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Analytic approach to semiclassical logarithmic potential theory

by

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0. Introduction. The object of this paper is to present a non-probabilistic, purely analytic method leading to the semiclassical potential theory on the plane in the sense of M. Kac and Z. Ciesielski. Kwapień [7] proposed such an approach to semiclassical potential theory on a Greenian domain. The theory for the plane and the line requires a separate treatment. It was M. Kac's idea to consider integral equations with two parameters and Bessel kernel. With this idea it was possible to obtain (cf. [5]) for an arbitrary compact set $B \subset R^2$ of positive Lebesgue measure, semiclassical analogs of such notions as the Dirichlet problem and the Green function for $R^2 - B$, as well as the equilibrium distribution and the Robin constant of B. For a large class of sets all these notions coincide with the classical ones. Roughly speaking, the main difference between semiclassical and classical approaches is that the polar sets are replaced by the sets of Lebesgue measure zero. Such replacement turns out to be advantageous as for all quantities mentioned above analytic formulas can be derived. All these results were established by probabilistic methods. In this note the same results are obtained in much simpler way; all considerations are based entirely on the potential theoretic background including some elements of the axiomatic potential theory. The potential theory corresponding to the differential operator $\frac{1}{2}\Delta - sI$ $(s > 0)$ is a main tool used in the present approach. A brief, by no means complete, survey of this theory is given in Section 1. The semiclassical solution of the Dirichlet problem as well as the other potential theoretic quantities are obtained in Section 2. Section 3 is devoted to discussion of the results of Section 2. Also the analytic formulas of Section 4 are immediate consequences of the considerations of Section 2.

We are going to deal with the plane only, for the semiclassical potential theory on the line is similar but much easier.

I wish to express my thanks to Professor Z. Ciesielski for suggesting the problem and for his valuable help and advices.

1. Potential theory for the operator $\frac{1}{2}\Delta - sI$. Let s be a fixed positive number. For any point $x = (x_1, x_2)$ of the 2-dimensional Euclidean space R^2 and for a function f, twice differentiable at x, we define

$$\Delta^s f(x) = \left(\frac{1}{2}\Delta - sI\right) f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(x) + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}(x) - sf(x).$$

A function f on an open subset U of R^2 is said to be *Δ^s-harmonic* on U if $\Delta^s f(x) = 0$ for every $x \in U$.

LEMMA 1.1. *Let K be a disc with center x_0 and with radius r. There exists a number $a_{r,s}$ independent of x_0 such that, for any function h Δ^s-harmonic on* Int K *and continuous on K we have*

$$h(x_0) = \frac{a_{r,s}}{2\pi r} \int_{\partial K} h(y)\, \sigma(dy), \tag{1.1}$$

where $\sigma(dy)$ is the Lebesgue measure on the circle ∂K.

Moreover, $a_{r,s} \nearrow 1$ as $r \searrow 0$ and $a_{r,s} \nearrow 1$ as $s \searrow 0$.

The proof follows immediately from the definition and it is omitted.

With harmonicity defined as above, R^2 is an elliptic, strongly harmonic space (cf. Bauer [2]). In the sequel all notions concerning potential theory for the operator Δ^s will be preceded by Δ^s (e.g. "Δ^s-superharmonic", "Δ^s-balayage"). The class of all Δ^s-superharmonic (resp. Δ^s-superharmonic and positive) functions on an open set $U \subset R^2$ will be denoted by $\mathscr{H}_s^\uparrow(U)$ (resp. $\mathscr{H}_{s+}^\uparrow(U)$). If $U = R^2$, we write simply $\mathscr{H}_s^\uparrow$, $\mathscr{H}_{s+}^\uparrow$ for $\mathscr{H}_s^\uparrow(U)$ and $\mathscr{H}_{s+}^\uparrow(U)$ respectively.

The following is an immediate consequence of Lemma 1.1:

COROLLARY 1.2. *If x_0, K and $a_{r,s}$ are the same as in Lemma 1.1, $v \in \mathscr{H}_s^\uparrow(U)$ and $K \subset U$, then*

$$v(x_0) \geqslant \frac{a_{r,s}}{2\pi r} \int_{\partial K} v(y)\, \sigma(dy). \tag{1.2}$$

Now, let us consider the equation $\Delta^s u = 0$. It is known that the function

$$K^s(\cdot) = \frac{1}{\pi} K_0(\sqrt{2s}\,|\cdot|)$$

is a fundamental solution of this equation, where $K_0(\cdot)$ is the Bessel function of second kind of zero order (cf. e.g. [1]). We have (in the sense of L. Schwartz distribution theory)

$$\Delta^s K^s = -\delta, \tag{1.3}$$

where δ denotes the Dirac distribution concentrated at 0; analogously, the Dirac distribution concentrated at x will be denoted by δ_x.

The function $K^s(\cdot)$ plays, as it is suggested by (1.3), essentially the same role here as the function $1/|\cdot|$ plays in the classical potential theory on R^3. Dealing with $K^s(\cdot)$ we shall refer frequently to the following properties of $K_0(\cdot)$ (cf. [11]):

$$0 < K_0(r) \leqslant +\infty, \quad r \geqslant 0, \tag{1.3}$$

$$K_0(\cdot) \text{ is continuous and decreasing on } \langle 0, +\infty), \tag{1.3$'$}$$

$$K_0(r) = 0\left(\frac{1}{r}e^{-r}\right) \quad \text{as } r \to +\infty, \tag{1.4}$$

$$K_0(r) = \log\frac{2}{r} - \gamma + o(1) \quad \text{as } r \to 0_+, \tag{1.5}$$

where γ is the Euler constant.

Making use of the properties (1.3), (1.3)′, (1.4) and Lemma 1.1 one can repeating almost literally the classical argument, prove the following proposition (cf. [3]):

PROPOSITION 1.3. *If T is a distribution such that*

$$\Delta^s T \leqslant 0, \tag{1.6}$$

then T is equivalent to a Δ^s-superharmonic function. In particular, for any Radon measure μ, the function

$$K^s\mu(\cdot) = \frac{1}{\pi}\int_{R^2} K_0(\sqrt{2s}\,|\cdot - y|)\, \mu(dy)$$

is either in $\mathscr{H}_{s+}^\uparrow$ or it is infinite at every point.

The next theorem we are going to formulate is the "Domination Principle".

THEOREM 1.4. *Let a function f be non-negative Lebesgue integrable on R^2; furthermore, assume that there is a bounded Borel set S such that $\int_{R^2 - S} f(y)\,dy = 0$. If $v \in \mathscr{H}_{s+}^\uparrow$ and*

$$\frac{1}{\pi}\int_{R^2} K_0(\sqrt{2s}\,|x-y|) f(y)\,dy \leqslant v(x)$$

for all $x \in S$, then this inequality holds everywhere on R^2.

Proof. It is known that the kernel K^s is regular (cf. [1]). Now, to obtain the desired result, it is sufficient to follow the Landkof's proof of the analogous theorem in the potential theory for the Newtonian kernel ([8], p. 149).

Also the semiclassical potential theory in the Kac-Ciesielski sense can be built on R^2 for the operator Δ^s. For this theory on a Greenian domain in the Laplace operator case cf. [4] and [7]. Let $v \in \mathscr{H}_{s+}^{\dagger}$ and A be a subset of R^2. *The strong Δ^s-balayage* of v onto A is defined as

$$S_{s,v}^A(x) = \inf\{u(x):\ u \in \mathscr{H}_{s+}^{\dagger},\ u \geqslant v \text{ a.e. on } A\}$$

(a.e. — almost everywhere in Lebesgue sense). It has essentially the same properties as in the Laplace operator case; in particular, $S_{s,v}^A \in \mathscr{H}_{s+}^{\dagger}$ and there exists $A_0 \subset A$ such that $|A - A_0| = 0$ and $S_{s,v}^A =$ "ordinary" Δ^s-balayage of v onto A_0.

Again we can define a set A to be *strongly Δ^s-thin at x_0* if either $|A \cap U| = 0$ for some neighbourhood U of x_0 or if there exists $v \in \mathscr{H}_{s+}^{\dagger}$ such that $v(x_0) < \operatorname{ess\,lim\,inf}_{x \in A, x \to x_0} v(x)$. It follows immediately from this definition that A is strongly Δ^s-thin at x_0 if and only if there is A^0 such that $|A - A^0| = 0$ and A^0 is Δ^s-thin at x_0.

PROPOSITION 1.5. *A is strongly Δ^s-thin at x_0 if and only if it is strongly thin at x_0 in the sense of the Laplace operator theory.*

Proof. It suffices to combine the preceding remark with the fact that A is Δ^s-thin at x_0 if and only if it is thin at x_0 (cf. [6]).

As usual, the set of all points at which A is not strongly thin will be denoted by A^*. If A is a Borel set then $|A - A^*| = 0$.

An argument similar to that in [4] leads to Lemma 1.6 and Proposition 1.7:

LEMMA 1.6. *If $x_0 \in A^*$, then, for any $v \in \mathscr{H}_{s+}^{\dagger}$ we have $S_{s,v}^A(x_0) = v(x_0)$; if, moreover, we assume that v is continuous at x_0, then $S_{s,v}^A$ is also continuous at x_0.*

PROPOSITION 1.7. *$x_0 \in A^*$ if and only if $S_{s,1}^A(x_0) = 1$.*

Let B be a fixed compact subset of R^2 such that $|B| > 0$. We shall use the following notation:

$L^2(B)$ is the Hilbert space of all square integrable functions f on B with $(f, g) = \int_B f(y) g(y)\, dy$;

$C(B)$ is the Banach space of all continuous functions on B with the norm $\|f\| = \sup_{x \in B} |f(x)|$;

$\mathscr{B}(R^2)$ is the class of all bounded and measurable functions on R^2.

For $f \in L^2(B)$ we write

$$K_B^s f(x) = \frac{1}{\pi} \int_B K_0(\sqrt{2s}\,|x-y|) f(y)\, dy. \tag{1.7}$$

LEMMA 1.8. *The operator $K_B^s: L^2(B) \to L^2(B)$ is self-adjoint completely continuous and positive. Moreover, for $f \in L^2(B)$ the function defined by (1.7) is bounded and continuous on R^2.*

Proof. Positiveness of K_B^s is easily derived from the following identity:

$$\frac{1}{\pi} K_0(\sqrt{2s}\,|x-y|) = \frac{1}{2\pi} \int_0^\infty e^{-st} \frac{1}{t} e^{-|x-y|^2/2t} dt$$
$$= \frac{1}{(2\pi)^2} \int_0^\infty e^{-st} \int_{R^2} e^{-i(x-y)z - \frac{t}{2} z^2} dz\, dt.$$

The proof of the other properties is based on (1.3), (1.4), and (1.5). As an immediate corollary we get that for any $u > 0$, $f \in L^2(B)$ there exists exactly one function $q_{u,s} \in L^2(B)$ such that

$$q_{u,s} + u K_B^s q_{u,s} = f. \tag{1.8}$$

If f is defined everywhere on R^2 (and not only almost everywhere on B), then

$$q_{u,s}(x) = f(x) - u K_B^s q_{u,s}(x) \tag{1.9}$$

defines a function on R^2 such that (1.8) holds identically; moreover, $q_{u,s} \in \mathscr{B}(R^2)$ if only $f \in \mathscr{B}(R^2)$.

Now, the results of Kwapień [7] can be adopted to the case of the kernel K^s (cf. also [9]).

First of all the following proposition can be derived from Theorem 1.4:

PROPOSITION 1.9. *Let u be an arbitrary positive number and f be a Δ^s-superharmonic and positive function such that the restriction of f to B is in $L^2(B)$. Then the unique solution of (1.8), $q_{u,s} \in L^2(B)$, extended to the whole R^2 by (1.9) is positive on R^2.*

This proposition combined with the properties of $S_{s,v}^B$ gives the following two theorems:

THEOREM 1.10. *Let $f \in \mathscr{B}(R^2) \cap \mathscr{H}_{s+}^{\dagger}$ and let, for given $u > 0$, $q_{s,u}$ be the bounded solution of*

$$q_{s,u} + u K_B^s q_{s,u} = u K_B^s f. \tag{1.10}$$

Then $q_{s,u}(x) \nearrow S_{s,f}^B(x)$ as $u \nearrow +\infty$, $x \in R^2$.

THEOREM 1.11. *Let $f \in C(B)$ and, for $u > 0$, let $q_{s,u}$ be the bounded solution of (1.10). Then there exists*

$$\lim_{u \to +\infty} q_{s,u}(x) \stackrel{\text{def}}{=} D_{s,f}^B(x)$$

for every $x \in R^2$ and the function $D_{s,f}^B(\cdot)$ is Δ^s-harmonic on $R^2 - B$, is equal to $f(x)$ and is continuous at each $x \in B^*$; moreover,

$$\sup_{x \in R^2} |D_{s,f}^B(x)| \leqslant \sup_{x \in B} |f(x)|.$$

2. Passage to the limit with the parameters s and u. Throughout the rest of this paper, B is an arbitrary but fixed compact subset of R^2 of positive Lebesgue measure.

Fix $s, u > 0$, $x \in R^2$ and let $Q(s, u; x, \cdot)$ be the square integrable on B solution of the equation

$$Q(s, u; x, \cdot) + uK_B^s Q(s, u; x, \cdot) = \frac{1}{\pi} K_0(\sqrt{2s}\,|x-\cdot|). \tag{2.1}$$

The function $Q(s, u; x, \cdot)$ is defined and continuous everywhere on $R^2 - \{x\}$ (it is continuous in "the wide sense" on R^2).

We are going to pass in (2.1) with s to zero, and then with u to infinity. Before that some properties of $Q(s, u; x, \cdot)$ have to be established.

Lemma 2.1. 1° *For $u, s > 0$ and $x, y \in R^2$ the function $Q(s, u; x, y)$ is positive.*

2° *For $s > 0$ and $x \neq y$, $Q(s, u; x, y)$ is decreasing in u.*

3° *For $u > 0$ and $x \neq y$, $Q(s, u; x, y)$ is decreasing in s.*

Proof. 1° follows immediately from Proposition 1.9.

To prove 2° assume that $u' > u > 0$. It is sufficient to notice that

$$Q(s, u; x, \cdot) - Q(s, u'; x, \cdot) + uK_B^s[Q(s, u; x, \cdot) - Q(s, u'; x, \cdot)] = (u'-u)K_B^s Q(s, u'; x, \cdot),$$

and then to apply Proposition 1.9.

Now, let $s > s' > 0$. (2.1) implies that

$$Q(s', u; x, \cdot) - Q(s, u; x, \cdot) + uK_B^s[Q(s', u; x, \cdot) - Q(s, u; x, \cdot)]$$
$$= \frac{1}{\pi} K_0(\sqrt{2s'}\,|x-\cdot|) - \frac{1}{\pi} K_0(\sqrt{2s}\,|x-\cdot|) - u[K_B^{s'} Q(s, u; x, \cdot) - K_B^s Q(s, u; x, \cdot)].$$

Notice that the function on the right-hand side of this equality can be extended to a continuous bounded function on the whole plane; it stems from (1.5) and from Lemma 1.8. Let it be denoted by $\Phi(\cdot)$. It will be shown that $\Phi \in \mathscr{H}_{s+}^{\dagger}$.

We know that

$$\Delta^{s'}\left(\frac{1}{\pi} K_0(\sqrt{2s'}\,|x-\cdot|)\right) = -\delta_x$$

and this gives

$$\Delta^{s'}\left(\frac{1}{\pi} K_0(\sqrt{2s}\,|x-\cdot|)\right) = -\delta_x + (s-s')\frac{1}{\pi} K_0(\sqrt{2s}\,|x-\cdot|).$$

Now, it can be easily seen that

$$\Delta^{s'}\Phi = -(s-s')\left[\frac{1}{\pi} K_0(\sqrt{2s}\,|x-\cdot|) - uK_B^s Q(s, u; x, \cdot)\right] = -(s-s')\,Q(s, u; x, \cdot).$$

Thus, by Proposition 1.3, $\Phi \in \mathscr{H}_s^{\dagger}$. Furthermore, since $\lim_{|y|\to\infty} \Phi(y) = 0$, we see, by Minimum Principle (cf. [2], p. 26) that $\Phi \in \mathscr{H}_{s+}^{\dagger}$. To complete the proof we apply Proposition 1.9.

Lemma 2.2. *For any $s, u > 0$ and $x, y \in R^2$,*

$$Q(s, u; x, y) = Q(s, u; y, x).$$

Proof. Let us write (2.1) as follows:

$$Q(s, u; x, y) = -\frac{u}{\pi}\int_B K_0(\sqrt{2s}\,|z-y|)Q(s, u; x, z)\,dz + \frac{1}{\pi}K_0(\sqrt{2s}\,|x-y|)$$

and

$$\frac{1}{\pi}K_0(\sqrt{2s}\,|z-y|) = Q(s, u; y, z) + \frac{u}{\pi}\int_B K_0(\sqrt{2s}\,|z-\zeta|)Q(s, u; y, \zeta)\,d\zeta.$$

Combining these equalities, we get

$$Q(s, u; x, y) = \frac{u}{\pi}\int_B Q(s, u; y, z)Q(s, u; x, u)\,dz - \frac{u^2}{\pi^2}\int_B\int_B K_0(\sqrt{2s}\,|z-\zeta|)Q(s, u; y, \zeta)Q(s, u; x, z)\,d\zeta\,dz + \frac{1}{\pi}K_0(\sqrt{2s}\,|x-y|).$$

The latter expression is symmetric with respect to x and y, so the lemma is proved.

Theorem 2.3. *For any $y \in R^2$ there exists a Radon measure μ_y such that*

1° *μ_y is concentrated on B and $\mu_y(B) = 1$;*

2° *for any function $f \in C(B)$*

$$\lim_{s\to 0+,\,u\to+\infty}\ \lim\ u\int_B f(z)Q(s, u; z, y)\,dz = \lim_{u\to+\infty}\ \lim_{s\to 0+}\ u\int_B f(z)Q(s, u; z, y)\,dz = \int_B f(z)\,\mu_y(dz);$$

3° *if we define*

$$D_f^B(y) = \int_B f(z)\,\mu_y(dz) \quad \text{for } f \in C(B),$$

then $D_f^B(\cdot)$ is harmonic on R^2-B and it is equal to f and continuous at each $y_0 \in B^*$.

Proof. First of all, notice that for any $f \in C(B)$ and $u, s > 0$ the function $u \int_B f(z) Q(s, u; z, \cdot) dz$ is a bounded solution of (1.10). Hence, by Theorem 1.11,

$$\lim_{u\to+\infty} u \int_B f(z) Q(s, u; z, y) dz = D_{s,f}^B(y). \tag{2.2}$$

Furthermore, Theorem 1.10 gives

$$u \int_B Q(s, u; z, y) dz \nearrow S_{s,1}^B(y) \quad \text{as } u \nearrow +\infty,\ s > 0,\ y \in R^2, \tag{2.3}$$

and

$$0 \leqslant u \int_B Q(s, u; z, y) dz \leqslant 1, \quad u, s > 0,\ y \in R^2. \tag{2.4}$$

For any $s, u > 0$, $y \in R^2$ and a Borel set $A \subset R^2$ we define

$$\mu_{s,u,y}(A) = u \int_{A\cap B} Q(s, u; z, y) dz.$$

Thus, by (2.2) there is a Radon measure $\mu_{s,y}$ such that $\mu_{s,u,y}$ is weakly convergent to $\mu_{s,y}$ (written $\mu_{s,u,y} \Rightarrow \mu_{s,y}$) as $u \to +\infty$, and $D_{s,f}^B(y) = \int_B f(z) \mu_{s,y}(dz)$.

Then, by virtue of Lemma 2.1 and (2.4), for each $y \in R^2$ there exists a Radon measure μ_y such that

$$\mu_{s,y} \Rightarrow \mu_y \quad \text{as } s \to 0_+. \tag{2.5}$$

We shall prove that $\mu_y(R^2) = 1$. Since $\mu_{s,u,y}(R^2) \nearrow S_{s,1}^B(y)$ as $u \nearrow +\infty$ and $S_{s,1}^B(y) \nearrow \mu_y(R^2)$ as $s \searrow 0$, we see, by (2.4), Lemma 1.1 and Corollary 1.2 that the function $\mu(R^2)$ is positive and superharmonic on the whole plane. Therefore $\mu_y(R^2) = c = \text{const}$ and, by (2.3), $c = 1$.

Now, for $u > 0$ and $x \neq y$ we write

$$Q(u; x, y) = \lim_{s\to 0_+} Q(s, u; x, y) \tag{2.6}$$

and, for any Borel set $A \subset R^2$,

$$\mu'_{u,y}(A) = u \int_{A\cap B} Q(u; z, y) dz.$$

Clearly, $\mu_{s,u,y} \Rightarrow \mu'_{u,y}$ as $s \to 0_+$ and the set of measures $\{\mu'_{u,y}\}_{u>0}$ is conditionally weakly compact in $C^*(B)$ ($C^*(B)$ denotes the dual space of $C(B)$). Hence, there exists a sequence of measures $\{\mu'_{u_m,y}\}_{m=1,2,\ldots}$ ($u_m \nearrow +\infty$) which is weakly convergent, say, to a measure μ'_y. For any positive function $f \in C(B)$ we have, by Lemma 2.1 and (2.5),

$$\mu_y(f) \leqslant \mu'_y(f).$$

Since $\mu'_y(1) \leqslant 1$ and $\mu_y(1) = 1$, we obtain $\mu'_y = \mu_y$. Thus, $\mu'_{u,y} \Rightarrow \mu_y$ as $u \to +\infty$.

It remains to prove 3° of the theorem. By Theorem 1.11, Lemma 1.1 and (2.5), we see that for $f \in C(B)$ the function $D_f^B(\cdot)$ is harmonic on R^2-B. Furthermore, Theorem 1.11 and (2.5) imply that $D_f^B(y) = f(y)$ at every point $y \in B^*$.

Let y_0 be an arbitrary point of B^* and $\{y_m\}_{m=1,2,\ldots}$ be a sequence of points such that $y_m \to y_0$ as $m \to \infty$. It is to be shown that $\mu_{y_m} \Rightarrow \delta_{y_0}$ as $m \to \infty$. We argue like before. The set $\{\mu_{y_m}\}_{m=1,2,\ldots}$ is weakly compact in $C^*(B)$, so there exist a subsequence $\{\mu_{y_{m_j}}\}_{j=1,2,\ldots}$ and a measure μ such that $\mu_{y_{m_j}} \Rightarrow \mu$ as $j \to \infty$.

We know that for any positive function $f \in C(B)$ and $s > 0$

$$\mu_{s,y_{m_j}}(f) \leqslant \mu_{y_{m_j}}(f), \quad j = 1, 2, \ldots$$

Hence, by Theorem 1.11,

$$\mu_{s,y_0}(f) = \delta_{y_0}(f) \leqslant \mu(f).$$

This, combined with the inequality $\mu(1) \leqslant 1$ gives $\mu = \delta_{y_0}$. It is clear now that $\mu_{y_m} \Rightarrow \delta_{y_0}$ as $m \to \infty$ and the proof is complete.

Theorem 2.3 gives the semiclassical solution of the Dirichlet problem for R^2-B.

Now, we are ready to pass to the limits with s and u in (3.1). By (1.5) we have for small s

$$\frac{1}{\pi} K_0(\sqrt{2s}\,|x-y|) = \frac{1}{\pi}\log\frac{1}{|x-y|} + \frac{1}{\pi}\left(\log\sqrt{\frac{2}{s}}-\gamma\right) + o(1), \tag{2.7}$$

where $o(1)$ is uniform in x and y ranging over an arbitrary bounded subset of R^2. Thus, if we write

$$Q(s, u; y) = \frac{1}{\pi}\left(\log\sqrt{\frac{2}{s}}-\gamma\right)\left[1-u\int_B Q(s, u; z, y) dz\right], \tag{2.8}$$

then (2.1) can be written as

$$Q(s, u; x, y) = \frac{1}{\pi}\log\frac{1}{|x-y|} - \frac{u}{\pi}\int_B \log\frac{1}{|x-z|}\, Q(s, u; z, y)\, dz + Q(s, u; y) + o(1), \tag{2.9}$$

where $o(1)$ is uniform in u.

THEOREM 2.4. *For any $x, y \in R^2$ and $x \neq y$, the following limits exist and they are finite:*

$$\lim_{u\to\infty}\lim_{s\to 0+} Q(s, u; x, y) \overset{\text{def}}{=} Q_0(x, y), \tag{2.10}$$

$$\lim_{u\to\infty}\lim_{s\to 0+} Q(s, u; y) \overset{\text{def}}{=} Q_0(y). \tag{2.11}$$

Moreover, we have

$$Q_0(x, y) = \frac{1}{\pi}\log\frac{1}{|x-y|} - \frac{1}{\pi}\int_B \log\frac{1}{|x-z|}\,\mu_y(dz) + Q_0(y). \tag{2.12}$$

Proof. Notice that, by Lemma 2.1, $Q_0(x, y)$ is well defined.

We shall prove that for any $u > 0$ and $y \in R^2$ there exists $\lim\limits_{s\to 0+} Q(s, u; y)$ and it is finite. Let $y \in R^2$ and $u > 0$ be fixed. (2.4) implies that $Q(u; x, y) < +\infty$ for almost all $x \in B$, where $Q(u; x, y)$ is given by (2.6). Furthermore, it is easy to see that

$$\lim_{s\to 0+}\int_B \log\frac{1}{|x-z|}\,Q(s, u; z, y)\,dz = \int_B \log\frac{1}{|x-z|}\,Q(u; z, y)\,dz \overset{\text{def}}{=} \varphi(x)$$

and the function $\varphi(\cdot)$ is superharmonic, so it is finite almost everywhere on R^2. Thus, there exists a point $x \in B$ such that both $\varphi(x) < +\infty$ and $Q(u; x, y) < +\infty$. Now, (2.9) implies that $\lim\limits_{s\to 0+} Q(s, u; y)$ exists and it is finite. Write

$$Q(u; y) = \lim_{s\to 0+} Q(s, u; y). \tag{2.13}$$

Letting, in (2.9), $s \to 0_+$ we obtain

$$Q(u; x, y) = \frac{1}{\pi}\log\frac{1}{|x-y|} - \frac{u}{\pi}\int_B \log\frac{1}{|x-z|}\,Q(u; z, y)\,dz + Q(u; y). \tag{2.14}$$

It is convenient to use the following notation: for any locally integrable function f, $x \in R^2$ and $r > 0$ we write

$$\mathfrak{M}(x, r, f) = \frac{1}{\pi r^2}\int_{\{z:\,|z-x|<r\}} f(y)\,dy.$$

Formula (2.14) gives

$$\mathfrak{M}\big(x, r, Q(u; \cdot, y)\big) = \mathfrak{M}\Big(x, r, \frac{1}{\pi}\log\frac{1}{|\cdot - y|}\Big) - \frac{u}{\pi}\int_B \mathfrak{M}\Big(x, r, \log\frac{1}{|\cdot - z|}\Big)\,Q(u; z, y)\,dz + Q(u; y), \qquad u, r > 0,\ x \neq y.$$

By (2.8) and (2.4) it is clear that there exists

$$Q_0(y) = \lim_{u\to\infty} Q(u; y). \tag{2.15}$$

Thus, by Theorem 2.3 and (2.10), letting $u \to +\infty$ we get

$$\mathfrak{M}\big(x, r, Q_0(\cdot, y)\big) = \mathfrak{M}\Big(x, r, \frac{1}{\pi}\log\frac{1}{|\cdot - y|}\Big) - \frac{1}{\pi}\int_B \mathfrak{M}\Big(x, r, \log\frac{1}{|\cdot - z|}\Big)\,\mu_y(dz) + Q_0(y).$$

Formula (2.14) and Lemma 2.1 imply that $Q_0(\cdot, y)$ is subharmonic on $R^2 - \{y\}$, therefore, letting $r \to 0_+$ we obtain (2.12) and the proof is complete.

Now, we are going to deal with the function $Q(s, u, \cdot)$. We know that for $s, u > 0$ the function

$$q_{s,u}(\cdot) = u\int_B Q(s, u; z, \cdot)\,dz$$

is a bounded solution of the equation

$$q_{s,u} + uK_B^s q_{s,u} = uK_B^s 1. \tag{2.16}$$

Thus, if we write

$$R(s, u) = \frac{1}{\pi}\Big(\log\sqrt{\frac{2}{s}} - \gamma\Big)\Big[1 - u\int_B Q(s, u; z)\,dz\Big], \tag{2.17}$$

then by (2.8), for small s we have

$$Q(s, u; y) = R(s, u) - \frac{u}{\pi}\int_B \log\frac{1}{|y-z|}\,Q(s, u; z)\,dz + o(1)\int_B Q(s, u; z)\,dz. \tag{2.18}$$

THEOREM 2.5. *There exists $\lim\limits_{u\to\infty}\lim\limits_{s\to 0+} R(s, u) \overset{\text{def}}{=} R_0$ and it is finite. Moreover, there exists a Radon measure μ_0 such that $\mu_0(R^2) = \mu_0(B) = 1$ and, for $y \in R^2$,*

$$Q_0(y) = R_0 - \frac{1}{\pi}\int_B \log\frac{1}{|y-z|}\,\mu_0(dz). \tag{2.19}$$

Proof. The proof is due to Z. Ciesielski and M. Kac and we are going only to outline its idea. For any $u, r > 0$, $y \in R^2$ and small s (2.18) gives

$$\mathfrak{M}(y, r, Q(s, u; \cdot))$$
$$= R(s, u) - \frac{u}{\pi} \int_B \mathfrak{M}\left(y, r, \log\frac{1}{|\cdot - z|}\, Q(s, u; z)\right) dz + o(1) \int_B Q(s, u; z)\, dz.$$

It can easily be proved that for any $u_0 > 0$ there exists $s_0 > 0$ and a function f such that f is integrable on B and $Q(s, u; y) \leqslant f(y)$ for all $y \in B$, $u \geqslant u_0$, $0 < s \leqslant s_0$. Therefore

$$\lim_{s \to 0_+} R(s, u) \overset{\text{def}}{=} R(u) \tag{2.20}$$

must exist. Thus, letting $s \to 0_+$ we get

$$\mathfrak{M}(y, r, Q(u, \cdot)) = R(u) - \frac{u}{\pi} \int_B \mathfrak{M}\left(y, r, \log\frac{1}{|\cdot - z|}\right) Q(u; z)\, dz.$$

Now, it is easy to see that letting $r \to 0_+$ we obtain

$$Q(u; y) = R(u) - \frac{u}{\pi} \int_B \log\frac{1}{|y-z|}\, Q(u; z)\, dz, \qquad u > 0,\ y \in R^2. \tag{2.21}$$

Applying Lemma 1.8 and equation (2.16), one can express $\int_B Q(s, u; y)\, dy$ by means of the eigenvalues and eigenfunctions of the operator K_B^s; then it is seen that $u \int_B Q(s, u; y)\, dy$ increases as $u \nearrow +\infty$ (for s sufficiently small). Hence, clearly

$$\lim_{u \to +\infty} R(u) \overset{\text{def}}{=} R_0$$

exists and it is finite.

For any $u > 0$ and a Borel set $A \subset R^2$ write

$$\nu_u(A) = u \int_{A \cap B} Q(u; z)\, dz.$$

By (2.17) and (2.20) we have

$$\nu_u(R^2) = u \int_B Q(u; z)\, dz = 1, \qquad u > 0. \tag{2.22}$$

Using the weak compactness of the set of measures $\{\nu_u\}_{u>0}$, it is not difficult to prove that there exists a Radon measure μ_0 such that

$$\nu_u \Rightarrow \mu_0 \quad \text{as } u \to +\infty$$

and then to derive (2.19) from (2.21), q.e.d.

3. The semiclassical potential theory on the plane. We are going to discuss some properties of $Q_0(\cdot, \cdot)$, $Q_0(\cdot)$, μ_y, μ_0 and R_0. A particular stress will be laid on analogies between classical and semiclassical potential theories.

THEOREM 3.1. *The function $Q_0(\cdot)$ has the following properties:*
1° it is non-negative and subharmonic on R^2, harmonic on $R^2 - B$;
2° $\lim_{|y| \to \infty}\left(Q_0(y) - \frac{1}{\pi}\log|y|\right) = R_0$;
3° $Q_0(\cdot)$ is continuous at each $y_0 \in B^*$ and

$$Q_0(y_0) = 0. \tag{3.1}$$

Proof. Only (3.1) needs to be proved. Let $y_0 \in B^*$. By Theorem 2.3, $\mu_{y_0} = \delta_{y_0}$, whence using (2.12) we obtain $Q_0(x, y_0) = Q_0(y_0)$ for all $x \in R^2$. Observe that there exists a point x such that $Q_0(x, y_0) = 0$ (in fact, (2.8) and (2.11) give $Q_0(\cdot, y_0) = 0$ a.e. on B). Thus $Q_0(y_0) = 0$.

THEOREM 3.2. *The function $Q_0(\cdot, \cdot)$ has the following properties:*
1° for any $x, y \in R^2$, $Q_0(x, y) \geqslant 0$ and $Q_0(x, y) = Q_0(y, x)$;
2° for every $x \in R^2$, $Q_0(x, \cdot)$ is subharmonic on $R^2 - \{x\}$ and superharmonic on $R^2 - B$;
3° for $y \in R^2$

$$Q_0(y) = \lim_{|x| \to \infty} Q_0(x, y);$$

4° for $x \in R^2$ and $y_0 \in B^*$

$$Q_0(x, y_0) = 0;$$

moreover, $Q_0(x, \cdot)$ is continuous at y_0.

Proof. The proof follows immediately from the assertions of Section 2.

It is suggested by Theorems 3.1, 3.2 and by (2.12) to call $Q_0(\cdot)$ and $Q_0(x, \cdot)$ the semiclassical Green function for $R^2 - B$ with the pole at infinity and the semiclassical Green function for $R^2 - B$ with the pole at x, respectively.

Now, for any Radon measure μ with compact support write

$$I(\mu, \mu) = \frac{1}{\pi} \int_{R^2} \int_{R^2} \log\frac{1}{|x-y|}\, \mu(dx)\, \mu(dy).$$

Moreover, assume

$$\mathscr{E}_B = \{\lambda\colon \lambda = \mu_1 - \mu_2,\ I(\mu_i, \mu_i) < +\infty,\ i = 1, 2 \text{ and support } \lambda \subset B\},$$
$$\mathscr{E}_B^* = \{\lambda \in \mathscr{E}_B\colon \lambda(\{y\colon Q_0(y) > 0\}) = 0\}.$$

The following theorem is due to Z. Ciesielski and M. Kac:

THEOREM 3.3. *The measure* μ_0 *is in* $\mathcal{E}_B^*$ *and*

$$R_0 = I(\mu_0, \mu_0) = \inf\{I(\lambda, \lambda):\ \lambda \in \mathcal{E}_B^*, \lambda(B) = 1\}.$$

Moreover, if $\lambda \in \mathcal{E}_B^*$, $\lambda(B) = 1$, $I(\lambda, \lambda) = I(\mu_0, \mu_0)$, *then* $\lambda = \mu_0$.

It is clear now that μ_0 and R_0 should be called the *semiclassical equilibrium distribution for* B and the *semiclassical Robin constant for* B, respectively.

Next, we turn to the discussion of μ_y, μ_0 and B^*. We are going just to state the results as they can be rather easily derived from the previous results, with the help of Proposition 1.6.

PROPOSITION 3.4. $\mu_y \Rightarrow \mu_0$, *as* $|y| \to +\infty$.

THEOREM 3.5. *All measures* μ_y *as well as* μ_0 *are concentrated on* B^*. *Moreover, if* $y \in \operatorname{Int} B$, *then*

$$\mu_y(\partial B) = \mu_0(\partial B) = 1.$$

THEOREM 3.6. *The following conditions are equivalent:*

1° $y_0 \in B^*$;

2° *for any* $f \in C(B)$, $D_f^B(y_0) = f(y_0)$;

3° $Q_0(x; y_0) = 0$ *for every* $x \in R^2 - \{y_0\}$;

4° *either* $y_0 \in \operatorname{Int} B$ *or, for any* $f \in C(B)$,

$$\lim_{\substack{y \to y_0 \\ y \in R^2 - B}} D_f^B(y) = f(y_0);$$

5° *either* $y_0 \in \operatorname{Int} B$ *or, for any* $x \in R^2$,

$$\lim_{\substack{y \to y_0 \\ y \in R^2 - B}} Q_0(x, y) = 0.$$

4. Analytic results. For $f \in L^2(B)$ write

$$G_B f(x) = \frac{1}{\pi} \int_B \log \frac{1}{|x-y|} f(y)\, dy.$$

It is well known that G_B: $L^2(B) \to L^2(B)$ is a self-adjoint, completely continuous operator but it need not to be positive definite. Combining M. Kac theorem [5] with J.L. Troutman's result [10], we see that if $R_0 \geqslant 0$, then all eigenvalues of G_B are positive but if $R_0 < 0$, then there exists exactly one simple negative eigenvalue of G_B and all other eigenvalues are positive.

Let $\{\varphi_m\}_{m=1,2\ldots}$ be an orthonormal set of all eigenfunctions of G_B and $\{\lambda_m\}_{m=1,2\ldots}$ be the corresponding eigenvalues.

THEOREM 4.1. *For* $u > 0$

$$\frac{1}{R(u)} = \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} (1, \varphi_m)^2, \tag{4.1}$$

$$Q(u; y) = R(u) \left[1 - \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} (1, \varphi_m) G_B \varphi_m(y)\right], \quad y \in R^2, \tag{4.2}$$

$$Q(u; x, y) = \frac{1}{\pi} \log \frac{1}{|x-y|} + \frac{1}{R(u)} Q(u; x) Q(u; y) - \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} G_B\varphi_m(x) G_B \varphi_m(y), \quad x, y \in R^2, x \neq y. \tag{4.3}$$

Proof. By (2.21), we have in $L^2(B)$

$$Q(u; \cdot) = R(u) \sum_{m=1}^{\infty} \frac{1}{1+u\lambda_m} (1, \varphi_m) \varphi_m. \tag{4.4}$$

Applying (2.22) we get (4.1). Furthermore, (4.4) gives

$$u G_B Q(u; y) = R(u) \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} (1, \varphi_m) G_B \varphi_m(y),$$

whence, by (2.21), we obtain (4.2).

Now, by (2.14) we have

$$\big(Q(u; \cdot, y), \varphi_m\big) = \frac{G_B \varphi_m(y) + (1, \varphi_m) Q(u, y)}{1+u\lambda_m},$$

whence

$$Q(u; \cdot, y) = \sum_{m=1}^{\infty} \frac{1}{1+u\lambda_m} G_B \varphi_m(y) \varphi_m + Q(u; y) \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} (1, \varphi_m) \varphi_m$$

in $L^2(B)$; therefore

$$\frac{u}{\pi} \int_B \log \frac{1}{|x-z|} Q(u; z, y)\, dz = \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} G_B \varphi_m(x) G_B \varphi_m(y) + Q(u; y) \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m} (1, \varphi_m) G_B \varphi_m(x).$$

Using (2.14) and (4.2) we get (4.3).

THEOREM 4.2. *The following formulas hold:*

$$\frac{1}{R_0} = \sum_{m=1}^{\infty} \frac{1}{\lambda_m} (1, \varphi_m)^2$$

(we accept the convention that $1/R_0 = \infty$ *for* $R_0 = 0$*);*

$$(4.5)\qquad Q_0(y) = R_0 - \lim_{u\to+\infty} R(u) \sum_{m=1}^{\infty} \frac{u}{1+u\lambda_m}(1, \varphi_m) G_B \varphi_m(y), \qquad y \in R^2;$$

$$Q_0(x, y) = \frac{1}{\pi}\log\frac{1}{|x-y|} + $$

$$+ \lim_{u\to+\infty}\left[\frac{1}{R(u)} Q(u; x) Q(u; y) - \sum_{m=1}^{\infty}\frac{u}{1+u\lambda_m} G_B\varphi_m(x) G_B\varphi_m(y)\right], \qquad x \neq y.$$

Moreover, for $f \in C(B)$

$$(4.6)\qquad \mu_0(f) = \lim_{u\to+\infty} R(u) \sum_{m=1}^{\infty}\frac{u}{1+u\lambda_m}(1, \varphi_m)(f, \varphi_m)$$

and, provided $R_0 \neq 0$,

$$(4.7)\qquad D_f^B(y) = \mu_0(f)\frac{Q_0(y)}{R_0} + \lim_{u\to+\infty}\sum_{m=1}^{\infty}\frac{u}{1+u\lambda_m}(f, \varphi_m) G_B\varphi_m(y), \qquad y \in R^2.$$

Proof. This theorem follows immediately from Theorem 4.1 and from the following equalities:

$$\mu_0(f) = \lim_{u\to+\infty} u \int_B Q(u; y) f(y)\, dy,$$

$$D_f^B(y) = \lim_{u\to+\infty} u \int_B f(x) Q(u; x, y)\, dx$$

(cf. Theorems 2.3 and 2.5).

COROLLARY 4.3. *The semiclassical Robin constant of* B *is zero if and only if*

$$\sum_{m=1}^{\infty}\frac{1}{\lambda_m}(1, \varphi_m)^2 = +\infty.$$

It is possible now to obtain one more characterization of the set B^* provided $R_0 \neq 0$.

THEOREM 4.4. *Assume* $R_0 \neq 0$. *If* $f \in C(B)$, *then*

$$(4.8)\qquad f(y) = \lim_{u\to+\infty}\sum_{m=1}^{\infty}\frac{u_m}{1+u\lambda_m}(f, \varphi_m)\varphi_m(y)$$

at every point $y \in B^*$. *Conversely, if* (4.8) *holds for all functions* $f \in C(B)$, *then* $y \in B^*$.

Proof. The first part of the theorem follows directly from (4.7) and (3.1).

For the proof of the second part, notice that putting $f \equiv 1$ in (4.8) and then applying (4.5), we obtain $Q_0(y) = 0$. Thus, by (4.7) and (4.8)

$$D_f^B(y) = f(y), \qquad f \in C(B),$$

Now, to complete the proof, it suffices to apply Theorem 3.6.

Remark. The assumption $R_0 \neq 0$ cannot be dropped. It can be verified (Z. Ciesielski and M. Kac) that for the unit disc and $f \equiv 1$ formula (4.8) does not hold for all points of the boundary.

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