textstyle\sum|z_i|\,\|a_i\|+\sum|w_j|\,\|b_j\|\colon y_1=\sum a_i\otimes z_i,\ y_2=\sum b_j\otimes w_j\}\subset\{\sum|z_k|\,\|c_k\|\colon y_1+y_2=\sum c_k\otimes z_k\}$$

implies the triangle inequality $\|y_1+y_2\|^\wedge\leqslant\|y_1\|^\wedge+\|y_2\|^\wedge$. We verify the equality $|c|\,\|y\|^\wedge=\|cy\|^\wedge$ observing that for $c\neq0$ we have

$$\{\textstyle\sum|z_i|\|a_i\|\colon cy=\sum a_i\otimes z_i\}=|c|\{\sum|w_j|\,\|b_j\|\colon y=\sum b_j\otimes w_j\}.$$

The mapping

$$(*)\qquad E^2\ni(x_1,x_2)\to x_1\otimes1+x_2\otimes i\in\hat E$$

is an isomorphism of (real) vector spaces. Note that if $x_1\otimes1+x_2\otimes i=\sum y_j\otimes z_j$, then $x_1=\sum y_j\mathrm{Re}z_j$, $x_2=\sum y_j\mathrm{Im}z_j$; so $\max\{\|x_1\|,\|x_2\|\}\leqslant\sum|z_j|\,\|y_j\|$ or $\max\{\|x_1\|,\|x_2\|\}\leqslant\|x_1\otimes1+x_2\otimes i\|^\wedge$. By this and the obvious inequality $\max\{\|x\otimes1\|^\wedge,\|x\otimes i\|^\wedge\}\leqslant\|x\|$ we obtain $\|x\otimes1\|^\wedge=\|x\otimes i\|^\wedge=\|x\|$. This implies that $\|x_1\otimes1+x_2\otimes i\|^\wedge\leqslant\|x_1\|+\|x_2\|$, and that $(*)$ is a topological isomorphism; thus $(\hat E,\|\ \|^\wedge)$ is a complete normed space.

(1) Another formula for a norm is given in [19].

Remark. In what follows we do not distinguish between E and $E\otimes 1$. We shall write $x\otimes z = zx$ for $x\in E$, $z\in \boldsymbol{C}$. Then $\hat{E}$ is the direct sum $E+iE$ (over $\boldsymbol{R}$).

LEMMA 2. *Let E be a real and F a complex Banach space. For every n-linear mapping $\varphi\in \mathrm{Hom}^n(E, F)$ there exists exactly one n-linear extension $\hat{\varphi}\in\mathrm{Hom}^n(\hat{E}, \hat{F})$. If φ is continuous, then $\|\varphi\| = \|\hat{\varphi}\|$.*

Proof. It is obvious that the following map, given by the formula

$$\hat{\varphi}(x_1^{(0)}+ix_1^{(1)}, \ldots, x_n^{(0)}+ix_n^{(1)}) = \sum_{\substack{\varepsilon_1=0,1\\ \cdots\cdots\\ \varepsilon_n=0,1}} i^{\varepsilon_1+\ldots+\varepsilon_n}\varphi(x_1^{(\varepsilon_1)}, \ldots, x_n^{(\varepsilon_n)}),$$

is the symmetric extension of φ, n-linear over $\boldsymbol{R}$. The simple observation that $\hat{\varphi}(u, ix) = i\hat{\varphi}(u, x)$, $u\in\hat{E}^{n-1}$, $x\in\hat{E}$, implies that $\hat{\varphi}$ is n-linear over $\boldsymbol{C}$.

The extension $\hat{\varphi}$ is unique because $\hat{E}$ is generated by E.

Now suppose that φ is continuous, i.e.

$$\|\varphi\| = \sup_{\|x_i\|=1}\|\varphi(x_1, \ldots, x_n)\| < \infty.$$

If $y_j = \sum_{i_j=1}^{k_j} z_{i_j}^{(j)}x_{i_j}^{(j)}$, then

$$\|\hat{\varphi}(y_1, \ldots, y_n)\| \leqslant \sum_{\substack{i_1=1,\ldots,k_1\\ \cdots\cdots\\ i_n=1,\ldots,k_n}} |z_{i_1}^{(1)}|\ldots|z_{i_n}^{(n)}|\,\|\varphi(x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)})\|$$

$$\leqslant \sum_{i_1,\ldots,i_n} |z_{i_1}^{(1)}|\ldots|z_{i_n}^{(n)}|\,\|\varphi\|\,\|x_{i_1}^{(1)}\|\ldots\|x_{i_n}^{(n)}\|$$

$$\leqslant \|\varphi\|\Big(\sum_{i_1=1}^{k_1}|z_{i_1}^{(1)}|\,\|x_{i_1}^{(1)}\|\Big)\ldots\Big(\sum_{i_n=1}^{k_n}|z_{i_n}^{(n)}|\,\|x_{i_n}^{(n)}\|\Big),$$

which implies that $\|\hat{\varphi}(y_1, \ldots, y_n)\| \leqslant \|\varphi\|\,\|y_1\|^\wedge\ldots\|y_n\|^\wedge$ or $\|\hat{\varphi}\|\leqslant\|\varphi\|$. By the trivial inequality $\|\hat{\varphi}\|\geqslant\|\varphi\|$ we have $\|\varphi\| = \|\hat{\varphi}\|$.

COROLLARY. *For every homogeneous polynomial $f\in Q^n(E, F)$ there exists exactly one "complex" polynomial $\hat{f}\in Q^n(\hat{E}, \hat{F})$ such that $f\subset\hat{f}$* (see also [18]).

For any continuous homogeneous polynomial $f\in P^k(E, F)$ we set

$$\|f\| = \sup_{\|x\|=1}\|f(x)\|.$$

LEMMA 3. *If V is a finite-dimensional real vector space, with a euclidean norm, F a real Banach space, and $f\in P^k(V, F)$, then $\|f\| = \|\bar{f}\|$.*

Proof. We start with the case(²) $F = \boldsymbol{R}$. It suffices to consider the case $\|\bar{f}\| = 1$.

(²) which is proved in [2]; the present version is due to S. Łojasiewicz.

Let $k = 2$. Suppose that $|\bar{f}(x, y)| = 1$ for $(x, y)\in S^2$ $(S = \{x\in V: \|x\| = 1\})$ and $x+y\neq 0$. Then

$$\left|\bar{f}\left(\frac{x+y}{\|x\|+\|y\|}\right)\right| = 1$$

(indeed, from the equality $\bar{f}(x, y) = \frac{1}{4}(f(x+y)-f(x-y))$ and the inequality $|f(x\mp y)|\leqslant\|x\mp y\|^2$ we get

$$1 = |\bar{f}(x, y)|\leqslant\tfrac{1}{4}|f(x+y)|+\tfrac{1}{4}|f(x-y)|\leqslant\tfrac{1}{4}\|x+y\|^2+\tfrac{1}{4}\|x-y\|^2 = 1,$$

i.e. $|f(x+y)| = \|x+y\|^2$).

For $k\in\boldsymbol{N}$ given arbitrarily, suppose that $\bar{f}(\tilde{x}_1, \ldots, \tilde{x}_k) = 1$ $(\tilde{x}_i\in S)$. It is clear that for an $a\in S$ the inner product $\langle a, \tilde{x}_i\rangle\neq 0$ $(i = 1, \ldots, k)$. Then, for an $\varepsilon > 0$, the set

$$A = \{(x_1, \ldots, x_k)\in S^k\colon \langle a, x_i\rangle\geqslant\varepsilon;\ |\bar{f}(x_1, \ldots, x_k)| = 1\}$$

is compact and A is not empty. Assume that the maximum of the function

$$A\ni(x_1, \ldots, x_k)\to\sum_{i=1}^{k}\langle a, x_i\rangle\in\boldsymbol{R}$$

is attained at the point $(x_1^*, \ldots, x_k^*)$. It is sufficient to prove that $\varepsilon_1x_1^* = \ldots = \varepsilon_kx_k^*$, where $\varepsilon_j = \pm 1$. If this is not true, i.e. $x_p^*\neq x_q^*$ and $x_p^*\neq -x_q^*$ for a pair of indices p, q, then at the point $(x_1', \ldots, x_k')\in A$, where $x_i' = x_i^*$ $(i\neq p, q)$,

$$x_p' = x_q' = \frac{x_p^*+x_q^*}{\|x_p^*+x_q^*\|},$$

we have

$$\sum_{i=1}^{k}\langle a, x_i'\rangle > \sum_{i=1}^{k}\langle a, x_i^*\rangle$$

(because $\|x_p^*+x_q^*\| < 2$), which is a contradiction.

The general case easily follows from the above. Let $\xi\colon F\to\boldsymbol{R}$ be a continuous linear functional such that $\|\xi\| = 1$ and $\|\xi\circ\bar{f}\| = 1$. In view of the previous case, there exists an element $x^*\in V$ such that $\|x^*\| = 1$ and $\xi\circ\bar{f}(x^*, \ldots, x^*) = 1$. This equality implies that $\|\bar{f}(x^*, \ldots, x^*)\| = \|f(x^*)\| = 1$, i.e. $\|f\| = 1$.

Remark. Lemma 3 in general is not true if the norm on V is not euclidean.

The following lemma is a version of the important Polynomial Lemma ([11], [16]):

LEMMA 4. *Let E be a complex Banach space, $A = A_1\times\ldots\times A_n\subset\boldsymbol{C}^m$, where A_i are compact, connected subsets of $\boldsymbol{C}$, not reduced to a single point* *and $\omega > 1$. Then there exist $\delta > 0$ and $n^* \in N$, so that for any $s \geqslant n^*$ and any (not necessarily homogeneous) polynomial $f\colon C^n \to E$ of degree $\leqslant s$ the following implication holds:*

$$\big(\|f(x)\| \leqslant M \text{ for } x \in A\big) \Rightarrow \big(\|f(x)\| \leqslant M\omega^s \text{ for } \varrho(x, A) < \delta\big).$$

Proof (by induction)(3). Let $n = 1$ and $\omega > 1$. Denote by d the diameter of the set $A = A_1$. Let $a \in A$ and $s \in N$. Then there exist $s+1$ points $z_0, \ldots, z_s$ of A such that $|a - z_k| = dk^2/s^2$ $(k = 0, \ldots, s)$. Let

$$w_k(z) = \prod_{\substack{j=0\\ j\neq k}}^{s} (z - z_j).$$

For $\delta > 0$ fixed, assume that $|z - a| < \delta$; then

$$|z - z_k| \leqslant \delta + \frac{dk^2}{s^2} = d\left(\alpha^2 + \frac{k^2}{s^2}\right), \quad \text{where } \alpha^2 = \frac{\delta}{d};$$

thus

$$|w_k(z)| \leqslant d^s \prod_{\substack{j=0\\ j\neq k}}^{s} \left(\alpha^2 + \frac{j^2}{s^2}\right) \leqslant d^s \prod_{j=1}^{s} \left(\alpha^2 + \frac{j^2}{s^2}\right),$$

$$|z_j - z_k| \geqslant \frac{d|k^2 - j^2|}{s^2} \quad \text{or} \quad |w_k(z_k)| \geqslant d_s \prod_{\substack{j=0\\ j\neq k}}^{s} \frac{|j^2 - k^2|}{s^2} \geqslant \frac{d^s}{2} \prod_{j=1}^{s} \frac{j^2}{s^2}\ (^4),$$

whence

$$\left|\frac{w_k(z)}{w_k(z_k)}\right| \leqslant 2I_s(\alpha)^s, \quad \text{where} \quad I_s(\alpha) = \left[\prod_{j=1}^{s} \frac{\alpha^2 + j^2/s^2}{j^2/s^2}\right]^{1/s}.$$

Since

$$\lim_{s\to\infty} I_s(\alpha) = e^{\int_0^1 \ln(1+\alpha^2/t^2)\,dt} \quad \text{and} \quad \lim_{\alpha\to 0} \int_0^1 \ln\left(1 + \frac{\alpha^2}{t^2}\right) dt = 0,$$

(3) We repeat here the argument of Leja (case $n = 1$) and Siciak (induction step) transferred to the case of Banach spaces.

(4) Since $\prod_{\substack{j=0\\ k\neq j}}^{s} |j^2 - k^2| \geqslant \frac{1}{2} \prod_{j=1}^{s} j^2$ (indeed,

$$\frac{\prod_{j=0}^{k-1} (k^2 - j^2)}{\prod_{j=1}^{k} j^2} = \prod_{j=0}^{k} \frac{k+j}{j+1} \quad \text{and} \quad \prod_{k+1}^{s} \frac{j^2 - k^2}{j^2} = \left(\prod_{j=1}^{s-k} \frac{j}{j+k}\right)\left(\prod_{j=k+1}^{s} \frac{k+j}{j}\right) \geqslant \prod_{j=1}^{k} \frac{j}{j+k};$$

the product of the right-hand sides being $\geqslant \frac{1}{2}$).

we may choose $\delta > 0$ and $\tilde{n} \in N$ such that $I_s(\alpha) < \sqrt{\omega}$ if $s \geqslant \tilde{n}$ $(\alpha^2 = \delta/d)$. The Lagrange interpolation formula (for the Banach space-valued polynomials) implies that for any polynomial $f\colon C \to E$ of degree $\leqslant s$:

$$f(z) = \prod_{k=0}^{s} f(z_k) \frac{w_k(z)}{w_k(z_k)}.$$

If $\|f(z)\| \leqslant M$ on the set A, then

$$\|f(z)\| \leqslant 2(s+1)\, M \cdot I_s(\alpha)^s \quad \text{if} \quad |z - a| < \delta,$$

which, together with the fact that $2(s+1) < \sqrt{\omega^s}$ for $s \geqslant \tilde{k}$, implies $\|f(z)\| \leqslant M\omega^s$ for $|z - a| < \delta$ and $s > \max\{\tilde{k}, \tilde{n}\}$; the choice of $\delta, \tilde{k}, \tilde{n}$ is independent of a.

Now assume that the lemma is true for the polynomials of $n-1$ variables. Let $\omega > 1$ and let $\delta > 0$, $n^* \in N$ be chosen to $\sqrt{\omega}$, as in the lemma, simultaneously in the case of one and $n-1$ variables. Let $s \geqslant n^*$ and suppose that $f\colon C^n \to E$ is a polynomial of the degree $\leqslant s$, so that $\|f(z', z)\| \leqslant M$ if $z' \in A' = A_1 \times \ldots \times A_{n-1}$, $z \in A_n$. Then (according to one variable case) $\|f(z', z)\| \leqslant M\sqrt{\omega^s}$ if $z' \in A'$, $\varrho(z, A_n) < \delta$; thus, according to the inductive assumption,

$$\|f(z', z)\| \leqslant M\omega^s \quad \text{if} \quad \varrho(z', A') < \delta, \quad \varrho(z, A_n) < \delta.$$

Corollary. *If $\{f_n\}_{n\in N}$ is a sequence of polynomials having a common bound on the set A (as in Lemma 4) and $\deg f_n = n$, then for every $\omega > 1$ there exist a constant $\tilde{M}$ and a neighbourhood U of the set A such that $\|f_n(z)\| \leqslant \tilde{M}\omega^n$ if $z \in U$.*

Lemma 5. *If E and F are complex Banach spaces, $f\colon E \to F$ is a homogeneous polynomial and $\|f(z)\| \leqslant M$ for $\|z - z_0\| \leqslant r$, then $\|f(z)\| \leqslant M$ for $\|z\| \leqslant r$.*

Proof. Let $\|h\| \leqslant r$, $h \in E$. The principle of maximum applied to the holomorphic map $\varphi\colon C \ni \eta \to f(\eta z_0 + h) \in F$ implies that there exists a complex number η_0, $|\eta_0| = 1$, such that $\|\varphi(\eta)\| \leqslant \|\varphi(\eta_0)\|$ for $|\eta| \leqslant 1$. Hence

$$\|f(h)\| = \|\varphi(0)\| \leqslant \|\varphi(\eta_0)\| = \|f(\eta_0 z_0 + h)\| = \|f(z_0 + \eta_0^{-1} h)\| \leqslant M.$$

Lemma 6. *The closure of any λ-neighbourhood A of zero in the Banach space E has interior points.*

Proof. Let K_δ be a ball with centre at zero and radius δ, contained in the space $\boldsymbol{R}$ or $\boldsymbol{C}$ (according to whether E is real or complex), $S = \{x \in E\colon \|x\| = 1\}$, $A_n = \{x \in S\colon K_{1/n} \cdot x \subset \bar{A}\}$. The sets A_n are closed and cover the sphere S. Hence Baire's theorem shows that, for some $n^* \in N$, A_{n^*} has an interior point (in S). Clearly, the interior of the set $K_{1/n^*} A_{n^*} \subset \bar{A}$ is not empty.

Remark. The interior of a λ-neighbourhood of zero may be void. Zero is not necessarily the interior point of $\bar{A}$.

Proof of Theorem 1. For any x in the unit sphere S, there exists a $\delta_x > 0$ such that $\|f_n(tx)\| \leqslant 1/2^n$ for all t belonging to the ball K_{δ_x}, centred at zero with radius δ_x. By the continuity of the polynomials f_n, the inequality $\|f_n(u)\| \leqslant 1/2^n$, true for $u \in A = \bigcup_{x \in S} K_{\delta_x} x$, holds also on the set $\bar{A}$, which contains a ball with radius $r > 0$ (Lemma 6).

Case I, of E and F being complex Banach spaces, is simple. Since $\|f_n(x)\| \leqslant 1/2^n$ in a ball with radius r, by Lemma 5 the same estimations hold in the ball centred at zero with the same radius r. But this means that $\sum_{n=0}^{\infty} f_n$ is normally convergent.

Case II, of E and F being real Banach spaces and $\dim E < \infty$, follows from Lemma 4. Indeed, without loss of generality we may assume $E = \boldsymbol{R}^p$. The set $\bar{A}$ contains some p-dimensional cube L; Lemma 4 implies that for complex extensions $\hat{f}_n$ of the polynomials f_n the estimations $\|\hat{f}_n(u)\| \leqslant M \cdot 2^n$ hold true, provided that u belongs to a sufficiently small neighbourhood of the set L in the space $\boldsymbol{C}^p$. In particular, $\|\hat{f}_n(u)\| \leqslant M \cdot 2^n$ in a ball with radius r contained in $\boldsymbol{C}^p$; thus $\|\hat{f}_n(u)\| \leqslant M \cdot 2^n$ for $|u| \leqslant r$ (Lemma 5). This fact implies that $\sum_{n=0}^{\infty} \hat{f}_n$, and then obviously $\sum_{n=0}^{\infty} f_n$ are normally convergent.

We shall deduce case III (E, F real Banach spaces) from cases I, II and the following Lemma 7.

We say that a formal series $\sum_{n=0}^{\infty} f_n \in Q(E, F)$ is *strongly quasi-convergent* if for every finite-dimensional vector subspace $V \subset E$ the series $\sum_{n=0}^{\infty} f_{nV} \in Q(V, F)$ is convergent ($f_{nV} = f_n | V$).

Lemma 7. *If the formal series $\sum_{n=0}^{\infty} f_n \in Q(E, F)$ of ("real") polynomials is strongly quasi-convergent, then the series of their complexifications $\sum_{n=0}^{\infty} \hat{f}_n \in Q(\hat{E}, \hat{F})$ is also strongly quasi-convergent (in $\hat{E}$).*

Proof. It is sufficient to assume that $\dim E < \infty$ and show that the convergence of $\sum_{n=0}^{\infty} f_n$ implies the convergence of $\sum_{n=0}^{\infty} \hat{f}_n$. (Indeed, every finite-dimensional vector subspace of $\hat{E}$ is contained in $\boldsymbol{C}V$ ($= V \otimes \boldsymbol{C}$ by identification) for some finite-dimensional subspace V of E.)

Next, we may assume that the norm on E is euclidean. Let $\bar{f}_n \in L^n(E, F)$ be an n-linear, symmetric mapping corresponding to $f_n \in P^n(E, F)$; denote by $\hat{f}_n \in P^n(\hat{E}, \hat{F})$ and $\hat{\bar{f}}_n \in L^n(\hat{E}, \hat{F})$ the complexification of f_n and $\bar{f}_n$, respectively. In virtue of case II,

$$\sum_{n=0}^{\infty} \sup \{\|\hat{f}_n(x)\| : \|x\| \leqslant r\}$$

is convergent for an $r > 0$; this means that $\sum_{n=0}^{\infty} r^n \|\hat{f}_n\| < \infty$. According to Lemmas 3 and 2 and the trivial inequality $\|\hat{\bar{f}}_n\| \geqslant \|\hat{f}_n\| \geqslant \|f_n\|$ we have

$$\|f_n\| = \|\bar{f}_n\| = \|\hat{\bar{f}}_n\| \geqslant \|\hat{f}_n\| \geqslant \|f_n\|;$$

thus the equality $\|f_n\| = \|\hat{f}_n\|$ must hold. But this implies that $\sum_{n=0}^{\infty} r^n \|\hat{f}_n\| < \infty$, i.e. $\sum_{n=0}^{\infty} \hat{f}_n$ is convergent.

Case III (E and F are real Banach spaces). According to case II the series $\sum_{n=0}^{\infty} f_{nV}$ is convergent for every finite-dimensional vector subspace $V \subset E$; thus $\sum_{n=0}^{\infty} f_n$ is strongly quasi-convergent. In virtue of Lemma 7, $\sum_{n=0}^{\infty} \hat{f}_n$ is also strongly quasi-convergent, whence, in particular, quasi-convergent. According to case I, this implies that the series $\sum_{n=0}^{\infty} \hat{f}_n$, and then the series $\sum_{n=0}^{\infty} f_n$ are normally convergent. The proof of Theorem 1 is complete.

Corollary. $\mathscr{K} \cap P = \mathscr{H} \cap P$.

II. GATEAUX-DIFFERENTIABLE MAPPINGS

In this section we will consider mappings defined on an open subset U of a Banach space E with values in a Banach space F.

A mapping $f\colon U \to F$ is said to have a *Gateaux derivative of order k at $x \in U$* if:

(a) For every $h \in E$ the mapping

$$f_h\colon K_h \ni t \to f(x + th) \in F$$

(defined in some neighbourhood K_h of zero in $\boldsymbol{R}$ or $\boldsymbol{C}$) is k times differentiable at zero.

(b) The mapping

$$\delta_x^k f\colon\ E \ni h \to \frac{d^k}{dt^k} f_h(0) \in F$$

is a homogeneous polynomial of degree s $(s = 1, \ldots, k)$.

We say that $f\colon U \to F$ *is of class* G^k if it has a Gateaux-derivative of order k at each point of U. The homogeneous polynomial $\delta_x^k f$ is called the *k-th Gateaux derivative of f at x*.

Let K be a neighbourhood of zero in $\boldsymbol{R}$ or $\boldsymbol{C}$.

LEMMA 8. *Assume that* $\varphi\colon K \to F$ *is n times differentiable at zero. Then the n-th derivative* φ *at zero can be expressed by the formula*

$$\frac{d^n\varphi}{dt^n}(0) = \lim_{t\to 0} \frac{1}{t^n} \sum_{s=0}^{n} (-1)^{n-s}\binom{n}{s} \varphi(st).$$

The proof of this lemma will be omitted(5) (see [5a], Ch. I, § 3, ex. 6 a).

THEOREM 2. *Let* $f\colon U \to F$ *be a continuous mapping of class* G^n *in* U. *Then the n-th Gateaux derivative* $\delta_x^n f$ *is — for each* $x \in U$ *— a continuous homogeneous polynomial (i.e.* $\delta_x^n f \in P^n(E, F)$*).*

Proof. Let $x \in U$ and

$$(*) \qquad \bar{\delta}_x^n f = \frac{1}{n!} \sum_{\substack{\varepsilon_1 = 0,1 \\ \cdots\cdots \\ \varepsilon_n = 0,1}} (-1)^{\varepsilon_1 + \cdots + \varepsilon_n} \delta_x^n f \circ \sigma_\varepsilon,$$

be a uniquely determined n-linear, symmetric mapping such that $\bar{\delta}_x^n f(h, \ldots, h) = \delta_x^n f(h)$. It suffices to show that $\bar{\delta}_x^n f$ is continuous.

Fix $(h_1^*, \ldots, h_n^*) \in E^n$ and a sequence $\{t_k\}_{k\in\boldsymbol{N}}$ of the real numbers $(t_k \neq 0)$ converging to zero; put $\Omega = \{0, 1\}^n$. Choose a neighbourhood V_ε of the element

$$\sigma_\varepsilon(h_1^*, \ldots, h_n^*) = \sum_{i=1}^{n} \varepsilon_i h_i^*, \qquad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \Omega,$$

such that $x + st_k V_\varepsilon \subset U$ for $s = 1, \ldots, n$ and k sufficiently large.

(5) Hint. Put $I_h\varphi\colon t \to \dfrac{\varphi(t+h) - \varphi(t)}{h}$, and note that

$$I_h^n\varphi(0) = \frac{1}{h^n} \sum_{s=0}^{n} (-1)^{n-s}\binom{n}{s} \varphi(st).$$

The proof of Lemma 8 is a simple consequence of the following observations: (a) $I_h(\varphi') = (I_h\varphi)'$. (b) If φ is k times differentiable in K, then $\|\varphi^{(k)}(t) - a\| \leqslant r$ in $K \Rightarrow \|I_h^k\varphi(t) - a\| \leqslant r$ if $t, t+kh \in K$. (c) If $\varphi'(0)$ exists, then $I_h\varphi(t) \to \varphi'(0)$ if $0 \leqslant |t| \leqslant 2h \to 0$.

The set $W = \bigcap_{\varepsilon\in\Omega} \sigma_\varepsilon^{-1}(V_\varepsilon)$ is a neighbourhood of $(h_1^*, \ldots, h_n^*) \in E^n$. Let

$$\psi_k\colon \bigcup_{\varepsilon\in\Omega} V_\varepsilon \ni h \to \frac{1}{t_k^n} \sum_{s=0}^{n} (-1)^{n-s}\binom{n}{s} f(x + st_k h) \in F.$$

In virtue of Lemma 8,

$$(**) \qquad \delta_x^n f(h) = \frac{d^n}{dt^n} f(x+th)(0) = \lim_{k\to\infty} \psi_k(h).$$

Setting

$$\varphi_k = \frac{1}{n!} \sum_{\varepsilon\in\Omega} (-1)^{n-(\varepsilon_1 + \cdots + \varepsilon_n)} \psi_k \circ (\sigma_\varepsilon | W),$$

we note (formulae $(*)$ and $(**)$) that

$$\bar{\delta}_x^n f(h_1, \ldots, h_n) = \lim_{k\to\infty} \varphi_k(h_1, \ldots, h_n), \qquad (h_1, \ldots, h_n) \in W.$$

This implies that $\bar{\delta}_x^n f | W$, as a pointwise limit of the sequence of continuous mappings, has a point of continuity. Our theorem follows, because an n-linear mapping in a Banach space having a point of continuity is continuous(6).

We say that a mapping $f\colon U \to F$ *satisfies the condition* $r^k(U)$ if for each affine subspace $V \subset E$, $\dim V \leqslant k+1$, the mapping $f | U \cap V$ is of class C^k.

LEMMA 9. *Any mapping* $f\colon U \to F$ *which satisfies the condition* $r^1(U)$ *is of class* G^1 *in* U.

Proof. For each $h \in E$ there exist

$$\delta_x f(h) = \frac{d}{dt} f(x+th)\Big|_{t=0},$$

and the restriction of $\delta_x f$ to every 2-dimensional vector subspace contained in E is linear; hence $\delta_x f$ is linear.

THEOREM 3. *The mapping* $f\colon U \to F$ *satisfying condition* $r^k(U)$ *is of class* G^k *in* U.

(6) Any multilinear mapping which is bounded in a neighbourhood of a point is continuous. Indeed, the statement is trivial if $n = 1$. Suppose that it is true for $n-1$ and let $\varphi\colon E^n \to F$ be an n-linear mapping, bounded in a neighbourhood of $(x^*, y^*) \in E^{n-1} \times E$. It is easy to see that by the inductive hypothesis we have $\|\varphi(x, y)\| \leqslant M_y$, for $\|x\| \leqslant 1$ and y fixed $(x \in E^{n-1}, y \in E$; M_y is independent of $x)$. The Banach-Steinhaus theorem [7] implies that, for some $M \geqslant 0$, $\|\varphi(x, y)\| \leqslant M$ provided that $\|x\| \leqslant 1$ and $\|y\| \leqslant 1$.

Proof. Set

$$\delta_x^p f(h) = \frac{d^p}{dt^p} f(x+th)|_{t=0} \quad (p \leqslant k).$$

It suffices to show that $\delta_x^p f$ are homogeneous polynomials, i.e. to find $\bar{\delta}_x^p f \in \mathrm{Hom}^p(E, F)$ such that $\bar{\delta}_x^p f(h, \ldots, h) = \delta_x^p f(h)$ $(p \leqslant k)$.

Let

$$\bar{\delta}_x^p f(h_1, \ldots, h_p) = \frac{\partial^p}{\partial t_1 \ldots \partial t_p} f\Big(x + \sum_{i=1}^{p} t_i h_i\Big)\Big|_{t_1 = \ldots = t_p = 0} \quad (p \leqslant k).$$

It is clear that $\bar{\delta}_x^p f$ are symmetric. It suffices to prove that $\bar{\delta}_x^p f$ are multilinear. If $p=1$, this is true by Lemma 9. Let $1 < p \leqslant k$ and fix $h_1, \ldots, h_{p-1}$; the hypothesis implies that the mapping

$$\varrho\colon\ U \ni x \to \bar{\delta}_x^{p-1} f(h_1, \ldots, h_{p-1}) \in F$$

satisfies the condition $r^1(U)$, and so, by Lemma 9, it is of class G^1 in U.

Hence

$$\delta_x \varrho(h_p) = \frac{d}{dt_p} \bar{\delta}^{p-1}_{x+t_p h_p} f(h_1, \ldots, h_{p-1})\Big|_{t_p=0}$$

$$= \frac{d}{dt_p}\left(\frac{\partial^{p-1}}{\partial t_1 \ldots \partial t_{p-1}} f\Big(x + t_p h_p + \sum_{i=1}^{p-1} t_i h_i\Big)\Big|_{t_1 = \ldots = t_{p-1} = 0}\right)\Big|_{t_p=0}$$

$$= \frac{\partial^p}{\partial t_1 \ldots \partial t_p} f\Big(x + \sum_{i=1}^{p} t_i h_i\Big)\Big|_{t_1 = \ldots = t_p = 0} = \bar{\delta}_x^p f(h_1, \ldots, h_p),$$

which shows that $\bar{\delta}_x^p f$ is linear in h_p, and hence (by symmetry) p-linear.

III. ANALYTIC FUNCTIONS IN BANACH SPACES

Recall the definition of an analytic mapping $f\colon U \to F$ from an open subset U of a Banach space E to a Banach space F. Suppose that f is of class C^∞ (we refer the reader to Dieudonné [7]). Recall shortly that under the assumption of the existence of the $(k-1)$-th (Fréchet) derivative $D^{k-1}f(x)$ in a neighbourhood of the point x_0 and the existence of the derivative of the map $x \to D^{k-1}f(x)$ at x_0, $D^k f(x_0)$ is a k-linear, continuous mapping (and symmetric, as can be proved), identified with this derivative by the following canonical isomorphism of Banach spaces:

$$L(E, M^{k-1}(E, F)) \ni \varphi \to \{E^k \ni (h_1, \ldots, h_k) \to \varphi(h_1)(h_2, \ldots, h_k) \in F\} \in M^k(E, F);$$

$M^k(E, F)$ is the Banach space of all k-linear continuous mappings. Thus the (Fréchet) derivatives $D^k f(x_0)$ of f at $x_0 \in U$ are the elements of $L^k(E, F)$.

$D^k f(x_0)(h, \ldots, h)$ is written shortly as $D^k f(x_0)(h)$. It is obvious that the existence of the k-th Fréchet derivative $D^k f(x)$ implies the existence of the k-th Gateaux derivative $\delta_x^k f$, and we have the equality $D^k f(x)(h) = \delta_x^k f(h)$.

If, for every $x_0 \in U$,

$$f(x_0 + h) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0)(h)$$

for $\|h\| \leqslant \varepsilon_{x_0}$ $(\varepsilon_{x_0} > 0)$, then f is called an *analytic mapping*. An analytic mapping in the complex Banach spaces is called *holomorphic*.

LEMMA 10. *Let E and F be complex Banach spaces. If the mapping*

$$g\colon D \ni z \to g^z \in L(E, F),$$

defined on an open subset $D \subset C$, has the property that for each $h \in E$

$$D \ni z \to g^z(h) \in F$$

is holomorphic, then g is holomorphic.

Proof. Fix $z_0 \in D$. There exists an $r > 0$ (independent of $h \in E$) such that for each $h \in E$

$$g^z(h) = \sum_{n=0}^{\infty} a_n(h)(z - z_0)^n \quad \text{if } |z - z_0| < r.$$

Obviously, the mappings $a_n\colon E \to F$ are linear. We shall prove that they are continuous. By the Cauchy integral formula we have

$$a_n(h) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{g^z(h)}{(z-z_0)^{n+1}}\, dz;$$

thus a_n, as the pointwise limit of the sequence of linear continuous mappings (Riemann sums), is continuous (Banach-Steinhaus theorem). The continuity of a_n, the estimations

$$\|r^n a_n(h)\| \leqslant \sup_{|z-z_0|=r} \|g^z(h)\|,$$

and the Banach-Steinhaus theorem imply that for a constant M and all $n \in N$,

$$\|r^n a_n(h)\| \leqslant M\|h\|,$$

i.e. $\|a_n\| \leqslant M/r^n$. Hence the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent in the disc $|z - z_0| < r$ and its sum is equal to g. The mapping g is therefore holomorphic.

THEOREM 4. *Let $f\colon U \to F$ be a locally bounded mapping from an open subset U of a complex Banach space E to a complex Banach space F. If, for every 1-dimensional affine subspace $\lambda \subset E$, the restriction $f|U\cap\lambda$ is holomorphic, then f is holomorphic.*

Proof. It is a simple consequence of the Hartogs theorem that f satisfies the condition $r^k(U)$ for each $k \in N$. Theorem 3 implies that f is of class G^∞. It is easy to see that, at each point $x \in U$, f can be expanded in the convergent formal series

$$(*)\qquad f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\delta_x^k f(h), \qquad \|h\| < \varepsilon_x,$$

where $\varepsilon_x > 0$. It remains to show that f is of class C^∞ (then, necessarily, $D^k f(x)(h) = \delta_x^k f(h)$ and the theorem follows).

Note that the integral Cauchy theorem implies the equality

$$\delta_x^k f(h) = \frac{k!}{2\pi i}\int_{|z|=1} f(x+zh)z^{-k-1}\,dz \quad \text{ if } x+\{z\in C\colon |z|\leqslant 1\}h \subset U.$$

Fix $x_0 \in U$ and choose an $r_0 > 0$ such that $\|f(x)\| \leqslant M$ for $\{\|x-x_0\| \leqslant 2r_0\} \subset U$. In view of the above integral formula, $\|\delta_x^k f(h)\| \leqslant Mk!$ if $\|x-x_0\| \leqslant r_0$ and $\|h\| \leqslant r_0$. These inequalities together with the homogeneous property of $\delta_x^k f$ give the following estimations:

$$(**)\qquad \|\delta_x^k f(h)\| \leqslant Mk!\,r_0^{-k}\|h\| \quad \text{ if } \|x-x_0\| \leqslant r_0.$$

Combining this estimate with $(*)$, we obtain for $\|x-x_0\| \leqslant r_0$ (and $\|h\| \leqslant r_0$)

$$\|f(x+h)-f(x)\| \leqslant \frac{Mr_0^{-1}\|h\|}{1-r_0^{-1}\|h\|}$$

and

$$\|f(x+h)-f(x)-\delta_x f(h)\| \leqslant \frac{Mr_0^{-2}\|h\|}{1-r_0^{-1}\|h\|}.$$

The first inequality gives the continuity of f and — as a consequence — the continuity of Gateaux derivatives $\delta_x^k f$ (Theorem 2). The second one proves that $\delta_x f$ is the Fréchet derivative $Df(x)$ of f at x.

If we can prove that the mapping

$$Df\colon U \ni x \to Df(x) \in L(E, F)$$

is locally bounded and, for each affine line $\lambda \subset E$, $Df|U\cap\lambda$ is holomorphic, the proof of the theorem will be complete. Indeed, this implies by induction that all the $D^k f$ exist and satisfy the assumptions of the theorem, whence (by the first part of the proof) they are continuous.

Now, the local boundedness of Df follows from $(**)$. Next, for every $h \in E$ and the affine line $\lambda \subset E$ fixed, the mapping

$$U\cap\lambda \ni x \to Df(x)(h) \in F$$

is obviously holomorphic, whence, by Lemma 10, $Df|U\cap\lambda$ is holomorphic. This proves the theorem.

COROLLARY 1. *If a sequence of holomorphic mappings $\{f_n\colon U\to F\}_{n\in N}$ is locally uniformly convergent to f, then f is holomorphic.*

Proof. The mapping f is obviously continuous. Applying Theorem 4 and the classical Weierstrass theorem about uniform convergence of the sequence of holomorphic maps (with values in a Banach space) of one complex variable [7], the theorem follows.

COROLLARY 2. *A mapping $f\colon U \to F$ is holomorphic if and only if at any $x\in U$ it can be expanded in a convergent continuous formal series*

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h).$$

Then $D^n f(x) = n!\,\bar{f}_n$.

In particular, every continuous polynomial is a holomorphic mapping. Corollary 1 and Theorem 1 imply (see also [1])

THEOREM 5. *Every analytic mapping $f\colon U\to F$ from an open subset U of a real Banach space E, to a complex Banach space F may be extended to a holomorphic mapping $\hat f$, defined on some open subset $\hat U \subset \hat E$ containing U. Then (at the points $x\in U$) the development of $\hat f$ in the formal series is a complexification of the development of f: $D^n\hat f(x) = \widehat{D^n f(x)}$.*

COROLLARY 2′. *Corollary 2 holds with "holomorphic" replaced by "analytic".*

We say that $f\colon U\to F$ satisfies the *condition* $r^\infty(U)$ (resp. $\omega(U)$) if and only if for each finite-dimensional affine subspace $V\subset E$ (resp. for each 1-dimensional affine subspace $\lambda\subset E$) the mapping $f|U\cap V\to F$ is of class C^∞ (resp. $f|U\cap\lambda\to F$ is analytic).

Let U be an open subset of a real Banach space, and F a real Banach space.

THEOREM 6. *A continuous mapping $f\colon U\to F$ satisfying the conditions $r^\infty(U)$ and $\omega(U)$ is analytic.*

Proof. From Theorems 3 and 2 it follows that f is of class G^∞ and $\delta_x^k f \in P^k(E, F)$ $(k = 1, 2, \ldots)$. The hypothesis implies that at each point $x_0\in U$

$$f(x_0+h) = \sum_{n=0}^{\infty}\frac{1}{n!}\,\delta_{x_0}^n f(h),$$

if h belongs to some λ-neighbourhood of zero in E.

Theorem 1 implies that

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, \delta_x^n f$$

is normally convergent; clearly,

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, \delta_x^n f$$

is convergent to f.

A simple consequence of the proof of Theorem 6 is

THEOREM 6'. *A continuous mapping $f\colon U \to F$ defined in some neighbourhood U of zero, satisfying condition $r^{\infty}(U)$ and analytic on the trace $\lambda \cap U$ of each 1-dimensional vector subspace λ on U is analytic in a neighbourhood of zero.*

We now have the tools to transfer the theorems dealing with analytic functions of one or of several variables (real or complex) to the case of analytic functions in Banach spaces.

Applications of Theorems 4, 5 and 6 will be given in the next section.

IV. APPLICATIONS

A. Weierstrass preparation theorem for analytic functions in Banach spaces. Let K be the field $\boldsymbol{R}$ of real or $\boldsymbol{C}$ complex numbers, I a neighbourhood of zero in K, and U a neighbourhood of zero in Banach space E (over K).

Any mapping of the form:

$$H\colon U \times K \ni (x, z) \to z^k + a_1(x) z^{k-1} + \ldots + a_k(x) \in K,$$

where $a_1, \ldots, a_k$ are analytic functions $U \to K$, and $a_j(0) = 0$, is called a *distinguished polynomial.*

THEOREM. *Let $f\colon U \times I \to K$ be an analytic function such that $f(0) = 0$ and $f|\{0\} \times I \neq 0$. Then there exist an analytic function φ defined in a neighbourhood of $0 \in E \times K$, $\varphi(0) \neq 0$, and a distinguished polynomial H such that $f = H\varphi$ in some neighbourhood of 0. Further, φ and H are uniquely determined by this conditions in a neighbourhood of 0.*

Proof. Uniqueness in the general case follows from uniqueness in the finite-dimensional case.

We shall deduce the theorem from the proof of the classical Weierstrass theorem [4], using Theorems 4 and 5.

Case I ($K = \boldsymbol{C}$). Choose $\varepsilon > 0$, so that f is holomorphic in the set $\{(x, z) \in E \times \boldsymbol{C}\colon \|x\| < \varepsilon, |z| < \varepsilon\}$ and $f(0, z) \neq 0$ if $0 < |z| \leqslant \varepsilon$. If r is the

multiplicity of the root of the function $z \to f(0, z)$ in zero, then for δ sufficiently small $(0 < \delta < \varepsilon)$, $f(x, z) \neq 0$ if $\|x\| < \delta$, $|z| = \varepsilon$, and — for x fixed — the equation $f(x, z) = 0$ has exactly r roots $z_1(x), \ldots, z_r(x)$ (not necessarily distinct) (theorem 9.17.4 of [7]). By the classical residuum theorem we have

$$b_j(x) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} z^j \frac{\dfrac{\partial f}{\partial z}(x, z)}{f(x, z)}\, dz = z_1(x)^j + \ldots + z_r(x)^j.$$

Hence, for each vector line $\lambda \subset E$, b_j is holomorphic in the set $\{x \in \lambda\colon \|x\| < \delta\}$. The continuity of b_j in $A_\delta = \{x \in E\colon \|x\| < \delta\}$ follows from the definition of b_j. Applying Theorem 4 we find that b_j is holomorphic in A_δ.

Set

$$H(x, z) = \big(z - z_1(x)\big) \ldots \big(z - z_r(x)\big) = z^r + a_1(x) z^{r-1} + \ldots + a_r(x)$$

and note that a_r are polynomials in b_j with complex coefficients such that $a_k(0) = 0$.

For $x \in A_\delta$ fixed, the functions $f(x, z)$ and $H(x, z)$ have the same zeros in $\{z \in \boldsymbol{C}\colon |z| \leqslant \varepsilon\}$ and $H(x, z) = 0$ if $z = \varepsilon$. The function

$$\varphi(x, z) = \frac{f(x, z)}{H(x, z)}$$

is well defined, different from zero, and holomorphic in z ($|z| \leqslant \varepsilon$, x fixed).

Hence φ may be expressed by the formula

$$\varphi(x, z) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{f(x, s)}{H(x, s)(s - x)}\, ds,$$

which implies the continuity of φ in $W = A_\delta \times \{z \in \boldsymbol{C}\colon |z| < \varepsilon\}$ and the fact that for each vector line $\lambda \subset E \times \boldsymbol{C}$, $\varphi | W \cap \lambda$ is holomorphic. Thus, by Theorem 4, φ is holomorphic in a neighbourhood of zero.

Case II ($K = \boldsymbol{R}$). Extend the analytic function $f\colon U \times I \to \boldsymbol{R}$ to a holomorphic function $\hat{f}\colon \hat{U} \times \hat{I} \to \boldsymbol{C}$, defined in a neighbourhood of zero in $\hat{E} \times \boldsymbol{C}$ (Theorem 5). Case I implies the existence of the holomorphic function $\hat{\varphi}$ and of the distinguished polynomial $\hat{H}$ defined in a neighbourhood of zero such that $f = \hat{H}\hat{\varphi}$. The uniqueness of the solution of our problem in the case of a finite-dimensional real Banach space implies that the restrictions of $\hat{H}$ and $\hat{\varphi}$ to a sufficiently small neighbourhood of zero in $E \times \boldsymbol{R}$ are real-valued functions. These restrictions, as analytic functions, are the required solution of the problem.

Remark. In the rest of the section, U will be an open subset of a real Banach space E, F an arbitrary real Banach space, and I an open subset of a 1-dimensional Banach space.

B. Generalization of a theorem of Malgrange.

THEOREM. *If U and I (as above) are connected, $f\colon U\times I\to F$ is an analytic mapping ($f\neq$ const), $\varphi\colon U\to I$ is a continuous mapping satisfying condition $r^\infty(U)$ and $f(x,\varphi(x))=0$ in U, then φ is analytic.*

Proof. The case $\dim E<\infty$, $\dim F=1$ was proved by Malgrange (other proofs of this case are due to Łojasiewicz (unpublished) and Siciak [17]).

Case I ($\dim F=1$). Without loss of generality we may assume that U is a convex set. In virtue of Theorem 6 it suffices to prove that, for each affine line $\lambda\subset E$, the mapping $\varphi|U\cap\lambda$ is analytic. For fixed $x_0\in U\cap\lambda$, choose a line $\mu\subset E$ such that $x_0\in\mu$ and $f|(\mu\cap U)\times I\neq$ const (the existence of such a line follows from the hypothesis). If π is the affine plane containing λ and μ, then $\tilde f=f|(\pi\cap U)\times I\neq$ const and $\tilde f(x,(\varphi|\pi\cap U)(x))=0$. According to the remark above, $\varphi|\pi\cap U$ is analytic. Obviously $\varphi|\lambda\cap U$ is also analytic.

Case II (F an arbitrary Banach space). Let a and b be different points belonging to $\operatorname{im} f$, and let $g\colon F\to \boldsymbol{R}$ be a continuous functional such that $g(a)\neq g(b)$. Then $g\circ f\colon U\times I\to\boldsymbol{R}$ is an analytic function, $g\circ f\neq$ const and $g\circ f(x,\varphi(x))=0$. Applying case I we infer that φ is analytic.

The following theorem is a simple consequence of Theorem B, but we give here also another proof, in order to illustrate the method of applying Theorem 6.

C. THEOREM. *If $f\colon I\to F$ is an analytic mapping ($f\neq$ const), $\varphi\colon U\to I$ is continuous and satisfies the condition $r^\infty(U)$, and $f\circ\varphi$ is analytic, then φ is analytic.*

Proof. Case I. First we prove the theorem in the special case, where U and I are neighbourhoods of zero in $\boldsymbol{R}$, $F=\boldsymbol{R}$, $\varphi(0)=f(0)=0$ and φ is analytic except perhaps the point 0. We can exclude the case $f\circ\varphi=0$ as a trivial one (then necessarily $\varphi=0$). In the other case we may assume that $\varphi(x)=x^k\psi(x)$ ($\psi(0)\neq 0$, ψ of class C^∞) and $f(x)=x^n g(x)$ ($g(0)\neq 0$, g analytic). Let $f\circ\varphi=h$. Since $f(\varphi(x))=f(x^k\psi(x))=(x^k\psi(x))^n g(x^k\psi(x))$, we have $h(x)=x^{nk}\tilde h(x)$ for some analytic function $\tilde h$ ($\tilde h(0)\neq 0$), or $(\psi(x))^n g(x^k\psi(x))=\tilde h(x)$. Now we can prove that ψ is analytic at 0. Indeed, the function $\Phi(x,y)=y^n g(x^k y)-\tilde h(x)$ is defined and analytic in some neighbourhood of $(0,\psi(0))\in\boldsymbol{R}^2$. Since

$$\frac{\partial\Phi}{\partial y}(0,\psi(0))=n\psi(0)^{n-1}g(0)\neq 0,$$

by the implicit function theorem ψ, as the solution of the equation $\Phi(x,\psi(x))=0$, is analytic in a neighbourhood of 0.

Case II ($F=\boldsymbol{R}$). For each affine line $\lambda\subset E$, the pairs of mappings f and $\varphi|U\cap\lambda$ satisfy (locally) the assumption of case I, and thus $\varphi|U\cap\lambda$ is analytic; Theorem 6 implies the analyticity of φ.

Case III, of F being an arbitrary Banach space, can be treated similarly to case II of Theorem B.

Remark. Theorems B and C are not true if $\dim I>1$.

D. Generalization of a theorem of Bernstein. A real function of one real variable having the derivatives of all orders non-negative is analytic. This theorem of Bernstein can be generalized to functions on Banach spaces as follows. Let A be a subset of the unit sphere $S\subset E$ such that $A\cup(-A)=S$.

THEOREM. *If $f\colon U\to\boldsymbol{R}$ is a continuous functions satisfying the condition $r^\infty(U)$ and for each point $x\in U$, $\delta_x^n f(a)\geqslant 0$ for $a\in A$ and $n=0,1,2,\ldots$, then f is analytic.*

Proof. The condition $\delta_x^n f(a)\geqslant 0$ ($a\in A$) implies that the derivatives of all orders of the function

$$(*)\qquad t\to f(x+ta)$$

defined in some neighbourhood of zero in $\boldsymbol{R}$ are non-negative. Applying the theorem of Bernstein, we obtain the analyticity of the mapping (*), i.e. the analyticity of $f|U\cap\lambda$ for every affine line $\lambda\subset E$. The theorem follows from Theorem 6.

Remark. A weaker version of the last theorem was proved in [20].

E. Generally we have the following principle:

If it is true that

If $f\colon I\to\boldsymbol{R}$ is the function of class C^∞, satisfying the conditions $W_1,\ldots,W_k$, then f is analytic,

then it is also true that

If $\varphi\colon U\to\boldsymbol{R}$ is a continuous mapping defined on an open subset U of a Banach space E, satisfying the condition $r^\infty(U)$, and for every affine line $\lambda\subset E$, $\varphi|U\cap\lambda$ satisfies the conditions $W_1,\ldots,W_k$, then φ is analytic.

References

[1] A. Alexiewicz and W. Orlicz, *Analytic operations in real Banach spaces*, Studia Math. 14 (1953), p. 57-78.

[2] S. Banach, *Über homogene Polynome in L^2*, ibidem, 7 (1937), p. 36-44.

[3] S. Bernstein, *Sur les fonctions absolument monotones*, Acta Math. 52 (1928), p. 56-66.
[4] S. Bochner and W. Martin, *Functions of several complex variables*, Princeton 1948.
[5] N. Bourbaki, *Variétés différentielles et analytiques (Fascicule de rèsultats)*, Paris 1967.
[5a] — *Fonctions d'une variable réelle*, Paris 1949.
[6] H. Cartan, *Sous-ensembles analytiques d'une variété banachique complexe* (d'après J.-P. Ramis), Séminaire N. Bourbaki, Février 1969.
[7] J. Dieudonné, *Eléments d'analyse*, tome I et II, Paris 1968.
[8] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[9] E. Hille, *Functional analysis and semigroups*, Amer. Math. Soc. Colloquium Publications 1948.
[10] K. Kuratowski, *Topology*, London and New York 1966.
[11] F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957.
[12] B. Malgrange, *Ideals of differentiable functions*, Oxford 1966.
[13] S. Mazur und W. Orlicz, *Grundlegende Eigenschaften ...*, Studia Math. 5 (1935), p. 50-68.
[14] J.-P. Ramis, *Sous-ensembles analytiques d'une variété analytique banachique*, to be published in Springer Verlag (Ergebnisse der Mathematik).
[15] R. Schatten, *A theory of cross-spaces*, Princeton 1950.
[16] J. Siciak, *Asymptotic behaviour ...*, Ann. Pol. Math. 20 (1968), p. 267-278.
[17] — *A characterization of analytic functions of n real variables*, this fasc., p. 293-297.
[18] A. E. Taylor, *Additions to the theory of polynomials in normed linear spaces*, The Tôhoku Math. Journal 44 (1938), p. 302-318.
[19] — *Analysis in complex Banach spaces*, Bull. Amer. Math. Soc., 49 (1943), p. 652-669.
[20] R. Welland, *Bernstein's theorem for Banach spaces*, Proc. Amer. Math. Soc. 19 (1968), p. 789-792.
[21] A. Donady, *Le problème des modules...*, Ann. Inst. Fourier 16, 1(1966), p. 1-98.

Reçu par la Rédaction le 16. 7. 1969

STUDIA MATHEMATICA, T. XXXV. (1970)

A characterization of analytic functions of n real variables

by

J. SICIAK (Kraków)

1. The main purpose of this note is to prove the following

THEOREM 1. *Assume that*

1° $f \in \mathscr{C}^\infty(D)$, *where* D *is a domain in* R^n,

2° *for every* $x \in D$ *there exists an* $r > 0$ *such that for every* $a \in R^n$, $\|a\| = 1$, *the function* $f(x+ta)$ *is analytic with respect to* $t \in (-r, r)$. *The number* r *may depend on* x *and* a.

Then the function f *is analytic in* D.

Example. The function $f(x_1, x_2) = x_1^2 x_2^2 (x_1^2 + x_2^4)^{-1}$, $f(0,0) = 0$, is continuous in R^2, analytic on every line $x = x_0 + ta$, $t \in R$ ($a \in R^2$), but f is not analytic at $(0,0)$ as a function of two real variables. Moreover, given any integer p ($0 < p < +\infty$), one may easily define a function $f \in \mathscr{C}^p(R^2)$ which is analytic on each line but is not analytic in R^2.

As a consequence of Theorem 1 and of the classical Weierstrass preparation theorem we shall get

THEOREM 2. *If* $H(x, y) = H(x_1, \ldots, x_n, y) \not\equiv 0$ *is analytic in a domain* $G \subset R^{n+1}$ *and* $H(x, f(x)) = 0$ *for* $x \in D$, *where* D *is a domain in* R^n *and* $f \in \mathscr{C}^\infty(D)$, *then* f *is analytic in* D.

Theorems 1 and 2 have been proved by Bochnak [1] also for functions in Banach spaces. Another proof of Theorem 2 (and also of a more general theorem) was earlier presented in [3]. Still another proof of Theorem 2, based on the theory of semianalytic sets, was given by S. Łojasiewicz.

2. Theorem 1 will easily follow from the following

LEMMA. *Let*

$$g(x) = \sum_{l=0}^{\infty} P_l(x), \quad x \in R^n, \tag{1}$$

be a series of homogeneous polynomials in n *variables of respective degrees* l. *Put* $S = \{a \in R^n: \|a\| = 1\}$ *and assume that there exists an open subset* Ω *of* S, $\Omega \neq \emptyset$, *such that for every* $a \in \Omega$ *one can find* $\varrho = \varrho_a > 0$ *such that series* (1) *is convergent at* $x = \varrho a$.