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One-dimensional point derivation spaces in Banach algebras

by

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1. Introduction. Let A be a commutative semisimple Banach algebra with identity, and let $M(A)$ be the collection of all multiplicative linear functionals on A. A *point derivation* D at a point $\varphi \in M(A)$ is defined to be a linear functional, not necessarily continuous, such that for all $g, h \in A$,

$$D(gh) = \varphi(g)D(h)+\varphi(h)D(g).$$

The study of point derivations has received increased attention in recent years, in connection with the study of analytic function properties in Banach algebras (see, for instance, [1] and [8]).

Our objective is to study the effect, as motivated by some of the important examples, of having the point derivation space at a point $\varphi \in M(A)$ to be one-dimensional. These examples include the *disk algebra* consisting of all continuous functions on the plane disk $\{z: |z| \leqslant 1\}$ which are analytic at interior points. The point derivation space at each interior point is one-dimensional. Some of the Dirichlet algebras [10] and log-modular algebras [3] are also examples. We shall comment more about these in Section 3. Also included as examples are the algebras in which a principal ideal is a maximal ideal (cf. [2]). A question left open in [2] is answered in Section 2.

A principal result obtained is that if there is a one-dimensional point derivation space at φ and if φ is not isolated in the norm topology of $M(A)$, then a point derivation at φ is bounded, and that "higher order" point derivations exist and are bounded. This leads to identification of a quotient algebra of A with a Banach algebra of power series, in which convergence of elements in the algebra implies convergence of coefficients of the power series.

Prior to the power series development, two sufficient conditions are given for an analytic disk in $M(A)$ to pass through the point φ, at which the point derivation space is one-dimensional. One condition is given in terms of a local maximum modulus principle relative to the norm topology of $M(A)$, and the other in terms of a removable singularity property.

It is further shown that in the event that a two-manifold $\mathcal{M}$ is embedded in $M(A)$ with the norm topology, and the derivation space at each $\varphi \in \mathcal{M}$ is one-dimensional, then $\mathcal{M}$ becomes a Riemann surface on which the Gelfand transform of any element $g \in A$ is analytic.

2. Analytic disk in $M(A)$. Given $\varphi \in M(A)$, let A_φ be the maximal ideal $\varphi^{-1}(0)$. Each $\varphi \in M(A)$ is a continuous linear functional, and thus $M(A) \subset A^*$. As indicated in the introduction, we shall primarily employ the relative norm topology of A^* for $M(A)$, i.e. the metric topology defined by the A^*-norm. For $\varphi_1, \varphi_2 \in M(A)$, we have the formula

$$\|\varphi_1 - \varphi_2\| = \sup_{\|g\| \leqslant 1} |\varphi_1(g) - \varphi_2(g)|.$$

It follows from a result of Singer and Wermer [9], p. 263, that the space of point derivations at φ is one-dimensional if and only if the space A_φ^2, defined to be the ideal generated by products of elements in A_φ, has codimension one in A_φ.

Throughout the discussion that follows, let C denote the field of complex numbers.

THEOREM 2.1. *Suppose the point derivation space at φ is one-dimensional. Then there is a norm neighborhood N of φ such that*

(a) *if $f \in A_\varphi - A_\varphi^2$ and $\|f\| \leqslant 1$, then there exist positive constants K_1 and K_2 such that for all $\varphi_1, \varphi_2 \in N$,*

$$K_1\|\varphi_1 - \varphi_2\| \leqslant |\varphi_1(f) - \varphi_2(f)| \leqslant K_2\|\varphi_1 - \varphi_2\|, \tag{1}$$

and

(b) *if D_ψ is a continuous point derivation at a point $\psi \in N$, then $D_\psi(f) \neq 0$ and $A_\psi = C\cdot\big(f - \psi(f)\big) + \overline{A_\psi^2}$.*

Proof. We extend an argument that Browder [1] used to prove that if there are no point derivations at φ, then φ is norm isolated.

Each $g \in A_\varphi$ can be expressed as a sum $af + h$, where $a \in C$ and $h \in A_\varphi^2$. Let

$$S = \Big\{\lambda f + \sum_{i\in F} \lambda_i g_i h_i:\ g_i, h_i \in A_\varphi,\ \|g_i\| = \|h_i\| = 1, \tag{2}$$
$$\lambda \text{ and } \lambda_i \in C,\ |\lambda| + \sum_{i\in F} |\lambda_i| \leqslant 1, \text{ and } F \text{ is finite}\Big\}.$$

Now $A_\varphi = \bigcup_{M=1}^{\infty} (nS)$, and since A_φ is a complete metric space, at least one of the sets $\overline{nS}$ has non-void interior. Since S is convex and symmetric, $\overline{S}$ must contain a neighborhood of zero, say of radius c.

Let $g \in A$, and $\|g\| \leqslant 1$. Then $g - \varphi(g) \in A_\varphi$ and $(c/2)\big(g - \varphi(g)\big) \in \overline{S}$. Let $h \in S$, say $h = \lambda f + \sum_{i\in F} \lambda_i f_i g_i$ as in (2), and let $\varphi_1, \varphi_2 \in M(A)$. Then

$$\begin{aligned}|\varphi_1(h) - \varphi_2(h)| &\leqslant |\varphi_1(f) - \varphi_2(f)| \\ &\quad + \Big|\sum_{i\in F} \lambda_i \varphi_1(f_i)[\varphi_1(g_i) - \varphi_2(g_i)] + \varphi_2(g_i)[\varphi_1(f_i) - \varphi_2(f_i)]\Big| \\ &\leqslant |\varphi_1(f) - \varphi_2(f)| + \|\varphi_1 - \varphi_2\|\,[\|\varphi_1 - \varphi\| + \|\varphi_2 - \varphi\|].\end{aligned}$$

Approximating $(c/2)\big(g - \varphi(g)\big)$ by elements of S, and taking the sup over $\{g\colon \|g\| \leqslant 1\}$, we obtain

$$(c/2)\|\varphi_1 - \varphi_2\| \leqslant |\varphi_1(f) - \varphi_2(f)| + \|\varphi_1 - \varphi_2\|\,[\|\varphi_1 - \varphi\| + \|\varphi_2 - \varphi\|]. \tag{3}$$

Let $N = \{\psi\colon \psi \in M(A) \text{ and } \|\psi - \varphi\| < c/8\}$. For $\varphi_1, \varphi_2 \in N$,

$$(c/4)\|\varphi_1 - \varphi_2\| \leqslant |\varphi_1(f) - \varphi_2(f)| \leqslant \|f\|\,\|\varphi_1 - \varphi_2\|,$$

so that with $K_1 = c/4$ and $K_2 = \|f\|$, we obtain (1).

Now suppose D_ψ is a bounded point derivation at a point $\psi \in N$. Then

$$\begin{aligned}\|D_\psi\| = \sup\{D_\psi(g)\colon g \in A \text{ and } \|g\| = 1\} &\leqslant 2\sup\{D_\psi(g)\colon g \in A_\varphi \text{ and } \|g\| = 1\} \\ &\leqslant (2/c)\sup\{|D_\psi(g)|\colon g \in S\}.\end{aligned}$$

Let $h \in S$, and let h have the representation $h = \lambda f + \sum_{i\in F} \lambda_i f_i g_i$ as in (2). Then

$$D_\psi(h) = \lambda D_\psi(f) + \sum_{i\in F} \lambda_i \psi(g_i) D_\psi(h_i) + \sum_{i\in F} \lambda_i \psi(h_i) D_\psi(g_i),$$

so that

$$|D_\psi(h)| \leqslant |D_\psi(f)| + 2\|D_\psi\|\,\|\psi - \varphi\|$$

and

$$\|D_\psi\| < (2/c)|D_\psi(f)| + (4/c)\|D_\psi\|\,\|\psi - \varphi\|.$$

Therefore, if $\|\psi - \varphi\| < c/8$, i.e. if $\psi \in N$, then

$$(1/2)\|D_\psi\| \leqslant (2/c)|D_\psi(f)|, \quad \text{and} \quad D_\psi(f) \neq 0.$$

It now follows that the linear span of A_ψ^2 and $f - \psi(f)$ is dense in A_ψ. For otherwise, there exists a bounded linear functional L on A which is zero on this span, and $L(e) = 0$, where e is the unit of A. But then L is a point derivation at ψ since $L|A_\psi^2 = 0$, and by the above argument we would have $L(f) \neq 0$, which is not true. Thus $D_\psi^{-1}(0) = \overline{A\psi^2}$, and $A_\psi = C\cdot\big(f - \psi(f)\big) + \overline{A_\psi^2}$. Q.E.D.

In the following discussion, we shall also use the weak-* topology on $M(A)$. This, of course, is the same as the Gelfand topology introduced on the collection of maximal ideals $\{\psi^{-1}(0)\colon \psi \in M(A)\}$. A fundamental result which we shall employ is the local maximum modulus theorem of Rossi [6], which holds for any commutative semisimple Banach algebra with identity, as well as for the sup-norm algebras considered in [6]. If Γ denotes the Šilov boundary of $M(A)$, and U is any weak-* open subset of $M(A)-\Gamma$, then, for any $g \in A$,

$$\sup_{\psi \in U} |\psi(g)| = \sup_{\psi \in \partial U} |\psi(g)|.$$

Also, following the customary notation, we shall define $\hat{g}\colon M(A) \to C$ by $\hat{g}(\psi) = \psi(g)$.

A subset E of $M(A)$ will be called an *analytic disk* if there is a one-to-one continuous mapping τ from an open disk in the plane onto E (with the norm topology), such that for each $g \in A$, $\hat{g}\circ\tau$ is analytic. It should be noted that if in this definition, E is given the weak-* topology, then τ is also continuous in the norm topology. In fact, the two topologies agree on an analytic disk.

For a subset E of $M(A)$, let $A(E)$ denote the sup-norm closure of the algebra $\{\hat{g}\,|\,E\colon g \in A\}$. Let $M(E)$ be the set of multiplicative linear functionals on $A(E)$, and let $\Gamma(E)$ be the Šilov boundary of $A(E)$.

THEOREM 2.2. *Let N be the norm neighborhood of φ described in Theorem 2.1, and let $E_r = \{\psi\colon \|\psi-\varphi\| \leqslant r\}$. A sufficient condition for φ to lie in an analytic disk is that $\varphi \notin \Gamma(E_r)$ for some $E_r \subset N$.*

Proof. First, we have $M(E_r) = E_r$, as follows essentially from Theorem 6.1 of Rossi's work [6]. Also, the weak-* topology of $M(E_r)$ as a subset of $A(E_r)^*$ is the same as the weak-* topology of E_r as a subset of A^*. Since it is assumed that $x \notin \Gamma(E_r)$, we have by (1) in Theorem 2.1

$$\varrho = \inf\{|f(t)|\colon t \in \Gamma(E_r)\} > 0.$$

By Theorem 3.3.23 of Rickart's treatise [5], the set $f(E_r)$ contains $W = \{z\colon |z| \leqslant \varrho\}$. Let $V = (f\,|\,E_r)^{-1}(W)$ and let $\tau = (f\,|\,V)^{-1}$. Denote by A_τ the algebra $\{\hat{g}\circ\tau\colon g \in A\}$. It follows from the local maximum modulus principle, as applied to the algebra $A(E_r)$, that each $\hat{g}\circ\tau \in A_\tau$ assumes its maximum absolute value on ∂W. Since f is one-to-one on E_r, A_τ contains z (i.e. the function g defined by $g(z) = z$). Rudin [7] has proved that every function in an algebra with these last two properties is analytic at interior points of the disk, and thus the conclusion of the theorem follows. Q.E.D.

For another sufficient condition to have x lie in an analytic disk, we consider a removable singularity problem. Let N be the norm neighborhood of φ described in Theorem 2.1, and let $f \in A_\varphi - A_\varphi^2$, and $\|f\| = 1$. Then f is one-to-one in N and for each $g \in A_\varphi$ there exists an $h \in A_\varphi^2$ and a constant a such that $g = af+h$. Now h has the form $\sum_{i\in F} \lambda_i f_i g_i$, where $f_i, g_i \in A_\varphi$, $\lambda_i \in C$ and F is finite. Therefore

$$|\hat{h}(\psi)| \leqslant \Big(\sum_{i\in F} |\lambda_i|\,\|f_i\|\,\|g_i\|\Big)\|\psi-\varphi\|^2, \qquad \psi \in M(A),$$

so that, for some constant $K > 0$,

$$|\hat{h}(\psi)| \leqslant K\|\psi-\varphi\|^2, \qquad \psi \in M(A). \tag{4}$$

By the inequality on the left-hand side of (1)

$$|\hat{f}(\psi)| \geqslant K_1^{-1}\|\psi-\varphi\|^{-1}, \qquad \psi \in N.$$

It follows that $(\hat{h}/\hat{f})(\psi) \to 0$ as $\|\psi-\varphi\| \to 0$, and hence that $\hat{g}/\hat{f}$ can be regarded as a continuous function on N.

Let $\langle\psi_n\rangle$ be a sequence in N such that $\|\psi_n-\varphi\| \to 0$. We call $\langle\psi_n\rangle$ a *removable singularity sequence*, or simply an RS-sequence, if for each $g \in A_\varphi$ there exists an $h \in A$ such that $(\hat{g}/\hat{f})(\psi_n) = \hat{h}(\psi_n)$ for each positive integer n. The significance of having an RS-sequence in N is that if I denotes the ideal in A of elements whose transforms are zero on $\langle\psi_n\rangle$ then the Banach algebra A/I has the property that the maximal ideal $M = \{g+I\colon \hat{g}(\varphi) = 0\}$ is the same as the principal ideal $(A/I)(f+I)$. The results of [2] are applicable, but in order to use them we need the following lemma which answer a question left open in [2]:

LEMMA 2.1. *Suppose A is a commutative semisimple Banach algebra with identity, and that φ is a multiplicative linear functional on A. If the maximal ideal $A_\varphi = \varphi^{-1}(0)$ is a principal ideal Af, then φ is not in the Šilov boundary Γ of $M(A)$, unless φ is weak-* isolated in $M(A)$.*

Proof. As observed in [2], the operator $T\colon A \to A_\varphi$ defined by $T(g) = gf$ is one-to-one, and has bounded inverse T^{-1}. Let

$$K = \|T^{-1}\| = \sup\{\|g\|\colon \|gf\| \leqslant 1\}. \tag{5}$$

Suppose $\varphi \in \Gamma$. Then there is a weak-* neighborhood U of φ such that $|\hat{f}(\psi)| < 1/2K$ for all $\psi \in U$, since $\hat{f}(\varphi) = 0$. There exists a $g \in A$ such that $g(\psi_0) = 1 = \|\hat{g}\|_\infty$ for some $\psi_0 \in U$ and $|\hat{g}| < 1$ in $M(A)-U$. For some positive integer n, $\|\hat{g}^n\hat{f}\|_\infty < 1/2K$. Let $H = 2g^n$, so that $\|\hat{H}\hat{f}\|_\infty < 1/K$. Let ϱ be chosen so that $\|\hat{H}\hat{f}\|_\infty < \varrho < 1/K$. By the spectral radius formula, $\|\hat{H}\hat{f}\|_\infty = \lim \sqrt[m]{\|H^m f^m\|}$, so that for sufficiently large m, $\|H^m f^m\| < \varrho^m$. By repeated application of (5), $\|H^m\| < \varrho^m K^m$, and $\sqrt[m]{\|H^m\|} < \varrho K < 1$ for large values of m. Since $\|\hat{H}\|_\infty = \lim \sqrt[m]{\|H^m\|}$, $\|\hat{H}\|_\infty \leqslant 1$. But $\|\hat{H}\|_\infty = 2\|\hat{g}^n\| = 2$, which is impossible.

With the aid of this lemma, the main result of [2] now becomes the following:

If the principal ideal Af is the maximal ideal $\varphi^{-1}(0)$, then $m = \min\{|\hat{f}(\psi)|: \psi \in \Gamma\} > 0$, and $\{\psi: |\hat{f}(\psi)| < m\}$ is an analytic disk in $M(A)$, unless φ is weak- isolated in $M(A)$.*

THEOREM 2.3. *Suppose that the point derivation space at φ is one-dimensional and that φ is the limit of an RS-sequence. Then there is an analytic disk passing through φ.*

Proof. We apply the theorem above to the algebra described before the lemma, and obtain that an analytic disk in $M(A/I)$ passes through φ. The space $M(A/I)$ can be identified with the set $hI = \{\psi: \hat{g}(\psi) = 0$ for all $g \in I\}$ by letting $\psi(g+I) = \hat{g}(\psi)$ for all $g \in A$. The correspondence is a weak-* homeomorphism, and thus the analytic disk in $M(A/I)$ also lies in $M(A)$.

3. Banach algebras of power series. We begin this section by considering properties of the ideal A_φ^n, where A_φ^n is defined to be the linear span of all products of n elements in A_φ. In an arbitrary Banach algebra A, the ideals A_φ^n need not be closed. For the situation under consideration, however, we have the following theorem, which shows the existence of "higher order" bounded point derivations at φ:

THEOREM 3.1. *Suppose that the point derivation space at φ is one-dimensional and that φ is not isolated in the norm topology of $M(A)$. Then each A_φ^n is closed and, moreover, $\dim A_\varphi^n / A_\varphi^{n+1} = 1$.*

Proof. Consider the case $n = 1$. Since $A_\varphi = \varphi^{-1}(0)$, A_φ is closed. Let f and N be as in Theorem 2.1. Let D_1 be a linear functional (i.e. point derivation) such that $D_1(f) = 1$ and $D_1(A_\varphi^2) = \{0\}$. Each $g \in A_\varphi$ can be expressed as $g = D_1(g)f + h$, where $h \in A_\varphi^2$. Following the discussion after inequality (4), we have $(\hat{h}/\hat{f})(\psi) \to 0$ as $\|\psi - \varphi\| \to 0$. Therefore

$$D_1(g) = \lim_{\|\psi-\varphi\|\to 0} (\hat{g}/\hat{f})(\psi)$$

and

$$|D_1(g)| \leqslant \frac{\|g\|\,\|\psi-\varphi\|}{K_1\|\psi-\varphi\|} = (1/K_1)\|g\|,$$

proving that D_1 is bounded and its kernel A_φ^2 is closed.

Now suppose that $A_\varphi, A_\varphi^2, \ldots, A_\varphi^n$ are closed, and that $\dim A_\varphi^k/A_\varphi^{k+1} = 1$, $k = 1, 2, \ldots, n-1$. Given $g \in A_\varphi^n$, then, by a simple calculation, g can be expressed in the form $af^n + h$, where $a \in C$ and $h \in A_\varphi^{n+1}$. Moreover, there is a constant $K_h > 0$ such that $|h(\psi)| \leqslant K_h\|\psi - \varphi\|^{n+1}$ for $\psi \in M(A)$. Since $(\hat{g}/\hat{f}^n)(\psi) = a + (\hat{h}/\hat{f}^n)(\psi)$ for $\psi \in N$ and $\psi \neq \varphi$, and since $|\hat{f}^n(\psi)| \geqslant K_1^n\|\psi-\varphi\|^n$ for $\psi \in N$, we have

$$(\hat{g}/\hat{f}^n)(\psi) \to a \quad \text{as } \|\psi - \varphi\| \to 0. \tag{6}$$

Since φ is not norm isolated, a is uniquely determined by g, and we let $D_n(g) = a$, so that $g = D_n(g)f^n + h$. Thus A_φ^{n+1} is the kernel of the non-trivial linear functional D^n on A_φ^n, and hence $\dim A_\varphi^n/A_\varphi^{n+1} = 1$.

We see that A_φ^n is closed, and by an argument analogous to the proof of Theorem 2.1, there is a constant K_n such that if $g \in A_\varphi^n$ and $\|g\| \leqslant 1$ then g can be approximated by elements of the form

$$K_n(\lambda f^n + \sum_{i\in F} \lambda_i f_{i,1} f_{i,2} \ldots f_{i,n+1}),$$

where $|\lambda| + \sum_{i\in F} |\lambda_i| \leqslant 1$, $\|f_{i,k}\| = 1$, $f_{i,k} \in A_\varphi$, and F is finite. It follows from this and (6) that $|D_n(g)| \leqslant K_n$, so that D_n is a bounded linear functional on A_φ^n. Thus $A_\varphi^{n+1} = D_n^{-1}(0)$ is closed. Q.E.D.

As an immediate consequence of Theorem 3.1, we obtain the following result. We think of equation (7) as an abstract Taylor formula.

COROLLARY 3.1. *Suppose the point derivation space at φ is one-dimensional and that φ is not norm isolated in $M(A)$. Let f and N be as in Theorem 2.1. For each $g \in A$ there is a uniquely defined sequence of complex constants $\langle a_n \rangle$ and sequence of functions $\langle h_n \rangle$, where $h_n \in A_\varphi^n$, such that*

$$g = a_0 + a_1 f + \ldots + a_n f^n + h_{n+1}. \tag{7}$$

Moreover, each of the linear mappings $g \to a_n$ is bounded.

Let $K(\zeta)$ be the class of all formal power series in the indeterminant ζ with complex coefficients. The operations on $K(\zeta)$ are the usual addition and multiplication by scalars, with product defined by the Cauchy product formula. By a *Banach algebra of power series* we mean a subalgebra of $K(\zeta)$ which is a Banach algebra under some norm and such that the norm convergence of a sequence in the algebra implies convergence of the coefficients in the power series.

It should be remarked that the definition of a *normed power series ring* given by Lorch [4] calls in addition for the algebra to be generated by ζ and the identity. We omit this requirement for purposes of the following theorem, and comment on the role of this requirement afterwards.

THEOREM 3.2. *Let $A_\varphi^\infty = \bigcap_{n=1}^{\infty} A_\varphi^n$. Suppose the derivation space at φ is one-dimensional and that φ is not norm isolated in $M(A)$. Then A_φ^∞ is a closed ideal, and A/A_φ^∞ is isomorphic to a Banach algebra of power series.*

Proof. Since each A_φ^n is closed by Theorem 3.1, then their intersection A_φ^∞ is also closed, and is certainly an ideal. Let f and N be as in Theorem 2.1. Let $g \in A$ and let $\langle a_n \rangle$ be the sequence of Taylor coefficients, i.e. the sequence occurring in the Taylor formula (7). If $\langle a_n \rangle$ is the zero sequence, then, by Corollary 3.1, $g \in A_\varphi^\infty$. On the other hand, if $g \in A_\varphi^\infty$, then $\langle a_n\rangle$ is the zero sequence. For if not, let a_k be the first nonzero term. Then $g = a_k f^k + h$, where $h \in A_\varphi^{k+1}$, and hence $\hat{h}(\psi)/\hat{f}^k(\psi) \to 0$ as $\|\psi-\varphi\| \to 0$. Therefore for $\psi \in N$ and $\|\psi-\varphi\|$ sufficiently small,

$$|\hat{g}(\psi)| = |\hat{f}^k(\psi)|\,|a_k + \hat{h}(\psi)/\hat{f}^k(\psi)| \geqslant (1/2)|a_k|\,|\hat{f}^k(\psi)| \geqslant \lambda_1\|\psi-\varphi\|^k$$

for some $\lambda_1 > 0$. But $g \in A_\varphi^{k+1}$, and there is a constant $\lambda_2 < \infty$ such that $|\hat{g}(\psi)| \leqslant \lambda_2\|\psi-\varphi\|^{k+1}$ for $\psi \in M(A)$. Therefore, for $\|\psi-\varphi\|$ sufficiently small, $\lambda_2\|\psi-\varphi\|^{k+1} \geqslant \lambda_1\|\psi-\varphi\|^k$, which is impossible since φ is not norm isolated. Thus A_φ^∞ can also be described as the set of all $g \in A$ whose associated Taylor coefficients (relative to a given $f \in A_\varphi - A_\varphi^2$) are all zero.

Consider the quotient algebra A/A_φ^∞. For $g \in A$, let $\tilde{g} = g + A_\varphi^\infty$. If g has Taylor coefficients $\langle a_k\rangle$ (relative to f), then so does $g+h$, where $h \in A_\varphi^\infty$. Let $\alpha\colon A/A_\varphi^\infty \to K(\zeta)$ be defined by $\alpha(\tilde{g}) = \sum_{k=0}^{\infty} a_k\zeta^k$. Then α is a well-defined, one-to-one, linear mapping into $K(\zeta)$. Moreover, as follows from the Taylor formula (7), α preserves multiplication, so that α is an algebra isomorphism into $K(\zeta)$. For $\alpha(\tilde{g}) = \sum_{k=0}^{\infty} a_k\zeta^k$, let us write

$$\Big\|\sum_{k=0}^{\infty} a_k\zeta^k\Big\| = \|\tilde{g}\|.$$

Under this norm, the image of A/A_φ^∞ under α is a Banach algebra. Now suppose $\|\alpha(\tilde{g}_n) - \alpha(\tilde{g})\| \to 0$. Then $\|\tilde{g}_n - \tilde{g}\| \to 0$ and there exists a sequence $\langle h_n\rangle$ in A_φ^∞ such that $\|g_n - g + h_n\| \to 0$. If g_n has Taylor coefficients $\langle a_{n,k}\rangle$ and g has Taylor coefficients $\langle a_k\rangle$, then $g_n - g + h_n$ has Taylor coefficients $\langle a_{n,k} - a_k\rangle$, and by the boundedness assertion in Corollary 3.1, $|a_{n,k} - a_k| \to 0$ as $n \to \infty$. Thus norm convergence of a sequence of power series implies convergence of the coefficients, so that A/A_φ^∞ is isomorphic to a Banach algebra of power series. Q.E.D.

As remarked before Theorem 3.2, Lorch imposed the additional requirement that the algebra of power series be generated by ζ and 1. In our case, this amounts to the requirement that A/A_φ^∞ be generated by $\tilde{f}$ and $\tilde{1}$. The space $M(A/A_\varphi^\infty)$ of multiplicative linear functionals on A/A_φ^∞ corresponds to the set $hA_\varphi^\infty = \{\psi\colon \psi \in M(A)$ and $\hat{g}(\psi) = 0$ for all $g \in A_\varphi^\infty\}$ by letting $\psi\colon A/A_\varphi^\infty \to C$ be defined by $\psi(\tilde{g}) = \psi(g)$ for $g \in A$. Thus the Gelfand transform of $\tilde{g}$ is the restriction of $\hat{g}$ to hA_φ^∞. Since A is semi-simple, then $\{\hat{g}\colon g \in A\}$ separates points in $M(A)$ and the restrictions of these functions to hA_φ^∞ separate points in hA_φ^∞. Since $\tilde{f}$ and $\tilde{1}$ generate A/A_φ^∞, we infer that $\hat{f}$ is one-to-one on hA_φ^∞. Let $X = \hat{f}(hA_\varphi^\infty)$ and let $P(X)$ be the uniform limits of polynomials on X. Then $\{\hat{g}\colon g \in A/A_\varphi^\infty\}$ is a subalgebra (not necessarily sup-norm closed) of $P(X)$. In particular, if zero is an interior point of X, then, in some neighborhood of zero, each function in $P(X)$ is analytic. It follows that an analytic disk exists in $M(A)$ passing through the point φ.

Example. A well-known example from the theory of Dirichlet algebras [10] applies to the situation at hand. Let $X = \{(t, z)\colon t$ is real, z is complex, $t \in [0, 1]$ and $|z| = 1\}$. Let A be the sup-norm algebra on X generated by the coordinate functions t and z. Then $M(A)$ can be identified with the solid cylinder $\{(t, z)\colon t \in [0, 1],\ |z| \leqslant 1\}$. Each $g \in A$, when restricted to a disk $S_{t_0} = \{(t, z)\colon t = t_0$ and $|z| < 1\}$, is analytic in z.

Let φ be the functional $\varphi(g) = g(0, 0)$ for $g \in A$. Then $A_\varphi = \{g\colon g \in A$ and $g(0, 0) = 0\}$. It is not difficult to show that the only point derivation at φ is the usual derivative at the origin, i.e.

$$D(g) = \lim_{z\to 0} z^{-1}[g(0, z) - g(0, 0)],$$

and similarly for the "higher" derivatives.

Then norm topology of $M(A)$ agrees on each disk S_{t_0} with the usual plane topology for a disk. Each disk S_{t_0} is open and closed in the norm topology. The ideal A_φ^∞ is seen to be $\{g\colon g \in A$ and $g\,|\,S_{t_0} = 0\}$. The algebra A/A_φ^∞ is isomorphic and isometric to the *disk algebra* defined in the introduction. Of course, this algebra is isomorphic to a Banach algebra of power series.

If X is a compact Hausdorff space, and A is a sup-norm closed algebra of functions on X (A is assumed to separate points in X and contain the constant functions), it may happen that an element φ of $M(A)$ has exactly one (positive Baire) representing measure on X. This happens in the case of Dirichlet algebras [10] and logmodular algebras [3]. An extensive theory is developed for such algebras, and it is known, for instance, that if $P_\varphi = \{\psi\colon \psi \in M(A)$ and $\|\psi-\varphi\| < 2\}$ contains more than the point φ, then P_φ is an analytic disk in $M(A)$. In addition, as shown by Sidney [8], $\dim (A_\varphi^n)^-/(A_\varphi^{n+1})^- = 1$. For $n = 1$ this says that there exists exactly one bounded point derivation at φ. Combining this result with Theorem 3.2, we obtain the following corollary:

Corollary 3.2. *Let A be a sup-norm algebra on the compact Hausdorff space X. Suppose that $\varphi \in M(A)$, that the representing measure for φ on X is unique, and that P_φ contains more than one point. If there are no unbounded point derivations at φ, then each A_φ^n is closed and $\dim A_\varphi^n/A_\varphi^{n+1} = 1$.*

4. Riemann surfaces in $M(A)$.

In the previous sections, information on the point derivation space at a point $\varphi \in M(A)$ was used to gain information on the behavior of the transforms $\hat{g}$ of elements $g \in A$ in a norm neighborhood of φ. Another type of problem is to consider the point derivation spaces at all points in some subset of $M(A)$. The next result is a theorem of this type.

THEOREM 4.1. *Suppose $\mathcal{M}$ is a 2-manifold embedded in $M(A)$ with the norm topology and that the point derivation space at each point in $\mathcal{M}$ is one-dimensional. Then there is an analytic structure in $\mathcal{M}$ making $\mathcal{M}$ into a Riemann surface. Moreover, for each $g\in A$, the restriction of $\hat{g}$ to $\mathcal{M}$ is analytic.*

Proof. Let $\varphi\in\mathcal{M}$ and let $f\in A_\varphi-A_\varphi^2$. By Theorem 2.1, there is a norm neighborhood N of φ in which $\hat{f}$ is one-to-one. We may assume $N\subset\mathcal{M}$. It follows from standard arguments (employing the Brouwer invariance of domain theorem) that the restriction of $\hat{f}$ to N is a homeomorphism and that $\hat{f}(N)$ is open in the plane.

Also by Theorem 2.1, we know that if D_ψ is a bounded point derivation at $\psi\in N$, then $D_\psi(f)\neq 0$, and that $A_\psi=C[f-\hat{f}(\psi)]+\overline{A_\psi^2}$. Since we are assuming that the point derivation space at each $\psi\in\mathcal{M}$ is one-dimensional, we infer by Theorem 3.1 that A_ψ^2 is closed, so that $A_\psi=C(f-\hat{f}(\psi))+A_\psi^2$.

Let $\tau=\hat{f}\,|N$ and let $g\in A$ be given. We show that $\hat{g}\circ\tau^{-1}$ is an analytic function on the plane open set $\tau(N)$. Let $\tau(\psi)=z$ and $\tau(\psi_0)=z_0$ for $\psi,\psi_0\in N$. By the above argument, there is a function $h\in A_{\psi_0}^2$ and a constant $a\in C$ such that $\hat{g}-\hat{g}(\psi_0)=a[\hat{f}-\hat{f}(\psi_0)]+\hat{h}$. Thus $\hat{g}\circ\tau^{-1}(z)-\hat{g}\circ\tau^{-1}(z_0)=a(z-z_0)+\hat{h}\circ\tau^{-1}(z)$, and

$$\left|\frac{\hat{g}\circ\tau^{-1}(z)-\hat{g}\circ\tau^{-1}(z_0)}{z-z_0}-a\right|=\left|\frac{\hat{h}\circ\tau^{-1}(z)}{z-z_0}\right|=\left|\frac{\hat{h}(\psi)}{\hat{f}(\psi)-\hat{f}(\psi_0)}\right|.$$

Now since τ is a homeomorphism, $\tau(\psi)\to\tau(\psi_0)$ as $z\to z_0$. By the argument following inequality (4),

$$\left|\frac{\hat{h}(\psi)}{\hat{f}(\psi)-\hat{f}(\psi_0)}\right|\to 0\quad \text{as } \|\psi-\psi_0\|\to 0,$$

and we conclude that

$$\frac{d}{dz}(\hat{g}\circ\tau^{-1})(z_0)=a.$$

Let us now denote the dependence of τ and N on φ by writing τ_φ instead of τ and N_φ instead of N. We observe that the collection of ordered pairs $\{(\tau_\varphi,N_\varphi)\colon \varphi\in\mathcal{M}\}$ constitutes an analytic structure on $\mathcal{M}$. For if $N_\varphi\cap N_\psi$ is non-empty, then $\hat{f}_\varphi|N_\psi$ has the property that $\hat{f}_\varphi\circ\tau_\psi^{-1}$ is analytic on $\tau_\psi(N_\psi)$. In particular, $\hat{f}_\varphi\circ\tau_\psi^{-1}$ is analytic on $\tau_\psi(N_\varphi\cap N_\psi)$, or $\tau_\varphi\circ\tau_\psi^{-1}$ is analytic on $\tau_\psi(N_\varphi\cap N_\psi)$. Thus $\mathcal{M}$ becomes a Riemann surface when given this structure.

Moreover, we have already seen that for each $g\in A$, the restriction of $\hat{g}$ to $\mathcal{M}$ is analytic, since each $\hat{g}\circ\tau_\varphi^{-1}$ is analytic on $\tau_\varphi(N_\varphi)$. Q.E.D.

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