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Mergelyan's theorem for vector-valued functions with an application to slice algebras

by

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In the first part of this note we state and prove an analogue of the Mergelyan Theorem for functions with values in a locally convex space. Although this extension has a proof that lies surprisingly near the proof of Mergelyan's Theorem (for example as presented in Rudin's book on real and complex analysis [4]) it appears that the vector-valued version of the theorem is not in the literature.

The second part contains an application of the extended Mergelyan's Theorem to a problem from the theory of function algebras. We refer to that part for details.

I. Mergelyan's Theorem for functions with values in a l. c. space.

MERGELYAN'S THEOREM. *Let X be a compact set in the complex plane whose complement is connected. If f is a continuous complex function on X which is holomorphic in the interior of X and if $\varepsilon > 0$, then there exists a polynomial P such that*

$$|f(z)-P(z)| < \varepsilon, \quad \forall z \in X.$$

We want to extend this theorem to the case where f maps X into a locally convex space B.

EXTENDED MERGELYAN'S THEOREM. *Let X be as above and f a continuous function on X with values in a locally convex space B, which is holomorphic in the interior of X. Then, if p is a continuous semi-norm on B and if $\varepsilon > 0$, there exists a polynomial $P\colon X \to B$ such that*

$$p\big(f(z)-P(z)\big) < \varepsilon, \quad \forall z \in X.$$

(By a *polynomial* $P\colon X \to B$ we mean a function of the form $P(z) = \sum_{i=1}^{k} z^i b_i$, $z \in X$, $b_i \in B$, $i = 1, \ldots, k$.)

Proof. The proof follows closely the proof of Mergelyan's Theorem given in [4], p. 386.

By $A(X, B)$ we denote the set of continuous functions from X into B, holomorphic in the interior of X.

Let $f \in A(X, B)$. We extend f to a continuous function in the plane with compact support, by first extending f to a continuous function from $C \to B$ (see [2]) and then multiplying this extension by a C_0^∞-function identically equal to 1 on X. We will also denote this extension by f. For any continuous seminorm p and any $\delta > 0$ put

$$\omega_p(\delta) = \sup\{p(f(z_1)-f(z_2)) \mid |z_1-z_2| < \delta\}.$$

Thus

$$\lim_{\delta\to 0} \omega_p(\delta) = 0.$$

Let δ be fixed. We shall prove that there exists an open subset Ω of C containing X and a function $F \in H(\Omega, B)$ (the space of holomorphic functions from Ω into B) such that

$$p(f(z)-F(z)) < K_0 \cdot \omega_p(\delta)$$

for all $z \in X$, where K_0 is a positive constant independent of δ.

We first construct a function $\Phi \in C_c'(R^2, B)$ (continuously differentiable functions with compact support from R^2 into B) with the following properties:

(1) $$p(f(z)-\Phi(z)) \leqslant \omega_p(\delta),$$

(2) $$p(\bar\partial\Phi(z)) < \frac{2\omega_p(\delta)}{\delta} \qquad \left(\bar\partial = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right),$$

(3) $$\Phi(z) = -\frac{1}{\pi}\iint_K \frac{\bar\partial\Phi(\zeta)}{\zeta - z}\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta) \text{ for every } z \in C,$$

$$K = \{\zeta \in \operatorname{supp}\Phi \mid \operatorname{dist}(\zeta, \complement X) \leqslant \delta\}.$$

We construct Φ as the convolution of f with a smoothing function A in the following manner. Put

$$a(r) = \begin{cases} \dfrac{3}{\pi\delta^2}\left(1-\dfrac{r^2}{\delta^2}\right)^2 & (0 \leqslant r \leqslant \delta), \\ 0 & (r > \delta), \end{cases}$$

and define

$$A(z) = a(|z|), \qquad \forall z \in C.$$

Then $A \in C_c'(R^2)$ and

(4) $$\iint_{R^2} A = 1,$$

(5) $$\iint_{R^2} \bar\partial A = 0,$$

(6) $$\iint_{R^2} |\bar\partial A| < \frac{2}{\delta}.$$

(For details see [4], p. 387.)

We now define

(7) $$\Phi(z) = \iint_{R^2} f(z-\zeta)A(\zeta)\, d\xi\, d\eta = \iint_{R^2} A(z-\zeta)f(\zeta)\, d\xi\, d\eta.$$

f is a continuous vector-valued function and A is a continuous scalar function with compact support, so the integrals are well-defined and $\Phi(z) \in B$, $\forall z \in C$. Also Φ has compact support. Now

$$\Phi(z) - f(z) = \iint_{R^2} (f(z-\zeta)-f(z))A(\zeta)\, d\xi\, d\eta$$

and $A(\zeta) = 0$ for $|\zeta| > \delta$. Thus (1) follows from (4).

Now, since $A \in C_c'(R^2)$, the difference quotients of A converge boundedly to the corresponding partial derivatives; also f is uniformly bounded, so the last expression in (7) may be differentiated under the integral sign to get

(8) $$\begin{aligned} \bar\partial\Phi(z) &= \iint_{R^2} \bar\partial A(z-\zeta)f(\zeta)\, d\xi\, d\eta \\ &= \iint_{R^2} f(z-\zeta)(\bar\partial A)(\zeta)\, d\xi\, d\eta \\ &= \iint_{R^2} [f(z-\zeta)-f(z)](\bar\partial A)(\zeta)\, d\xi\, d\eta \quad \text{(because of (5))}, \end{aligned}$$

so (6) and (8) give (2).

By writing (8) with Φ_x and Φ_y in place of $\bar\partial\Phi$ we see that Φ has continuous partial derivatives.

Thus for any $b^* \in B^*$ (the dual space of B)

$$b^*\Phi(z) = -\frac{1}{\pi}\iint_{R^2} \frac{b^*(\bar\partial\Phi)(z)}{\zeta - z}\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta).$$

([4], Lemma (20.3) applied to $b^*\Phi$).

Thus

$$\Phi(z) = -\frac{1}{\pi}\iint_{R^2} \frac{\bar\partial\Phi(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

and (3) will follow if we can show that $\bar\partial\Phi = 0$ in $G = \{z \in X \mid \operatorname{dist}(z, \complement X) > \delta\}$. We will do this by showing that

$$\Phi(z) = f(z) \qquad (z \in G).$$

($\bar\partial f = 0$ in G since f is holomorphic in G.)

If $z \in G$, then $z - \zeta$ is in the interior of X, $\forall \zeta$ with $|\zeta| < \delta$ so

$$\int_0^{2\pi} f(z - re^{i\theta})\, d\theta = 2\pi f(z) \qquad \text{for } r < \delta.$$

Thus by the first equation in (7):

$$\Phi(z) = \int_0^\delta a(r)r\,dr\int_0^{2\pi} f(z-re^{i\theta})\,d\theta$$
$$= 2\pi f(z)\int_0^\delta a(r)r\,dr = f(z)\iint_{R^2} A = f(z).$$

We have now proved (1), (2), and (3).

The definition of K shows that K is compact and that K can be covered by finitely many open discs $D_1, \ldots, D_n$ of radius 2δ, whose centers are not in X.

Since $C\setminus X$ is connected, the center of each D_j can be "joined to ∞" by a polygonal path in $C\setminus X$. It follows that each D_j contains a compact connected set E_j of diameter at least 2δ so that $C\setminus E_j$ is connected and $X\cap E_j = \emptyset$. (Take, for example, E_j = the intersection of the above mentioned path with $\bar{D}_j$.)

We now apply [4] (Lemma 20.2) with $r = 2\delta$. Thus there exist $g_j\in H(C\setminus E_j)$ (complex holomorphic functions on $C\setminus E_j$) and constants c_j so

(9) $$|Q_j(\zeta, z)| < \frac{50}{\delta}.$$

(10) $$\left|Q_j(\zeta, z) - \frac{1}{z-\zeta}\right| < \frac{4000\,\delta^2}{|z-\zeta|^3}$$

holds for $z\notin E_j$ and $\zeta\in D_j$ if

(11) $$Q_j(\zeta, z) = g_j(z) + (\zeta - c_j)g_j^2(z).$$

Put $\Omega = \mathbb{C}(E_1\cup\ldots\cup E_n)$; then Ω is an open set which contains X. Put $K_1 = K\cap D_1$ and $K_j = (K\cap D_j) - (K_1\cup\ldots\cup K_{j-1})$ for $2\leqslant j\leqslant n$.

Define

$$R(\zeta, z) = Q_j(\zeta, z) \qquad (\zeta\in K_j,\ z\in\Omega),$$

and

(12) $$F(z) = \frac{1}{\pi}\iint_K \bar\partial\Phi(\zeta)R(\zeta, z)\,d\xi\,d\eta \qquad (z\in\Omega).$$

Since

$$F(z) = \sum_{j=1}^n \frac{1}{\pi}\iint_{K_j}\bar\partial\Phi(\zeta)Q_j(\zeta, z)\,d\xi\,d\eta,$$

(11) shows that F is of the form $\sum_{j=1}^n a_j(z)b_j$, where $a_j\in H(\Omega)$ and $b_j\in B$.

By (12), (2), and (3)

$$p\big(F(z)-\Phi(z)\big) < \frac{2\omega_p(\delta)}{\pi\delta}\iint_K\left|R(\zeta, z) - \frac{1}{z-\zeta}\right|d\xi\,d\eta \qquad (z\in\Omega).$$

By the proof of Mergelyan's Theorem ([4], p. 389)

$$p\big(F(z)-\Phi(z)\big) < K_0\,\omega_p(\delta).$$

Now we only have to use Runge's theorem on the a_j's to get the statement of the theorem.

COROLLARY 1 (Runge's theorem for analytic vector-valued functions). *Suppose $G\subset C$ is an open set which does not separate the plane and does not contain ∞ and suppose E is a Banach space. Suppose further that $f\colon G\to E$ is analytic. Then, on every compact subset of G, f can be uniformly approximated by polynomials.*

II. Slice algebras. Let X and Y be compact Hausdorff spaces and $B\subseteq C(X)$, $C\subseteq C(Y)$ be sup norm algebras. Let $B\otimes_\lambda C$ be the uniform closure in $C(X\times Y)$ of the algebraic tensor product $B\otimes C$ by means of the usual identification

$$f\otimes g \leftrightarrow f(\cdot)g(\cdot)$$

and let $S(B, C)\subset C(X\times Y)$ be the slice algebra over B and C, [1], i.e.

$$S(B, C) = \{f\in C(X\times Y)\mid \forall x\in X, f(x,\cdot)\in B, \forall y\in Y, f(\cdot, y)\in A\}.$$

It is clear that in general we have

$$B\otimes_\lambda C\subseteq S(B, C).$$

It is not known whether $B\otimes_\lambda C = S(B, C)$, but we can give a proof of this equality if one of the algebras B or C is singly generated. This result was first obtained by Eifler [3].

PROPOSITION. *Suppose B and C are sup norm algebras and suppose B is singly generated. Then $S(B, C) = B\otimes_\lambda C$.*

Proof. We may suppose that X is a compact subset of the complex plane with connected complement and that $B = P(X)$, the continuous function on X uniformly approximable by polynomials on X.

Now $P(X)\otimes_\lambda C = P(X, C)$, the set of continuous functions from X into C uniformly approximable by polynomials in X with coefficients from C.

According to the extended Mergelyan's Theorem, $P(X, C) = A(X, C)$, the set of continuous functions from X into C analytic in the interior of X.

Thus we only have to prove that

$$S\big(P(X), C\big) = A(X, C).$$

Since $A(X, C) = P(X)\otimes_\lambda C$, $A(X, C)$ is trivially contained in $S\big(P(X), C\big)$.

Let $f(x, y) \in S(P(X), C)$. Then

$$x \to f(x, \cdot)$$

is a continuous map from X into C.

We want to prove that it is analytic at interior points of X. We will do that by proving that $c^*(f(x, \cdot))$ is analytic at interior points of X for $c^* \in C^*$.

Let x_0 be an interior point of X, and $c^* \in C^*$. We shall prove that

$$\int_\gamma c^* f(x, \cdot)\, dx = 0,$$

where γ is any circle around x_0 contained in the interior of X.

Now finite linear combinations of elements of the form ε_{y_i} (evaluation at y_i) are weak* dense in C^*, i.e.

$$\int_\gamma c^* f(x, \cdot)\, dx = c^* \int_\gamma f(x, \cdot)\, dx$$

can be approximated by

$$\sum_{i=1}^n a_i \varepsilon_{y_i} \int_\gamma f(x, \cdot)\, dx,$$

but the last expression is equal to

$$\sum_{i=1}^n a_i \int_\gamma f(x, y_i)\, dx$$

and this expression is 0 by the assumption on f.

References

[1] F. Birtel, *Products of maximal function algebras*, Proc. Int. Symp. Function Alg., Tulane 1965.

[2] J. Dugundji, *An extension of Tietze's theorem*, Pac. J. Math. 1 (1951), p. 353-367.

[3] L. Eifler, *The slice product of function algebras* (to appear).

[4] W. Rudin, *Real and complex analysis*, New York 1966.

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Sur la théorie semi-classique du potentiel pour les processus à accroissements indépendants

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0. Introduction. Soit A un ensemble compact, $A \subset R^d$ avec une frontière assez régulière, $X = \{x_t, P^x\}$ un mouvement brownien dans R^d (pour simplifier les notations nous supposons $d \geqslant 3$). Posons

$$\tau_0 = \inf\{t\colon \int_0^t I_A(x_s)\, ds > 0\}.$$

En 1951 Kac [9] a prouvé que

$$(0.1) \qquad B_1^A(x) = P^x(\tau_0 < +\infty)$$
$$= \lim_{t \downarrow 0} \sum_{j=1}^{+\infty} \frac{1}{\lambda_j} e^{-t/\lambda_j} \int_A G(x, y) \varphi_j(y)\, dy \cdot \int_A \varphi_j(y)\, dy,$$

où B_1^A est le potentiel capacitaire de A,

$$G(x, y) = \frac{\Gamma(d/2-1)}{2(\pi)^{d/2}} \cdot \frac{1}{|x-y|^{d-2}}$$

et $(\lambda_j, \varphi_j)_{j \geqslant 1}$ est le système des valeurs et des fonctions propres de la transformation G_A:

$$G_A f(x) = \int_A G(x, y) f(y)\, dy, \qquad x \in A,\ f \in L^2(A).$$

Ciesielski [2] a montré ensuite, que la formule (0.1) est vraie pour tout ensemble compact A, à condition qu'on y remplace le potentiel B_1^A par

$$S_1^A(x) = \inf\{v(x)\colon v \geqslant 1 \text{ p.p. sur } A,\ v \text{ surharmonique},\ v \geqslant 0\}$$

(p.p. signifie sauf sur un ensemble de mesure de Lebesgue nulle). En introduisant le potentiel S_1^A on peut développer une théorie du potentiel analogue à la théorie classique: la théorie semi-classique (voir [2], [3] et [4]). Ces résultats ont été étendus par Stroock [14]. Il a considéré les processus de diffusion (symétriques) et les équations qu'il a obtenues