STUDIA MATHEMATICA, T. XXXV. (1970)

On symmetry of group algebras of discrete nilpotent groups

by

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A Banach $*$-algebra $\mathscr{A}$ is symmetric if for every element x in $\mathscr{A}$ the spectrum of the element x^*x is real, non-negative. The aim of this paper is to prove that the l_1-group algebra of a discrete nilpotent group is symmetric.

It seems that the method employed can be used to prove symmetry of $l_1(G)$ for a slightly wider class of groups such as e.g. finite extensions of nilpotent groups in a similar way as in [2] we proved this for finite extensions of FC groups, but we shall not do it here. It should be mentioned also that solvable groups need not have symmetric l_1-group algebras, as it was recently proved by Jenkins [4], [5]; cf. also [3].

The author wants to express his gratitude to Professor B.H. Neumann for a very helpful information concerning the relation between free nilpotent groups and free nilpotent rings.

Let G be a discrete group. The l_1-group algebra of G with the usual norm, multiplication and involution

$$\|x\| = \sum_{s\in G} |x(s)|, \qquad xy(s) = \sum_{t\in G} x(t)y(t^{-1}s), \qquad x^*(s) = \overline{x(s^{-1})},$$

respectively, acts in a natural way on $l_2(G)$. To each $x\in l_1(G)$ we associate the operator

$$L_x\colon\ l_2(G)\ni y \to xy \in l_2(G).$$

The correspondence

$$x \to L_x$$

is a $*$-representation of l_1 into the algebra of bounded operators in $l_2(G)$, The natural question, which are the functions x on G for which the convolution xy is defined for every y in $l_2(G)$ and the operator L_x is bounded, was raised long ago. It is easy to verify that x must be in $l_2(G)$ but not all functions in l_2 have this property. If $x\in l_1(G)$, then

$$L_{(e+x^*x)} = I + L_x^* L_x$$

and, consequently, $L^{-1}_{(e+x^*x)}$ exists. Since $L^{-1}_{(e+x^*x)}$ commutes with all R_y, $y \in l_1(G)$, where

$$R_y: \ l_2(G) \ni z \to zy \in l_2(G),$$

we have

$$L^{-1}_{(e+x^*x)} = L_w$$

for a function w (in $l_2(G)$, of course). Thus

$$z(e+x^*x) = (e+x^*x)z = e$$

and we write

$$z = (e+x^*x)^{-1}.$$

It follows immediately from the definition that a Banach $*$-algebra is symmetric if and only if for every element x of it $(e+x^*x)^{-1}$ exists (in the algebra). This for the l_1-group algebra gives a version of the Wiener theorem:

If $l_1(G)$ is symmetric, then for every x in $l_1(G)$ the function $(e+x^*x)^{-1}$ is in $l_1(G)$.

Let

$$\lambda(x) = \|L_x\|.$$

Then, since $\|L_x\| \leqslant \|x\|$, we have

(1) $$\lambda(x) = \lambda(x^n)^{1/n} \leqslant \lim_{n\to\infty} \|x^n\|^{1/n} = \nu(x)$$

for all $x = x^*$ in $l_1(G)$. Suppose the equality

(2) $$\lambda(x) = \nu(x) \quad \text{for all } x = x^* \text{ in } l_1(G)$$

holds. Then, for every x in $l_1(G)$, the completion $\mathscr{A}^\lambda$ of the algebra $\mathscr{A}$ generated by e and x^*x in the norm λ is equal to the completion $\mathscr{A}^\nu$ of $\mathscr{A}$ in the norm ν. Therefore (2) implies that $(e+x^*x)^{-1} \in \mathscr{A}^\nu$, which, by the Gelfand theorem, results in $(e+x^*x)^{-1} \in \mathscr{A}^{\|\cdot\|} \subset l_1(G)$ and, consequently, $l_1(G)$ is symmetric.

Our first goal is to find a sufficient condition on the group G which implies (2) and, consequently, the symmetry of $l_1(G)$. Then we show that nilpotent groups satisfy this condition.

1. The condition. Let G be a discrete group. For a function x on G let $\operatorname{supp} x = \{s: \ x(s) \neq 0\}$. Let $A_1, \ldots, A_n$ be a family of finite subsets of G. By $A_1 \ldots A_n$ we denote the set of all the products $a_1 \ldots a_n$, where $a_i \in A_i$, $i = 1, \ldots, n$. We shall also use the abbreviated notation A^n for $A \ldots A$. For a finite subset A of G we denote by $|A|$ the number of the elements in A. Clearly

$$|A_1 \ldots A_n| \leqslant |A_1| \ldots |A_n|.$$

By a probability distribution on G we mean a function p on G such that

$$p(s) \geqslant 0 \text{ for all } s \text{ in } G \text{ and } \sum_{s\in G} p(s) = 1.$$

We write

$$G^\infty = \{(g_1, g_2, \ldots): \ g_i \in G\}.$$

For a finite set A in G and a sequence $\boldsymbol{t} = (t_1, t_2, \ldots)$ in G^∞ we write

$$f_n(A, \boldsymbol{t}) = |t_1 A \ldots t_n A|.$$

Condition (C). *Let P be a finite family of probability distributions on G and let $\boldsymbol{P}$ be the family of the (Borel) measures on G^∞ each of which is the direct product measure of a sequence $p_1, p_2, \ldots, p_j \in P$. A group is said to satisfy condition* (C) *if for every finite family P, a finite set A in G and number $c > 1$, there is an n_0 such that*

$$\int f_n(A, \boldsymbol{t})\, d\boldsymbol{p}(\boldsymbol{t}) < c^n$$

for all $n > n_0$ and $\boldsymbol{p} \in \boldsymbol{P}$.

Remark. Condition (C) as formulated above is a refinement of a condition considered in [2] and, again, it imposes a restriction on the increase of

$$f_n(A, \boldsymbol{t}) = |At_1 \ldots At_n|$$

as n tends to infinity. A stronger condition

$$\sup_{\boldsymbol{t}} f_n(A, \boldsymbol{t}) = o(c^n) \quad \text{as } n \to \infty \text{ for } c > 1$$

was proved to be satisfied by FC-groups but is not satisfied by nilpotent groups (even of class 2 and two generators). We shall prove that “the average” condition (C) is satisfied by nilpotent groups and, on the other hand, it implies, via (2), the symmetry of $l_1(G)$.

Theorem 1. *If a group G satisfies condition* (C), *then for every hermitian element x in $l_1(G)$ we have*

$$\lambda(x) = \nu(x).$$

Consequently, $l_1(G)$ is symmetric.

First we prove the following

Lemma. *Suppose G satisfies* (C). *Then for every probability distribution p on G, any $c > 1$ and a finite subset A of G there exists an n_0 such that if $n > n_0$ and $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ are non-negative integers with $u_1 + \ldots + u_r = m$ and $v_1 + \ldots + v_r = n - m$, we have*

(3) $$\sum |b_1 A^{v_1} \ldots b_r A^{v_r}| p(t_1) \ldots p(t_m) < c^n,$$

where the summation is all over the set of the sequences $t_1, \ldots, t_m$, $t_j \epsilon G$ *and* $b_j = t_{u_{j-1}+1} \ldots t_{u_j}$, $j = 1, \ldots, r$, $u_0 = 0, \ldots, u_j = u_1 + \ldots + u_j$.

Proof. Since the left-hand side of (3) increases when A is replaced by a superset of A, we may assume that the unit element e of G is in A. Then inserting A between the t's in all the b_j, we have

$$t_1 A \ldots A t_{u_1} A^{v_1} \ldots t_{u_{j-1}+1} A \ldots A t_m A^{v_r} \supset b_1 A^{v_1} \ldots b_r A^{v_r},$$

whence

$$\sum |b_1 A^{v_1} \ldots b_r A^{v_r}| p(t_1) \ldots p(t_m) \leqslant \int f_n(A, \boldsymbol{t}),\ d\boldsymbol{p}(\boldsymbol{t}),$$

where $P = \{p, \delta_e\}$ and $\boldsymbol{p}$ is the direct product measure of a sequence whose terms are p and δ_e in a suitable order.

Proof of Theorem 1. Let z be a hermitian element in $l_1(G)$. In virtue of (1) it is sufficient to prove

$$\nu(z) \leqslant \lambda(z). \tag{4}$$

For a positive number ε we write

$$z = x + y,$$

where $\operatorname{supp} x = A$ is finite and $\|y\| < \varepsilon$. We then have

$$z^n = (y+x)^n = \sum_{m=0}^{n} \sum y^{u_1} x^{v_1} \ldots y^{u_r} x^{v_r},$$

where the summation extends over all sequences $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ of non-negative integers such that $u_1 + \ldots + u_r = m$ and $v_1 + \ldots + v_r = n - m$. Hence

$$\|z^n\| \leqslant \sum_{m=0}^{n} \sum \|y^{u_1} x^{v_1} \ldots y^{u_r} x^{v_r}\|. \tag{5}$$

Now we fix $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ and we note that

$$y^{u_1} x^{v_1} \ldots y^{u_r} x^{v_r} = \sum_{\alpha, \beta} y(t_1) \ldots y(t_m) x(s_1) \ldots x(s_{n-m}) \delta_e(a_r^{-1} b_r^{-1} \ldots a_1^{-1} b_1^{-1} s),$$

where the summation is all over the set of the sequences

$$\alpha = (t_1, \ldots, t_m), \quad t_j \epsilon G,\ j = 1, \ldots, m,$$

$$\beta = (s_1, \ldots, s_{n-m}), \quad s_j \epsilon G,\ j = 1, \ldots, n-m,$$

$$a_j = s_{v_{j-1}+1} \ldots s_{v_j}, \quad v_0 = 0,\ v_j = v_1 + \ldots + v_j,\ j = 1, \ldots, r,$$
$$b_j = t_{u_{j-1}+1} \ldots t_{u_j}, \quad u_0 = 0,\ u_j = u_1 + \ldots + u_j,\ j = 1, \ldots, r.$$

For every j ($j = 1, \ldots, r$) we have

$$a_j^{-1} b_j^{-1} = s_{v_j}^{-1} \ldots s_{v_{j-1}+2}^{-1} (s_{v_{j-1}+1}^{-1} b_j^{-1}),$$

whence

$$\|y^{u_1} x^{v_1} \ldots y^{u_r} x^{v_r}\| \leqslant \sum_{s \epsilon G} \sum_{\alpha} |y(t_1)| \ldots |y(t_m)|,$$

$$\Big| \sum_{\beta} x(b_1^{-1} s_1) x(s_2) \ldots x(s_{v_1}) x(b_2^{-1} s_{v_1+1}) \ldots x(s_{v_2}) \ldots \tag{6}$$
$$\leqslant \ldots x(b_r^{-1} s_{v_{r-1}+1}) x(s_{v_{r-1}+2}) \ldots x(s_{n-m}) \delta_e(s_{n-m}^{-1} \ldots s_1^{-1} s) \Big|$$
$$= \sum_{\alpha} |y(t_1)| \ldots |y(t_m)|\, \|x_{b_1} x^{v_1 - 1} \ldots x_{b_r} x^{v_r - 1}\|,$$

where $x_{b_j}(s) = x(b_j^{-1} s)$, $j = 1, \ldots, r$. But, since $\operatorname{supp} x = A$, we have $\operatorname{supp} x_{b_j} = b_j A$ and hence

$$\operatorname{supp}(x_{b_1} x^{v_1 - 1} \ldots x_{b_r} x^{v_r - 1}) \subset b_1 A A^{v_1 - 1} \ldots b_r A A^{v_r - 1}$$
$$= b_1 A^{v_1} \ldots b_r A^{v_r} = M(t_1, \ldots, t_m).$$

Therefore, by Schwarz inequality,

$$\|x_{b_1} x^{v_1 - 1} \ldots x_{b_r} x^{v_r - 1}\| \leqslant |M(t_1, \ldots, t_m)|^{1/2} \|x_{b_1} x^{v_1 - 1} \ldots x_{b_r} x^{v_r - 1}\|_2$$
$$\leqslant |M(t_1, \ldots, t_m)|^{1/2} \lambda(x_{b_1}) \lambda(x)^{v_1 - 1} \ldots \lambda(x_{b_r}) \lambda(x)^{v_r - 1}$$
$$= |M(t_1, \ldots, t_m)|^{1/2} \lambda(x)^{n-m},$$

because, clearly, $\|x\|_2 \leqslant \lambda(x)$, $\lambda(xy) \leqslant \lambda(x)\lambda(y)$ and $\lambda(x_b) = \lambda(x)$ for all x, y in $l_1(G)$.

Now let $p(t) = |y(t)|\,\|y\|^{-1}$. Then p is a probability distribution on G and, by (6),

$$\|y^{u_1} x^{v_1} \ldots y^{u_r} x^{v_r}\| \leqslant \varepsilon^m \lambda(x)^{n-m} \sum_{\alpha} |M(t_1, \ldots, t_m)| p(t_1) \ldots p(t_m).$$

Let $c > 1$ and let n_0 be selected by the lemma for c, A and p. Then for all $n > n_0$ and arbitrary $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ we have

$$\|y^{u_1} x^{v_1} \ldots y^{u_r} x^{v_r}\| \leqslant c^n \varepsilon^m \lambda(x)^{n-m}.$$

Consequently, by (5), for $n > n_0$ we have

$$\|z^n\| \leqslant c^n \sum_{m=0}^{n} \binom{n}{m} \varepsilon^m \lambda(x)^{n-m} = c^n (\varepsilon + \lambda(x))^n$$

and so

$$\nu(z) \leqslant c(\varepsilon + \lambda(x)),$$

which, by virtue of the arbitrary choice of c greater than 1, since $|\lambda(x) - \lambda(z)| \leqslant \lambda(x-z) \leqslant \|x-z\| = \|y\| < \varepsilon$, implies

$$\nu(z) \leqslant \lambda(z) + 2\varepsilon,$$

which completes the proof.

2. Nilpotent groups. Let $\mathcal{R}$ be a nilpotent ring of class $\leqslant r$, i.e. an associative ring for which

(7) $$\gamma_0 \ldots \gamma_r = 0$$

for every sequence $\gamma_0, \ldots, \gamma_r$ of elements of $\mathcal{R}$. The operation

$$\alpha \circ \beta = \alpha + \beta + \alpha\beta$$

in $\mathcal{R}$ is a group operation and the group $G = (\mathcal{R}, \circ)$ is a nilpotent group of class $\leqslant r$. It is clear that G can be identified with the set of elements $I+\gamma$, $\gamma \in \mathcal{R}$, where I plays the role of the unit element and

$$(I+\gamma)(I+\gamma') = I + \gamma + \gamma' + \gamma\gamma'.$$

If $\mathcal{R}$ is a free nilpotent ring of nilpotency class r with the free generators

$$\gamma_1, \gamma_2, \ldots,$$

then the elements $I+\gamma_i$ generate freely a nilpotent group of class r.

The aim of this section is to prove that if $G = (\mathcal{R}, \circ)$, where $\mathcal{R}$ is a nilpotent ring, then G satisfies condition (C) and so, by Theorem 1, $l_1(G)$ is symmetric. If H is a homomorphic image of a group G and $l_1(G)$ is symmetric, then, trivially, $l_1(H)$ is symmetric. Consequently, for every nilpotent group the l_1-group algebra is symmetric.

From now on we shall deal with a fixed ring $\mathcal{R}$ which is nilpotent of class $\leqslant r$. The group $(\mathcal{R}, \circ)$ is denoted by G. For a finite set A in $\mathcal{R}$ we write

$$nA = \{a_1 + \ldots + a_n\colon\ a_j \in A\}, \qquad A^n = \{a_1 \ldots a_n\colon\ a_j \in A\}.$$

We have

(9) $$|nA| \leqslant (n+1)^{|A|},$$

since

$$nA = \{k_1 a_1 + \ldots + k_{|A|} a_{|A|}\colon\ k_1 + \ldots + k_{|A|} = n,\ k_j \geqslant 0\}.$$

THEOREM 2. *If $\mathcal{R}$ is a nilpotent ring, then $G = (\mathcal{R}, \circ)$ satisfies condition (C); consequently, $l_1(G)$ is symmetric.*

COROLLARY. *If G is a nilpotent group, then $l_1(G)$ is symmetric.*

A decisive property of the measures in $\boldsymbol{P}$ which is used in the proof that the group G has property (C) is the following:

For a finite subset T of G and a positive number k we write

$$M_n(k, T) = \{(t_1, t_2, \ldots) \in G^\infty\colon\ t_j \in T \text{ for at least } (1-k)n \text{ of } j = 1, \ldots, n\}.$$

A family $\boldsymbol{p}$ of Borel measures on G^∞ is said to have *property* (*) if

(*) *For every $\varepsilon > 0$ and $k > 0$ there exists a finite subset T of G such that*

$$1 - \boldsymbol{p}(M_n(k, T)) < \varepsilon^n \quad \textit{for all } n = 1, 2, \ldots \textit{ and all } \boldsymbol{p} \textit{ in } \boldsymbol{P}.$$

LEMMA 1. *For a finite family of probability distributions P on G let $\boldsymbol{P}$ consists of the direct product measures of sequences $p_1, p_2, \ldots$ with $p_j \in P$. Then $\boldsymbol{P}$ has property (*).*

PROOF. Let ξ be a positive number such that $2\xi^k < \varepsilon$ and let T be a finite subset of G such that

$$p(T) > 1 - \xi \quad \text{for all } p \text{ in } P.$$

If a $\boldsymbol{t} = (t_1, t_2, \ldots)$ does not belong to $M_n(k, T)$, then more than kn of the first n terms of $\boldsymbol{t}$ do not belong to T. Consequently, by the binomial formula, cf. e.g. [1], for any $\boldsymbol{p}$ in $\boldsymbol{P}$ we have

$$1 - \boldsymbol{p}(M_n(k, T)) \leqslant \sum_{m \geqslant kn} \binom{n}{m} \big(\max\{p(T)\colon\ p \in P\}\big)^{n-m} \xi^m$$
$$\leqslant \xi^{kn} \sum_{m \geqslant kn} \binom{n}{m} \leqslant \xi^{kn} 2^n = (2\xi^k)^n < \varepsilon^n.$$

We now introduce some notation. Let s be a positive integer $\leqslant r$ and let $S = \{1, \ldots, s\}$. Let further

$$\boldsymbol{\Delta} = \{\Delta_0, \ldots, \Delta_u\},$$

where

$$\Delta_j = \{t \in S\colon\ a_j \leqslant t \leqslant b_j\} \text{ and } b_j < a_{j+1}, \qquad j = 0, \ldots, u-1.$$

We let

$$\Delta_j' = \{t \in S\colon\ b_j < t < a_{j+1}\}.$$

The segments Δ_j and Δ_j' $(j = 0, \ldots, u)$ cover together S. Let

$$\Delta = \bigcup_j \Delta_j \quad \text{and} \quad \Delta' = \bigcup_j \Delta_j'.$$

Let $N = \{1, \ldots, n)$ and let α and τ be functions:

$$\alpha\colon\ N \to \mathcal{R}, \qquad \tau\colon\ N \to \mathcal{R}.$$

We identify a and τ with elements $(a(1), \ldots, a(n), 0, \ldots)$ and $(\tau(1), \ldots, \tau(n), 0, \ldots)$ of $\mathcal{R}^\infty$, respectively. Let Φ be the set of strictly increasing functions

$$\varphi\colon\ S \longrightarrow N.$$

For a fixed $\boldsymbol{\Delta}$ and a φ in Φ we write

$$(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}} = \prod_{j\in\Delta_0} a(\varphi(j)) \prod_{j\in\Delta_0'} \tau(\varphi(j)) \ldots \prod_{j\in\Delta_u} a(\varphi(j)) \prod_{j\in\Delta_u'} \tau(\varphi(j))$$

and for a subset Ψ of Φ let

$$f_{\boldsymbol{\Delta}\Psi}(a, \tau) = \sum_{\varphi\in\Psi} (a\varphi, \tau\varphi)_{\boldsymbol{\Delta}}.$$

Let, finally,

$$\boldsymbol{A}_n = \{a\colon\ N \to A\}.$$

Clearly,

$$|\boldsymbol{A}_n| = |A|^n.$$

Moreover, if $f_1, \ldots, f_k$ are functions, $f_j\colon\ \boldsymbol{A}_n \to \mathcal{R}$, and g is a function of k variables from $\mathcal{R}\times\ldots\times\mathcal{R}$ into $\mathcal{R}$, then

$$\left|g\left(f_1(\boldsymbol{A}_n), \ldots, f_k(\boldsymbol{A}_n)\right)\right| \leqslant |f_1(\boldsymbol{A}_n)| \ldots |f_k(\boldsymbol{A}_n)|.$$

For a fixed $\boldsymbol{\Delta}$ we say that a subset Ψ of Φ has property $(\boldsymbol{\Delta})$ if for every function ψ in Ψ all possible extensions of $\psi|\Delta'$ to increasing functions $S \to N$ belong to Ψ.

Lemma 2. *For every $c < 1$ and a finite subset A of $\mathcal{R}$ and a positive integer v there exits a n_0 such that*

$$\int |f_{\Psi\boldsymbol{\Delta}}(\boldsymbol{A}_n, \tau)|^v\, d\boldsymbol{p}(\tau) < c^n$$

for all $n > n_0$ and $\boldsymbol{p}\in\boldsymbol{P}$, where $\boldsymbol{P}$ is a family of Borel measures on $\mathcal{R}^\infty$ which has property $(*)$, $\boldsymbol{\Delta}$ is an arbitrary sequence of segments in S, $s \leqslant r$, and Ψ is a subset of Φ which has property $\boldsymbol{\Delta}$).

Proof. First we note that it suffices to prove the lemma for the $\boldsymbol{\Delta}$ with $|\Delta_0| = \ldots = |\Delta|_u = 1$, since, clearly,

$$\{(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}}\colon\ a\in\boldsymbol{A}_n\} \subset \{(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}_1}\colon\ a\colon\ S \to A^s\},$$

where $\boldsymbol{\Delta}_1 = \{\Delta_0^1, \ldots, \Delta_u^1\}$ and Δ_j^1 are $\boldsymbol{\Delta}_j'$ one-element sets. In fact $\prod_{j\in\Delta_l} a(\varphi(j)) \in A^{|\Delta_l|} \subset A^s$.

The proof is by inducton on $|\Delta'|$. If $|\Delta'| = 0$, then for each $\varphi\in\Phi$ the function $(a\varphi, \tau\varphi)$ does not depend on τ and so

$$(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}} \in \bigcup_{t\leqslant s} A^t = B.$$

Thus

$$f_{\boldsymbol{\Delta}\Psi}(a, \tau) \in B + \ldots + B, \tag{10}$$

where the number of the summands on the right-hand side of (10) is less or equal to the number of the increasing functions $\varphi\colon\ S \to N$, this being $\leqslant n^r$. On the other hand, $|B| \leqslant r|A|^r$, whence, by (9),

$$|B + \ldots + B| \leqslant (n^r+1)^{r|A|^r},$$

which shows that

$$\int |f_{\boldsymbol{\Delta}\Psi}(\boldsymbol{A}_n, \tau)|^v\, d\boldsymbol{p}(\tau) \leqslant (n^r+1)^{rv|A|^r},$$

and this completes the proof for the case $|\Delta'| = 0$.

Now suppose Lemma 2 is proved for all $\boldsymbol{\Delta}$ with $|\Delta'| = m$. Let $\boldsymbol{\Delta}$ be such that

$$|\Delta_0| = \ldots = |\Delta_u| = 1 \quad \text{and} \quad |\Delta'| = m+1.$$

For a $c > 1$ we select a positive number k such that

$$(k^{-1}+2)^{3k|A|v} < \sqrt{c}. \tag{11}$$

(Such a k exists, since $\lim\limits_{k\to 0}(k^{-1}+2)^k = 1$.) Let further

$$0 < \varepsilon < |A|^{-v}. \tag{12}$$

Since $\boldsymbol{P}$ has property $(*)$, there exists a finite subset T of $\mathcal{R}$ such that

$$1 - \boldsymbol{p}\left(M_n(k, T)\right) < \varepsilon^n \quad \text{for all } n = 1, 2, \ldots \text{ and } \boldsymbol{p}\in\boldsymbol{P}.$$

Then, for every positive integer n, we have

$$\begin{aligned}\int |f_{\boldsymbol{\Delta}\Psi}(\boldsymbol{A}_n, \tau)|^v\, d\boldsymbol{p}(\tau) &= \int_{M_n(k,T)} + \int_{\mathcal{R}^\infty\setminus M_n(k,T)} \\ &\leqslant \int_{M_n(k,T)} |f_{\boldsymbol{\Delta}\Psi}(\boldsymbol{A}_n, \tau)|^v\, d\boldsymbol{p}(\tau) + |A|^{nv}\varepsilon^n \\ &\leqslant \int_{M_n(k,T)} |f_{\boldsymbol{\Delta}\Psi}(\boldsymbol{A}_n, \tau)|^v\, d\boldsymbol{p}(\tau) + 1,\end{aligned}$$

for all $\boldsymbol{p}$ in $\boldsymbol{P}$.

Now for a fixed τ in $M_n(k, T)$ let

$$\Phi_0 = \{\varphi\in\Psi\colon\ \varphi(\Delta') \cap \tau^{-1}(T) = \emptyset\}$$

and let $\{\Phi_j\}$, $j \in \Delta'$, be a position into disjont subsets of $\Psi\setminus\Phi_0$ such that φ in Φ_j we have $\varphi(j)\in\tau^{-1}(T)$ and all the Φ_j have property Δ_0.

Then, of course,

$$f_{\boldsymbol{\Delta}\Psi}(a,\tau) = \sum_{j\in\Delta'\cup\{0\}} \sum_{\varphi\in\Phi_j} (a\varphi, \tau\varphi)_{\boldsymbol{\Delta}}$$

and so

(13) $$|f_{\boldsymbol{\Delta}\Psi}(\boldsymbol{A},\tau)| \leqslant \prod_{j\in\Delta'\cup\{0\}} \Big|\sum_{\varphi\in\Phi_j}(\boldsymbol{A}_n\varphi, \tau\varphi)_{\boldsymbol{\Delta}}\Big|.$$

But for a fixed j in Δ' for all a such that $a(N)\subset A$ we have

$$\sum_{\varphi\in\Phi_j}(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}} \in \{\sum_{\varphi\in\Xi}(\beta\varphi, \tau\varphi)_{\boldsymbol{\Delta}_j}\colon\ \beta\in\boldsymbol{B}_n\},$$

where Ξ is the set of increasing functions of the set S_j obtained from S by amalgamation of $j-1$, j, $j+1$ (or only $j-1$, j, or j, $j+1$ at the end points) into N and agree the functions of Φ_j on $S\setminus\{j-1, j, j+1\}$, $\boldsymbol{\Delta}_j$ is obtained from $\boldsymbol{\Delta}$ by the above amalgamation and $B_n = \{\beta\colon \beta\colon N\to B\}$ with $B = ATA$.

Hence

$$\Big|\sum_{\varphi\in\Phi_j}(\boldsymbol{A}_n\varphi, \tau\varphi)_{\boldsymbol{\Delta}}\Big| \leqslant \Big|\sum_{\varphi\in\Xi}(\boldsymbol{B}_n\varphi, \tau\varphi)_{\boldsymbol{\Delta}_j}\Big|.$$

But, $|(\Delta_j)'| = m$. Consequently, by the inductive hypothesis with $ATA = B$ in place of A, $2^{j+1}v$ in place of v, where $\boldsymbol{j}$ is the ordinal number of j in the natural order of Δ' and $\sqrt{c}$ in place of c, we see that there exists an n_0 such that

(14) $$\int_{M_n(k,T)} \Big|\sum_{\varphi\in\Phi_j}(\boldsymbol{B}_n\varphi, \tau\varphi)_{\boldsymbol{\Delta}_j}\Big|^{2^{j+1}v} d\boldsymbol{p}(\tau) \leqslant \int |f_{\boldsymbol{\Delta}_j\Xi}(\boldsymbol{B}_n,\tau)|^{2^{j+1}v} d\boldsymbol{p}(\tau) < c^{n/2}$$

for all $n > n_0$, all $\boldsymbol{p}\in\boldsymbol{P}$ and all j in Δ'.

To evaluate

$$\Big|\sum_{\varphi\in\Phi_0}(\boldsymbol{A}_n\varphi, \tau\varphi)_{\boldsymbol{\Delta}}\Big|$$

we fix again a τ in $M_n(k, T)$ and we put $N_\tau = N\setminus\tau^{-1}(T)$. By the definition of $M_n(k,T)$, we have $|N_\tau| < kn$. We let

(15) $$\Delta_0 = \{i_0\}, \quad \ldots, \quad \Delta_u = \{i_u\}.$$

Let Ψ_τ be the set of increasing functions

$$\psi\colon\ \Delta'\to N_\tau.$$

For a ψ in Ψ_τ let Φ_ψ be the set of all extensions of ψ to an increasing function from S into N which belongs to Φ_0. Then

(16) $$\sum_{\varphi\in\Phi_0}(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}} = \sum_{\psi\in\Psi_\tau}\sum_{\varphi\in\Phi_\psi}(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}}.$$

For a fixed ψ, every function in Φ_ψ is uniquely defined by its values on Δ and, conversely, given a function $\varphi_\Delta\colon \Delta\to N$ with

$$\psi(i_l-1) < \varphi_\Delta(i_l) < \psi(i_l+1), \qquad l = 0, \ldots, u,\ \psi(o) = 0,\ \psi(s+1) = n+1,$$

it defines the unique function φ in Φ_ψ. Let

$$D_l(\psi) = \{t\in N\colon \psi(i_l-1) < t < \psi(i_l+1)\}, \qquad l = 0, \ldots, u.$$

We then have

$$\sum_{\varphi\in\Phi_0}(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}} = \sum_{\psi\in\Psi_\tau}\sum_{\varphi\in\Phi_\psi}\prod_{l=0}^{u}\Big\{a\big(\varphi(i_l)\big)\prod_{j\in\Delta'_l}\tau\big(\psi(j)\big)\Big\}$$

$$= \sum_{\psi\in\Psi_\tau}\prod_{l=1}^{u}\sum_{j\in D_j(\psi)} a(j)\prod_{j\in\Delta'l}\tau\big(\psi(j)\big).$$

Now, let

$$N_\tau = \{n_1, \ldots, n_q\}, \qquad \text{where } n_1 < n_2 < \ldots < n_q.$$

Clearly, $q < kn$. Then, since $\psi(j)\in N_\tau$, for each $l = 0, \ldots, u$ the set $D_l(\psi)$ is the union of intervals

$$(n_i, n_{i+1}) = I_i^1, \qquad [n_i, n_{i+1}) = I_i^2, \qquad (n_i, n_{i+1}] = I_i^3,$$

with the convention $n_0 = 0$, $n_{q+1} = n+1$, that is

$$D_l(\psi) = \bigcup_{\eta=1,2,3}\ \bigcup_{i\in M_l^\eta} I_i^\eta.$$

Hence

$$\sum_{\varphi\in\Phi_0}(a\varphi, \tau\varphi) = \sum_{\psi\in\Psi_\tau}\prod_{l=0}^{u}\Big(\sum_{\eta=1,2,3}\sum_{i\in M_l^\eta}\sum_{j\in I_l^\eta} a(j)\Big)\prod_{j\in\Delta'_l}\tau\big(\psi(j)\big),$$

consequently, the function

$$f\colon \boldsymbol{A}_n\ni a\to\sum_{\varphi\in\Phi_0}(a\varphi, \tau\varphi)_{\boldsymbol{\Delta}}\in\mathscr{R}$$

can be factorized as follows:

(17) $$a\to\Big(\sum_{j\in I_1^1}a(j), \sum_{j\in I_1^2}a(j), \sum_{j\in I_1^3}a(j), \ldots, \sum_{j\in I_q^1}a(j), \sum_{j\in I_q^2}a(j), \sum_{j\in I_q^3}a(j)\Big)$$

$$\to\sum_{\psi\in\Psi}\prod_{l=0}^{u}\sum_{\eta=1,2,3}\sum_{i\in M_l^\eta}\sum_{j\in I_i^\eta}a(j)\prod_{j\in\Delta'_l}\tau\big(\psi(j)\big).$$

Since

$$\sum_{j\in I_i^\eta}a(j)\in|I_i^\eta|A.$$

(17) and (9) imply

$$\Big|\sum_{\varphi\in\Phi_0}(A_n\varphi,\varphi)_{\mathcal{A}}\Big|\leqslant\prod_{i=0}^{q}\prod_{\eta=1,2,3}\big||I_i^\eta|\mathcal{A}\big|$$

$$\leqslant\prod_{i=0}^{q}\prod_{\eta=1,2,3}(|I_i^\eta|+1)^{|\mathcal{A}|}\leqslant\prod_{i=0}^{q}(d_i+1)^{3|\mathcal{A}|},$$

where $d_i = n_{i+1}-n_i$ and, consequently,

$$\sum_{i=0}^{q} d_i\leqslant n.$$

But

$$\max\Big\{\prod_{i=0}^{q}(d_i+1)\colon \sum_{i=0}^{q} d_i\leqslant n,\ d_i \text{ real}\geqslant 0,\ q<kn\Big\}$$

is attained when $q = kn$ and $d_0 = \ldots = d_q = n/[kn]\leqslant k^{-1}+1$ if only $n>(k+1)^{-1} = n_0(0)$. Thus

$$\prod_{i=0}^{q}(d_i+1)\leqslant(k^{-1}+2)^{kn},$$

whence

$$\Big|\sum_{\varphi\in\Phi_0}(A_n\varphi,\tau\varphi)_{\mathcal{A}}\Big|^v\leqslant(k^{-1}+2)^{3kv|\mathcal{A}|n}\leqslant c^{n/2}\tag{18}$$

for all $n>n_0(0)$.

Now we return to inequality (13). We note that by an iterated application of the Schwarz inequality we have

$$\Big|\int f_1\ldots f_t\Big|\leqslant\Big(\int|f_1|^2\Big)^{1/2}\ldots\Big(\int|f_{t-1}|^{2^{t-1}}\Big)^{2^{-t+1}}\Big(\int|f_t|^{2^{t-1}}\Big)^{2^{-t+1}}.\tag{19}$$

We apply (19) to (13) and we complete the proof of Lemma 2 by (14) and (18). In fact,

$$\int|f_{\mathcal{A}}(A_n,\tau)|^v dp(\tau)\leqslant\int_{M_n(k,T)}|f_{\mathcal{A}}(A_n,\tau)|^v dp(\tau)+1$$

$$\leqslant 1+\int_{M_n(k,T)}\prod_{j\in\Delta'\cup\{0\}}\Big|\sum_{\varphi\in\Phi_j}(A_n\varphi,\tau\varphi)_{\mathcal{A}}\Big|^v dp(\tau)$$

$$\leqslant 1+\Big(\int\Big|\sum_{\varphi\in\Phi_0}(A_n\varphi,\tau\varphi)_{\mathcal{A}}\Big|^{2v}dp(\tau)\Big)^{1/2}\prod_{j\in\Delta'}\Big(\int\Big|\sum_{\varphi\in\Phi_j}(A_n\varphi,\tau\varphi)_{\mathcal{A}}\Big|^{2^{\mathbf{j}+1}}\Big)2^{-(\mathbf{j}+1)},$$

where $\mathbf{j}$ is the number of j in the natural order of Δ' for all j but the last one, and for the last one $\mathbf{j} = |\Delta'|-1$. By (18) and (14) we then have

$$\int|f_{\mathcal{A}\Psi}(A_n,\tau)|^v dp(\tau)\leqslant 1+c^{n/2}\Big(\prod_{j\in\Delta'}c^{2^{-\mathbf{j}}}\Big)^n\leqslant c^{2n}+1$$

for all $n>n_0 = \max\{n_0(0), n_0(j)\colon j\in\Delta'\}$, which in virtue of the fact that $c>1$ was chosen arbitrarily, completes the proof of Lemma 2.

Proof of Theorem 2 (conclusion). Let $\mathcal{A}$ be a finite set in $\mathcal{R}$ and let

$$\alpha_1,\ldots,\alpha_n,\qquad \tau_1,\ldots,\tau_n$$

be two sequences of elements of $\mathcal{R}$. Then

$$\alpha_1\circ\tau_1\circ\ldots\circ\alpha_n\circ\tau_n = \sum_{\mathcal{A}\in D}\sum_{\varphi\in\Phi}(\alpha\varphi,\tau\varphi)_{\mathcal{A}},$$

where $D = \bigcup_{s\leqslant r} D_s$ and D_s is the set of all the sequences of segments in $\{1,\ldots,s\} = S$ as defined just after the proof of Lemma 1, Φ the set of the increasing functions from S to $N = \{1,\ldots,n\}$ and α, τ the functions

$$\alpha\colon j\to\alpha_j,\qquad \tau\colon j\to\tau_j.$$

Thus

$$|\tau_1\circ A\circ\ldots\circ\tau_n\circ A|\leqslant\Big|\sum_{\mathcal{A}\in D}f_{\mathcal{A}}(A_n,\tau)\Big|\leqslant\prod_{\mathcal{A}\in D}|f_{\mathcal{A}}(A_n,\tau)|,$$

where $f_{\mathcal{A}} = f_{\mathcal{A}\Phi}$.

Hence, by iterated application of Schwarz inequality, for any $p\in P$, we have

$$\int|\tau_1\circ A\circ\ldots\circ\tau_n\circ A|\,dp(\tau)\leqslant\int\prod_{\mathcal{A}\in D}|f_{\mathcal{A}}(A_n,\tau)|\,dp(\tau)$$

$$\leqslant\prod_{\mathcal{A}\in D}\Big(\int|f_{\mathcal{A}}(A_n,\tau)|^v dp(\tau)\Big)^{v^{-1}},$$

where $v = 2^k$ with $k = k(\mathcal{A})$ is the number of $\mathcal{A}$ in an ordering of D. By Lemma 2, for each $c>1$ and $\mathcal{A}$ there is an n_0 such that

$$\int|f_{\mathcal{A}}(A_n,\tau)|^v dp(\tau)<c^n$$

for all $n>n_0$ and $p\in P$. Thus, since the number of the elements in D is fixed (D depends only on r), Theorem 2 follows.

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Reçu par la Rédaction le 2. 6. 1969