A note on rearrangements of series

by

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Let

\[ \sum_{n=1}^{\infty} u_n \]

be a convergent series of elements of a Hilbert space \( H \). Let \( U \) be the set of elements \( x \) of \( H \) for which there exists a rearrangement \( \sum u'_n \) of (1) such that \( \sum u'_n \) converges to \( x \). Let \( S \) be the set of elements of \( H \), for which there exists a rearrangement \( \sum u'_n \) of (1) such that some subsequence of partial sums of \( \sum u'_n \) converges to \( x \). If \( H \) is finite-dimensional, it was shown by Steinhaus in [2] that \( U = S \). Hadwiger in [1] showed that \( U = S \) properly when \( H \) is infinite-dimensional. The object of this note is to show that \( U = S \) in case (1) has the property \( \sum ||u_n|| < \infty \). More precisely:

**Theorem.** Let \( u_n \) be a sequence of elements in a real Hilbert space \( H \) such that

(A) \( \sum ||u_n|| < \infty \),

(B) \( \lim_{k \to \infty} (u_1 + u_2 + \ldots + u_n) = x \) for some increasing sequence of integers \( \{n_k\} \).

Then the series \( \sum u_n \) can be rearranged to converge to \( x \).

The proof is based on the following

**Lemma.** Let \( u_1, u_2, \ldots, u_n \) be elements of a real Hilbert space \( H \) and let \( a = u_1 + \ldots + u_n \). Then \( u_1, u_2, \ldots, u_n \) can be rearranged in a sequence \( u'_1, u'_2, \ldots, u'_n \) such that for \( p = 1, 2, \ldots, n \)

\[ ||u'_1 + u'_2 + \ldots + u'_p|| \leq ||a|| + \left( \frac{3}{2} ||u'_1||^2 + ||a|| (||a|| + 2M)^{1/2} \right), \]

where \( M = \max(||u'_1||, ||u'_2||, \ldots, ||u'_n||) \).

**Proof.** We assume first that \( a = 0 \). Since the case \( a \neq 1 \) presents no difficulty, we may assume \( n \geq 2 \). Let \( u'_1 = u_1 \). Since \( a = 0 \) this means that

\[ 0 = (a, u'_1) = \sum_{n=1}^{\infty} (u_n, u'_1) = (u'_1, u'_1) + \sum_{n=2}^{\infty} (u_n, u'_1), \]
where \((a, b)\) denotes the real inner product of the space \(H\). Since \((u_i', u_i') \geq 0\), (3) shows that for some \(i \geq 2\), \((u_i, u_i') \leq 0\). Let \(w_i'\) be such an element; then

\[
[(u_i' + w_i')^2 = (u_i' + u_i', u_i' + w_i') = ||u_i'||^2 + 2(u_i', w_i') + ||w_i'||^2 < ||u_i'||^2 + ||w_i'||^2.
\]

Writing next

\[
0 = a' + a'' + \sum_{i \neq \alpha \in I} w_i,
\]

we get

\[
0 = (u_i' + w_i', u_i' + w_i') + \sum_{i \neq \alpha \in I} (u_i, u_i' + w_i')
\]

and since \((u_i' + w_i', u_i' + w_i') > 0\), for some \(u_i \neq u_i', w_i'\), we must have \((u_i' + w_i', u_i) \leq 0\). Denote such \(u_i\) by \(w_i\). Then

\[
||(u_i' + w_i')^2 = (u_i' + w_i', u_i' + w_i' + u_i + u_i') = ||u_i'||^2 + ||w_i'||^2 + 2(u_i', w_i' + u_i) \leq ||u_i'||^2 + ||w_i'||^2 + ||w_i'||^2.
\]

Proceeding in this manner we get the desired result. Assume finally that \(a \neq 0\).

Applying the case \(a = 0\) to the sequence \(\{u_i - \frac{1}{n} a\}\) we can rearrange the \(w_i\) so that

\[
\left\| \sum_{i} \left( u_i - \frac{1}{n} a \right) \right\| < \left\| \sum_{i} \left( u_i - \frac{1}{n} a \right) \right\|^{1/2}.
\]

But then

\[
\left\| \sum_{i} u_i \right\| < ||a|| + \left\| \sum_{i} \left( u_i - \frac{1}{n} a \right) \right\| < ||a|| + \left( \sum_{i} ||u_i||^2 + ||a||^2 \right)^{1/2} \leq ||a|| + \left( \sum_{i} ||u_i||^2 + ||a||^2 \right)^{1/2},
\]

q.e.d.

The proof of the theorem is now immediate. Let

\[
S_k = u_1 + \ldots + u_k,
\]

\[
M_k = \sup \{ ||u_i|| : i \geq k \}.
\]