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A note on rearrangements of series

by

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Let

$$\sum_{n=1}^{\infty} u_n \tag{1}$$

be a convergent series of elements of a Hilbert space H. Let U be the set of elements x of H for which there exists a rearrangement $\sum u_n'$ of (1) such that $\sum u_n'$ converges to x. Let S be the set of elements of H, for which there exists a rearrangement $\sum u_n'$ of (1) such that some subsequence of partial sums of $\sum u_n'$ converges to x. If H is finite-dimensional, it was shown by Steinitz in [2] that $U = S$. Hadwiger in [1] showed that $U \subset S$ properly when H is infinite-dimensional. The object of this note is to show that $U = S$ in case (1) has the property $\sum \|u_n\|^2 < \infty$. More precisely:

THEOREM. *Let u_n be a sequence of elements in a real Hilbert space H such that*

(A) $\sum \|u_n\|^2 < \infty$,

(B) $\lim_{k\to\infty} (u_1 + u_2 + \ldots + u_{n_k}) = x$ *for some increasing sequence of integers* $\{n_k\}$.

Then the series $\sum u_n$ can be rearranged to converge to x.

The proof is based on the following

LEMMA. *Let $u_1, u_2, \ldots, u_n$ be elements of a real Hilbert space H and let $a = u_1 + \ldots + u_n$. Then $u_1, u_2, \ldots, u_n$ can be rearranged in a sequence $u_1', u_2', \ldots, u_n'$ such that for $p = 1, 2, \ldots, n$*

$$\|u_1' + u_2' + \ldots + u_p'\| \leqslant \|a\| + \Big(\sum_1^p \|u_i'\|^2 + \|a\|(\|a\| + 2M)\Big)^{1/2}, \tag{2}$$

where $M = \max(\|u_1\|, \|u_2\|, \ldots, \|u_n\|)$.

Proof. We assume first that $a = 0$. Since the case $n = 1$ presents no difficulty, we may assume $n \geqslant 2$. Let $u_1' = u_1$. Since $a = 0$, this means that

$$0 = (a, u_1') = \sum_1^n (u_i, u_1') = (u_1', u_1') + \sum_2^n (u_i, u_1'), \tag{3}$$

where (a, b) denotes the real inner product of the space H. Since $(u'_1, u'_1) \geqslant 0$, (3) shows that for some $i \geqslant 2$, $(u_i, u'_1) \leqslant 0$. Let u'_2 be such an element; then

$$(4) \qquad \|u'_1+u'_2\|^2 = (u'_1+u'_2,\ u'_1+u'_2)$$
$$= \|u'_1\|^2+2(u'_1, u'_2)+\|u'_1\|^2 \leqslant \|u'_1\|^2+\|u'_2\|^2.$$

Writing next

$$0 = u'+u'+\sum_{u_i \neq u'_1, u'_2} u_i,$$

we get

$$0 = (u'_1+u'_2,\ u'_1+u'_2)+\sum_{u_i \neq u'_1, u'_2} (u_i,\ u'_1+u'_2)$$

and since $(u'_1+u'_2, u'_1+u'_2) \geqslant 0$, for some $u_i \neq u'_1, u'_2$, we must have $(u'_1+u'_2, u_i) \leqslant 0$. Denote such u_i by u'_3. Then

$$\|u'_1+u'_2+u'_3\|^2 = (u'_1+u'_2+u'_3,\ u'_1+u'_2+u'_3)$$
$$= \|u'_1+u'_2\|^2+\|u'_3\|^2+2(u'_1+u'_2,\ u'_3) \leqslant \|u'_1\|^2+\|u'_2\|^2+\|u'_3\|^2.$$

Proceeding in this manner we get the desired result. Assume finally that $a \neq 0$.

Applying the case $a = 0$ to the sequence $\left\{u_i - \frac{1}{n}a\right\}$ we can rearrange the u's so that

$$\left\| \sum_1^p \left(u'_i - \frac{1}{n}a\right)\right\| \leqslant \left(\sum_1^p \left\| u'_1, \frac{1}{n}a\right\|^2\right)^{1/2}.$$

But then

$$\left\|\sum_1^p u'_1\right\| \leqslant \|a\| + \left\|\sum_1^p \left(u'_i - \frac{1}{n}a\right)\right\| \leqslant \|a\| + \left(\sum_1^p \left\| u'_i - \frac{1}{n}a\right\|^2\right)^{1/2}$$
$$\leqslant \|a\| + \left(\sum_1^p \|u_i\|^2 + \|a\|(\|a\|+2M)\right)^{1/2},$$

q.e.d.

The proof of the theorem is now immediate. Let

$$S_k = u_1+\ldots+u_k,$$
$$M_k = \sup\{\|u_i\|:\ i \geqslant k\}.$$

By the lemma, the terms $\{u_{n_k+1}, \ldots, u_{n_{k+1}}\}$ can be rearranged into $\{u'_{n_k+1}, \ldots, u'_{n_{k+1}}\}$ so that if $t_k = S_{n_{k+1}} - S_{n_k}$

$$\left\|\sum_{n_k+1}^{p} u'_i\right\| \leqslant t_k + \left(\sum_{n_k+1}^{p} \|u'_i\|^2 + t_k(t_k+2M_{n_k})\right)^{1/2}.$$

Since the right-hand side tends to 0 independently of what p is, and S_{n_k} converges to x, we get the result.

References

[1] H. Hadwiger, *Über das Umordnungsproblem im Hilbertschen Raum*, Math. Zeit. 46 (1940), p. 70-79.

[2] E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, Jour. reine u. angew. Math. 143 (1913), p. 128-175.

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