STUDIA MATHEMATICA, T. XXXV. (1970)

Ideals in subalgebras of the group algebras

by

LEONARD Y. H. YAP (Houston)

1. Introduction. Throughout this note G denotes a locally compact Abelian group. Let p be a real number such that $1 \leqslant p < \infty$, and let $L_p(G)$ denote the usual Lebesgue class with respect to λ, the Haar measure of G. Write $L_1(G) \cap L_p(G)$ as $A_p(G)$, and for $f \in A_p(G)$, define $|||f|||_p = \|f\|_1 + \|f\|_p$. It is easy (see (2.1) below) to verify that $A_p(G)$ is a Banach algebra with respect to $|||\cdot|||_p$ (multiplication in $A_p(G)$ is the usual convolution). It is plain that if $p = 1$ (and G is arbitrary) or G is discrete (and $1 \leqslant p < \infty$), then $A_p(G)$ is precisely the group algebra $L_1(G)$ and $|||\cdot|||_p$ is equivalent to $\|\cdot\|_1$; if G is compact, then $A_p(G) = L_p(G)$ and $|||\cdot|||_p$ is equivalent to $\|\cdot\|_p$. The purpose of this note is to present various properties of the algebra $A_p(G)$. Roughly speaking, our results say that some of the important known results of $L_1(G)$ can be extended to $A_p(G)$ while at the same time $A_p(G)$ lacks some of the useful properties possessed by $L_1(G)$. We have been motivated by the interesting papers of Porcelli-Collins [9, 10], Warner [15], and our earlier considerations of non-factorization theorems in [16]. The relationships of various results will be pointed out at the appropriate places.

Now we give a brief summary of the main results in the individual sections. In Section 2 we consider the factorization property of $A_p(G)$ (an algebra A is said to have the *factorization property* if every element in A can be written as $x \cdot y$ with x, y in A) and it is proved that $A_p(G)$ has the factorization property if and only if $p = 1$ or G is discrete; $A_p(G)$ has a bounded approximate unit if and only if $p = 1$ or G is discrete. In Section 3 we prove that $A_p(G)$ is a regular semi-simple Banach algebra satisfying Ditkin's condition D (as defined in Loomis [7]) and the general Tauberian theorem for $A_p(G)$: Let I be a closed ideal in $A_p(G)$, then I contains every element f in kernel (hull (I)) such that the set [boundary (hull (f)) $\cap$ hull (I)] contains no non-void perfect set. Warner [15] proves the results of Section 3 for $p = 2$. In Section 4 we prove that

(i) every maximal ideal in $A_p(G)$ is regular (closed) if and only if $p=1$ or G is discrete;

(ii) every positive functional on $A_p(G)$ is continuous if and only if $p=1$ or G is discrete;

(iii) if I is a proper prime ideal in $A_p(G)$, then I is regular maximal if and only if I is closed;

(iv) $A_p(G)$ contains a non-closed prime ideal if and only if G is infinite;

(v) every prime ideal of $A_p(G)$ is contained in a unique regular maximal ideal if and only if G is discrete.

Thus the results in Section 4 are either extensions or "counter-examples" of the corresponding results in Porcelli-Collins [9, 10] and they answer the $A_p(G)$ version of the two questions (see [9]) raised at a recent international symposium held in Sopot, Poland.

2. Factorization problems in $A_p(G)$. We begin with a very simple fact:

(2.1) THEOREM. *The function $|||\cdot|||_p$ is a norm for the linear space $A_p(G)$ and $A_p(G)$ is a Banach algebra with respect to $|||\cdot|||_p$ if multiplication in $A_p(G)$ is the usual convolution of functions.*

Proof. That $|||\cdot|||_p$ is a norm is obvious, while completeness easily follows from the definition of $|||\cdot|||_p$, the completeness of $L_r(G)$ $(1\leqslant r<\infty)$ and the fact: $\|f_n-f\|_r\to 0$ implies that $f_{n_k}\to f$ a.e. for some subsequence (f_{n_k}) of (f_n). Finally, to complete the proof, recall that $\|f*g\|_p\leqslant \|f\|_1\cdot\|g\|_p$ for all $f\in L_1(G)$ and $g\in L_p(G)$. Using this fact we immediately have $|||f*g|||_p\leqslant |||f|||_p\cdot|||g|||_p$ for all f,g in $A_p(G)$.

For the convenience of the reader, we now review briefly what we need from the theory of $L(p,q)$-spaces.

(2.2) DEFINITION. Let f be a complex-valued measurable function defined on (G,λ), where λ is the Haar measure of G. For $y\geqslant 0$, we define

$$m(f,y)=\lambda\{x\in G: |f(x)|>y\}.$$

Note that $m(f,\cdot)$ is a non-increasing, right-continuous function defined on $[0,\infty)$. For $x\geqslant 0$, we define

$$\begin{aligned} f^*(x)&=\inf\{y: y>0 \text{ and } m(f,y)\leqslant x\}\\ &=\sup\{y: y>0 \text{ and } m(f,y)>x\},\end{aligned}$$

with the conventions $\inf\emptyset=\infty$ and $\sup\emptyset=0$. We note that f^* is a non-increasing, right-continuous function. For $x>0$, we write

$$f^{**}(x)=\frac{1}{x}\int_0^x f^*(t)\,dt,$$

and let

$$\|f\|_{(p,q)}=\left\{\int_0^\infty [x^{1/p}f^{**}(x)]^q\frac{dx}{x}\right\}^{1/q},$$

where $1<p<\infty$, $1\leqslant q<\infty$. We say that $f\in L(p,q)(G)$ if $\|f\|_{(p,q)}<\infty$. A theorem of Hardy (see [17], p. 20) shows that (the case $q=1$ is obtained by passing to the limit)

$$\|f\|_{(p,q)}\leqslant p'\left\{\int_0^\infty [x^{1/p}f^*(x)]^q\frac{dx}{x}\right\}^{1/q}\leqslant p'\|f\|_{(p,q)},$$

where $1/p+1/p'=1$. Hardy's theorem can also be used to show that $\|f\|_p\leqslant\|f\|_{(p,p)}\leqslant p'\|f\|_p$, so that $L(p,p)=L_p$. The following fact (which is a special case of ([8], (2,6)) will be useful to us later. A simple proof is given in [16], (2.2).

(2.3) THEOREM. *If p,r,s are real numbers such that $1<r,s<\infty$, $1/r+1/s>1$ and $1/p=1/r+1/s-1$, then*

$$L_r(G)*L_s(G)\subseteq L(p,1)(G).$$

Recall that an algebra A is said to have the *factorization property* if $A\cdot A=A$. $A\cdot A$ will be written as A^2 throughout the rest of this note.

(2.4) THEOREM. *The Banach algebra $A_p(G)$ has the factorization property $\Leftrightarrow p=1$ (and G is arbitrary) or G is discrete (and $1\leqslant p<\infty$).*

Proof. The implication $\Leftarrow$ is the well-known factorization theorem of P.J. Cohen (see, for example, Cohen [1], Hewitt [4], Koosis [6]).

Next we suppose that $1<p<\infty$ and G is non-discrete. We will construct a function F in $A_p(G)$ such that F is *not* in $(A_p(G))^2$. Define $r=s=2p/(1+p)$ so that $1<r=s<\infty$, $r<p$, and $1/p=1/r+1/s-1$. By (2.3), we have

(a) $$(A_p(G))^2\subseteq A_r(G)*A_s(G)\subseteq L(p,1)(G).$$

Thus it suffices to define a function F in $A_p(G)$ such that F is not in $L(p,1)(G)$. Define $\beta=1/(2p)+1/2$, and observe that $0<\beta<1$ and $\beta p>1$. Next we choose a positive integer n_0 such that $n_0>e^{p\beta}$ and then choose a sequence $(V_n)_{n=n_0}^\infty$ of pairwise disjoint Haar measurable subsets of G such that

$$\lambda(V_n)=\frac{1}{n(n+1)},\qquad n=n_0,n_0+1,\ldots$$

Write

$$F=\sum_{n=n_0}^\infty a_n\xi_{V_n},$$

where $a_n = n^{1/p}(\log n)^{-\beta}$, and ξ_E denotes the characteristic function of E. It is clear that $F \in A_p(G)$ and it remains to show that

(i)
$$\int_0^\infty [x^{1/p}F^*(x)]\frac{dx}{x} = \infty.$$

But straight-forward computations show that

$$m(F, y) = \begin{cases} 1/n_0, & \text{if } 0 \leqslant y < a_{n_0}, \\ 1/(n+1), & \text{if } a_n \leqslant y < a_{n+1}, \end{cases}$$

(ii)
$$F^*(x) = a_n \quad \text{if } 1/(n+1) \leqslant x < 1/n.$$

Finally, an easy calculation (using (ii) and $\sum n^{-1}(\log n)^{-\beta} = \infty$) gives (i) immediately.

(2.5) Corollary. *The Banach algebra $A_p(G)$ has a bounded approximate unit $\Leftrightarrow p = 1$ or G is discrete.*

Proof. The implication $\Leftarrow$ is of course well-known. The implication $\Rightarrow$ immediately follows from the preceding theorem and Hewitt's factorization theorem [4]: let A be a Banach algebra and let V be a Banach A-module (that is, V is an A-module in the algebraic sense and $\|av\| \leqslant \|a\|\cdot\|v\|$ for all $a \in A$, $v \in V$) and suppose that there is a constant $M > 0$ such that for $a \in A$, $v \in V$ and $\varepsilon > 0$, there exists $e \in A$ such that $\|e\| \leqslant M$, $\|a - ae\| < \varepsilon$ and $\|v - ev\| < \varepsilon$. Then $A \cdot V = V$.

(2.6) Remark. Hewitt's factorization theorem implies that $L_1(G) * A_p(G) = A_p(G)$ for all G and $1 \leqslant p < \infty$. It is easy to verify that if A and V are as in Hewitt's theorem and suppose that A_0 and V_0 are dense subsets of A and V, respectively, then $A_0 \cdot V_0$ is dense in $A \cdot V = V$. Hence $(C_{00}(G))^2$ is dense in $A_p(G)$, where $C_{00}(G)$ denotes the set of all continuous functions defined on G with compact supports.

We need the following corollary in Section 4:

(2.7) Corollary. *If $1 < p < \infty$ and G is non-discrete, then $(A_p(G))^2$ is a dense proper subset of $A_p(G)$.*

The situation for unbounded approximate unit is much more pleasant. Let $\mathscr{V}$ denote the family of all precompact neighborhoods of 0, the identity element of G. Partially order $\mathscr{V}$ by set inclusion and denote it by $\{V_d\}$. Then $\{V_d\}$ is a directed family, and for each V_d, choose a non-negative continuous function v_d with support $(v_d) \subset V_d$ and $\int v_d\, d\lambda = 1$. Then we have (see Loomis [7], p. 124) the following which is needed in the next section:

(2.8) Theorem. *The net $\{v_d\}$ is an (probably unbounded) approximate unit for $A_p(G)$.*

(2.9) Remarks. (a) We will have occasions to apply the above results in Section 4 to obtain both "positive" and "negative" theorems related to those in Porcelli-Collins [9, 10].

(b) The usefulness of bounded approximate units is of course well known and many interesting theorems have been obtained for algebras with bounded approximate units. For more recent examples, see Varopoulos [14], Rieffel [12], and Porcelli-Collins [10], Theorem 1.

3. Tauberian theorems for $A_p(G)$. We continue to use G to denote a locally compact Abelian group with character group Γ. *The Haar measure of Γ is denoted by μ.* All terms and notation not explained here are as in Loomis [7].

(3.1) Theorem. *The maximal ideal space $M_p(G)$ of the commutative Banach algebra $A_p(G)$ can be identified with Γ.*

Proof. For $0 \neq f \in A_p(G)$, we have

$$|||f^n|||_p \leqslant \|f^{n-1}\|_1 \cdot \|f\|_1 + \|f^{n-1}\|_1 \|f\|_p \leqslant \|f\|_1^{n-1}\, |||f|||_p.$$

Hence

$$\lim_{n\to\infty} |||f^n|||_p^{1/n} \leqslant \lim_{n\to\infty} \|f\|_1^{(n-1)/n} \cdot |||f|||_p^{1/n} \leqslant \|f\|_1.$$

Therefore

$$\lim |||f^n|||_p^{1/n} \leqslant \|f\|_1 \quad \text{for all } f \in A_p(G).$$

Now if $\gamma \in M_p(G)$, then

$$|\gamma(f)|^n = |\gamma(f^n)| \leqslant |||f^n|||_p.$$

Hence

$$|\gamma(f)| \leqslant \lim_{n\to\infty} |||f^n|||_p^{1/n} \leqslant \|f\|_1,$$

and so γ is $\|\cdot\|_1$-bounded and hence can be extended in a unique fashion to a multiplicative linear functional γ_1 on $L_1(G)$ and, conversely, every multiplicative linear functional γ_1 on $L_1(G)$ determines a $\gamma \in M_p(G)$. Now recall that the maximal ideal space of $L_1(G)$ is Γ and observe that the Gelfand topology on $M_p(G)$ agrees with the usual topology of Γ.

The next two lemmas are extensions of (and will be substitutes for) the corresponding results in Rudin [13], 2.6.1 and 2.6.2, and will play the same role.

(3.2) Lemma. *Suppose C is a compact subset of Γ, $V \subset \Gamma$, and $0 < \mu(V) < \infty$, $\mu(C - V) < \infty$, where μ is the Haar measure of Γ. Then there exists $k \in A_p(G)$ such that*

(a) $\hat{k} \equiv 1$ *on* C, $\hat{k} \equiv 0$ *outside of* $C + V - V$ *and* $0 \leqslant \hat{k} \leqslant 1$;

(b) $|||k|||_p \leqslant (\mu(C - V)/\mu(V))^{1/2} + (\mu(C - V)^{1-1/(2p)})/\mu(V)^{1/(2p)}$.

Proof. Let g and h be the inverse Plancherel transforms of the characteristic functions of V and $C-V$, respectively, and define

$$k(x) = \mu(V)^{-1}g(x)h(x) \qquad (x \in G).$$

Now by Rudin [13], p. 49, we see that $k \in L_1(G)$ and it satisfies condition (a), as well as

$$\|k\|_1 \leqslant \big(\mu(C-V)/\mu(V)\big)^{1/2}.$$

But k is also in $L_p(G)$ (because $\hat{k} = \mu(V)^{-1}\hat{g} * \hat{h}$; see Edwards [3], 10.4.7, if necessary) and

$$\begin{aligned}\|k\|_p &= \mu(V)^{-1}\|gh\|_p = \mu(V)^{-1}\|gh\|_\infty^{1-1/p}\|gh\|_1^{1/p}\\ &\leqslant \mu(V)^{-1}(\|g\|_\infty\|h\|_\infty)^{1-1/p}\|g\|_2^{1/p}\|h\|_2^{1/p}\\ &\leqslant \mu(V)^{-1}(\|\hat{g}\|_1\|\hat{h}\|_1)^{1-1/p}\|g\|_2^{1/p}\cdot\|h\|_2^{1/p}\\ &\leqslant \mu(C-V)^{1-1/2p}\cdot\mu(V)^{-1/2p}.\end{aligned}$$

(3.3) Theorem. *If W is an open set in Γ which contains a compact set C, then there exists f in $A_p(G)$ such that $\hat{f} \equiv 1$ on C and $\hat{f} \equiv 0$ outside of W.*

Proof. Choose a neighborhood V of 0 in Γ such that $C+V-V \subset W$, $\mu(C-V) < \infty$ and then apply (3.2).

(3.4) Corollary. *The commutative Banach algebra $A_p(G)$ is a regular semi-simple Banach algebra.*

Proof. Immediate from (3.3) and Loomis [7], p. 57.

The main result in this section is the following Tauberian theorem. Some applications of this theorem will appear in the next section.

(3.5) Tauberian Theorem [1]. *Let I be a closed ideal in $A_p(G)$. Then I contains every element f in kernel (hull(I)) such that the intersection of the boundary of hull(f) with hull(I) contains no non-void perfect set.*

In view of Theorem 25F in Loomis [7], p. 86, and Corollary (3.4), it suffices to show that the algebra $A_p(G)$ satisfies Ditkin's condition D: If $f \in A_p(G)$ and $\gamma \in \Gamma$ such that $f(\gamma) = 0$, then there exists a sequence (f_n) in $A_p(G)$ such that $\hat{f}_n \equiv 0$ on some neighborhood V_n of γ and $|||f*f_n-f|||_p \to 0$; if Γ is non-compact, the condition must also be satisfied for the point at infinity, that is, for each f in $A_p(G)$, there exists a sequence (f_n) in $A_p(G)$ such that $\hat{f}_n$ has compact support and $|||f*f_n-f|||_p \to 0$. The case $p=1$ is of course well known (see, for example, [7]) and the standard proof makes use of the bounded approximate units in $A_1(G)$. Warner [15] has proved it for $p=2$ and also observed (in our notation) that

(1) Professor Edwin Hewitt has informed me (November 1968) that he and Professor K.A. Ross have also obtained (jointly) this theorem.

$A_p(G)$ lacks a bounded approximate unit if G is neither compact nor discrete and $1 < p \leqslant 2$. The following lemmas, which lead to a proof that $A_p(G)$ satisfies Ditkin's condition D, are modifications of the corresponding lemmas in Warner's paper. We have included all the details for the convenience of the reader.

(3.6) Lemma. *The algebra $A_p(G)$ satisfies Ditkin's condition D at infinity.*

Proof. It suffices to show that for $0 \neq f \in A_p(G)$ and $\varepsilon > 0$, there exists a function $g \in A_p(G)$ such that $\hat{g}$ has compact support and $|||g*f-f|||_p < \varepsilon$. First choose a function h (from a possibly unbounded approximate unit (see (2.8)) in $A_p(G)$ such that

$$|||f*h-f|||_p < \varepsilon/2.$$

Now by 2.6.6 of Rudin [13], there exists a k in $L_1(G)$ such that $\hat{k}$ has compact support and

$$\|h*k-h\|_1 < \varepsilon/2\,|||f|||_p.$$

Note that $g = h*k$ is in $A_p(G)$ and since $g*f-f = f*(g-h)+f*h-f$, we obtain $|||g*f-f|||_p < \varepsilon$ and note that $\hat{g} = \hat{h}\hat{k}$ has compact support.

Let $\mathscr{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be the family of all symmetric neighborhoods of the identity element 0 in Γ of measure $\leqslant 1$. Then $\mathscr{U}$ is a directed family under set inclusion. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ denote any net of symmetric precompact neighborhoods of 0 satisfying the following conditions:

(i) given U_λ in $\mathscr{U}$, $\bar{V}_\lambda \subseteq U_\lambda$ and $\mu(U_\lambda) < 4\mu(V_\lambda)$ (μ is the Haar measure on Γ);

(ii) given $U_\lambda \in \mathscr{U}$ and V_λ, there is a neighborhood W_λ of 0 such that $V_\lambda + W_\lambda \subseteq U_\lambda$.

(3.7) Lemma. *There exists a net $(k_\lambda)_{\lambda\in\Lambda}$ in $A_p(G)$ such that, for each $\lambda \in \Lambda$, we have* (a) $|||k_\lambda||| \leqslant 6$; (b) $\hat{k}_\lambda \equiv 1$ *on some neighborhood W_λ of 0.*

Proof. For given U_λ in $\mathscr{U}$, let V_λ be the corresponding set as above. Let g_λ, h_λ be the inverse Plancherel transforms of the characteristic functions of U_λ and V_λ, respectively. Defining $k_\lambda = \mu(V_\lambda)^{-1}g_\lambda h_\lambda$ and comparing k_λ with the function k defined in the proof of (3.2), we see (by (3.2.(b))) that $|||k_\lambda|||_p \leqslant 6$. To show (b), let W_λ be the neighborhood corresponding to U_λ and V_λ as described in (ii) above, and then use $\hat{k}_\lambda = \mu(V_\lambda)^{-1}\hat{g}_\lambda * \hat{h}_\lambda$ to see that $\hat{k}_\lambda \equiv 1$ on W_λ.

(3.8) Lemma. *For any compact set $C \subset G$ and $\varepsilon > 0$, there exists λ_0 in Λ such that if k is in $\{k_\lambda \mid \lambda > \lambda_0\}$, then $|||k-k_s|||_p < \varepsilon$ for every $s \in C$.*

Proof. Recall that the set $U(C, \delta) = \{\gamma \in \Gamma \mid |(x, \gamma)-1| < \delta$ for all $x \in C\}$ is a neighborhood of 0 in Γ ($\delta > 0$). Hence there exists a λ_0 such that if $\lambda > \lambda_0$, then $U_\lambda^2 \subseteq U(C, \delta)$, where $\delta = \min(\varepsilon/4, (\varepsilon/4)^p)$. Now let $k \in \{k_\lambda | \lambda > \lambda_0\}$ and suppose $k = \mu(V)^{-1} gh$, where g, h, V correspond to $g_\lambda, h_\lambda, V_\lambda$, respectively, in the proof of the preceding lemma. It follows that $\hat{k} \equiv 0$ outside of UV. It remains to show that $|||k - k_s|||_p < \varepsilon$ for every $s \in C$. Observe that

$$\|k - k_s\|_2^2 = \|\hat{k} - \hat{k}_s\|_2^2 = \int_{UV} |k(\gamma)|^2 \, |1 - (s, \gamma)|^2 d\gamma < \delta^2.$$

Thus $\|k - k_s\|_2 < \delta$ and similarly $\|g - g_s\|_2 < \delta\mu(U)^{1/2}$, $\|h - h_s\|_2 < \delta\mu(V)^{1/2}$. Hence

$$\begin{aligned}\|k - k_s\|_1 &\leqslant \mu(V)^{-1}\{\|g(h - h_s)\|_1 + \|h_s(g - g_s)\|_1\} \\ &\leqslant \mu(V)^{-1}\{\|g\|_2 \|h - h_s\|_2 + \|h_s\|_2 \|g - g_s\|_2\} \\ &\leqslant \delta\big(\mu(U)/\mu(V)\big)^{1/2} < 2\delta.\end{aligned}$$

Next we compute $\|k - k_s\|_p$ for $s \in C$. We have

$$\begin{aligned}\|k - k_s\|_p^p &\leqslant \|k - k_s\|_\infty^{p-1} \cdot \|k - k_s\|_1 \leqslant 2^{p-1} \|k\|_\infty^{p-1} \cdot 2\delta \\ &\leqslant 2^p \delta \|\hat{k}\|_1^{p-1} = 2^p \delta,\end{aligned}$$

since $\hat{k} = \mu(V)^{-1} \xi_U * \xi_V$ and the L_1-norm of $\xi_U * \xi_V$ is $\mu(U) \cdot \mu(V)$. Finally,

$$|||k - k_s|||_p \leqslant 2\delta + 2\delta^{1/p} < 2 \cdot (\varepsilon/4) + 2(\varepsilon/4) = \varepsilon.$$

(3.9) Corollary. *If $f \in A_p(G)$ and $\hat{f}(0) = 0$, then* $\lim |||f * k_\lambda|||_p = 0$.

Proof. Let $\delta > 0$ be given. Choose symmetric compact $C \subseteq G$ such that

$$\int_{C'} |f(x)| \, dx < \frac{\delta}{24}.$$

Put $\varepsilon = \delta/3\|f\|_1$ and choose λ_0 so that if k is in $\{k_\lambda | \lambda > \lambda_0\}$, then $|||k - k_s|||_p < \varepsilon$ for $s \in C$. Hence

$$\begin{aligned}(f * k)(t) &= \int f(s)\, k(t - s)\, ds - \hat{f}(0)\, k(t) \\ &= \int f(s)\, [k_{-s}(t) - k(t)]\, ds.\end{aligned}$$

Therefore

$$\begin{aligned}\|f * k\|_p &= \{\int_G |\int_G f(s)\, [k_{-s}(t) - k(t)]\, ds|^p \, dt\}^{1/p} \\ &\leqslant \int_G \{\int_G |f(s)[k_{-s}(t) - k(t)]|^p \, dt\}^{1/p} ds \qquad \text{(by [2], p. 530)} \\ &= \int_G |f(s)| \cdot \|k_{-s} - k\|_p \, ds.\end{aligned}$$

Similarly,

$$\|f * k\|_1 \leqslant \int_G |f(s)| \, \|k_{-s} - k\|_1 ds.$$

Therefore

$$(*) \qquad |||f * k|||_p \leqslant \int_C |f(s)| \cdot |||k_{-s} - k|||_p \, ds + \int_{C'} |f(s)| \cdot |||k_{-s} - k|||_p \, ds.$$

Now if $s \in C$, then $|||k_{-s} - k|||_p < \varepsilon$, therefore $(*)$ implies

$$|||f * k|||_p < \varepsilon \|f\|_1 + 2\,|||k|||_p \cdot \delta/24 < \delta.$$

(3.10) Theorem. *$A_p(G)$ satisfies Ditkin's condition* D *at all (finite) points.*

Proof. It suffices to show that $A_p(G)$ satisfies condition D at the identity element 0 of Γ. Let $\{v_d\}$ be the (probably unbounded) approximate unit described in (2.8) and let $(k_\lambda)_{\lambda \in \Lambda}$ be as in Lemma (3.7). Write

$$v_{(d,\lambda)} = v_d - k_\lambda * v_d.$$

Clearly, $v_{(d,\lambda)}$ is in $A_p(G)$. Now we order the pairs (d, λ) as follows: $(d_1, \lambda_1) > (d_2, \lambda_2) \Leftrightarrow d_1 > d_2$ and $\lambda_1 > \lambda_2$. Now let q run through this directed set; then (v_q) is a net and $\hat{v}_q = \hat{v}_d - \hat{k}_\lambda \hat{v}_d = \hat{v}_d(1 - \hat{k}_\lambda)$ which is identically zero on some neighborhood W_λ of 0 (see condition (b) in (3.7)). Finally for $f \in A_p(G)$ and $\hat{f}(0) = 0$, we have

$$\lim |||v_q * f - f|||_p \leqslant \lim(|||v_d * f - f|||_p + \|v_d\|_1 \cdot |||k_\lambda * f|||_p) = 0.$$

This completes the proof that $A_p(G)$ satisfies condition D and also the proof of the Tauberian Theorem (3.5).

We record two corollaries below for later application.

(3.11) Corollary. *If I is a closed ideal in $A_p(G)$ and if hull (I) is empty, then $I = A_p(G)$.*

(3.12) Corollary. *Every proper closed ideal in $A_p(G)$ is contained in a regular maximal ideal.*

4. Maximal, regular, and prime ideals in $A_p(G)$. Throughout this section, G will continue to denote a locally compact Abelian group with character group Γ. The purpose of this section is either to extend or to show the impossibility of extending the theorems in Porcelli-Collins [9,10] from $L_1(G)$ to $A_p(G)$. We begin by recalling some facts that will be needed later. Let $[R^2]$ denote the ideal generated by R^2 in the sequel.

(4.1) Theorem ([11], p. 88). (a) *If R is a commutative Banach algebra such that $[R^2] \neq \{0\}$, then R contains a non-prime maximal ideal $\Leftrightarrow [R^2] \nsubseteq R$. Each non-prime maximal ideal is a maximal subspace of R which contains $[R^2]$.*

(b) *If R is a commutative Banach algebra without identity and M is a maximal ideal in R, then M is regular $\Leftrightarrow$ M is prime.*

(4.2) Theorem. *Every maximal ideal in $A_p(G)$ is regular $\Leftrightarrow p = 1$ or G is discrete.*

Proof. The implication $\Leftarrow$ is proved in [10], Theorem 1, while $\Rightarrow$ immediately follows from the proof of (2.4) and (4.1).

(4.3) Corollary. *Every maximal ideal in $A_p(G)$ is closed $\Leftrightarrow p=1$ or G is discrete.*

Proof. The implication $\Leftarrow$ is proved in [10], Corollary 1, while $\Rightarrow$ follows from the last assertion in (4.1(a)), (4.2) and (2.7).

(4.4) Corollary. *Every positive functional on $A_p(G)$ is continuous $\Leftrightarrow p=1$ or G is discrete.*

(A linear functional F defined on a $*$-algebra is *positive* if $F(xx^*)\geqslant 0$ for all $x\in A$.)

Proof. The implication $\Leftarrow$ is well-known result of Varopoulos [14]. Now suppose $p>1$ and G is non-discrete. By (4.3) there is a maximal non-closed ideal M in $A_p(G)$ such that $[(A_p(G))^2]\subseteq M$. Thus there exists $f_0\in A_p(G)$ such that $A_p(G)=M+\{af_0\}$. Define F by $F(m+af_0)=a$; then F is a positive linear functional, but it is not continuous.

(4.5) Lemma. *If I is an ideal in $A_p(G)$ such that I is contained in exactly one regular maximal ideal, say M, then $\bar{I}=M$.*

Proof. Let γ be the character in Γ corresponding to M. Thus hull $(\bar{I})=\{\gamma\}$ and by the Tauberian theorem (3.5) we have $\bar{I}=M$.

(4.6) Lemma. *If a prime ideal I of $A_p(G)$ is contained in a regular maximal ideal, then I is contained in only one regular maximal ideal.*

Proof. Same proof as Lemma 2 in [10], use our Theorem (3.3) instead of [13], 2.6.2.

(4.7) Lemma. *If I is an ideal in $A_p(G)$ such that I is contained in no regular maximal ideal, then $\bar{I}=A_p(G)$.*

Proof. It follows from (3.5).

(4.8) Theorem. *If I is a proper prime ideal in $A_p(G)$, then I is regular maximal $\Leftrightarrow I$ is closed.*

Proof. Consider the implication $\Leftarrow$. By (3.12), I is contained in a regular maximal ideal M. Hence $I=M$ by (4.6) and (4.5). The converse is of course valid for all Banach algebras.

(4.9) Theorem. *$A_p(G)$ contains a non-closed prime ideal $\Leftrightarrow G$ is infinite.*

Proof. Only $\Leftarrow$ requires proof. We consider two cases:

(a) If G is discrete, then $A_p(G)=L_1(G)$ and we are done by Theorem 3 of [10].

(b) If G is non-discrete so that Γ is non-compact. Now argue as in the first half of the proof of Theorem 3 in [10] (apply (4.7) and (3.3) at the appropriate points).

(4.9) Theorem. *Every prime ideal of $A_p(G)$ is contained in a unique regular maximal ideal $\Leftrightarrow G$ is discrete.*

Proof. Similar to the proof of Theorem 4 in [10].

Added in proof. I wish to thank Professor Edwin Hewitt for informing me that some of our results in Section 3 overlap certain results in Hans Reiter's new monograph: *Classical harmonic analysis and locally compact groups*. Reiter deals with a large class of subalgebras of $L_1(G)$ called *Segal algebras*, of which the algebras $A_p(G)$ are examples. We have been able to prove that the Tauberian theorem (3.5) is valid for all Segal algebras.

References

[1] P. J. Cohen, *Factorization in group algebras*, Duke Math. J. 26 (1959), p. 199-205.
[2] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, New York 1958.
[3] R. E. Edwards, *Functional analysis*, New York 1965.
[4] E. Hewitt, *The ranges of certain convolution operators*, Math. Scand. 15 (1964), p. 147-155.
[5] — and K. A. Ross, *Abstract harmonic analysis I*, Berlin 1963.
[6] P. Koosis, *Sur un théorème de Paul Cohen*, C. R. Acad. Sc. Paris 259 (1964), p. 1380-1382.
[7] L. H. Loomis, *An introduction to abstract harmonic analysis*, Princeton — New Jersey 1953.
[8] R. O'Neil, *Convolution operators and $L(p,q)$-spaces*, Duke Math. J. 30 (1963), p. 129-142.
[9] P. Porcelli and H. S. Collins, *Ideals in grou palgebras*, Bull. Amer. Math. Soc. 75 (1969), p. 83-84.
[10] — *Ideals in group algebras*, Studia Math. 33 (1969) p. 223-226.
[11] P. Porcelli, *Linear spaces of analytic functions*, New York 1966.
[12] M. A. Rieffel, *On the continuity of certain intertwining operators, centralizers, and positive linear functionals*, Proc. Amer. Math. Soc. 20 (1969), p. 455-457.
[13] W. Rudin, *Fourier analysis on groups*, New York 1962.
[14] N. T. Varopoulos, *Sur les formes positive d'une algèbre de Banach*, C. R. Acad. Sc. Paris 258 (1964), p. 2465-2467.
[15] C. R. Warner, *Closed ideals in the group algebra $L^1(G)\cap L^2(G)$*, Trans. Amer. Math. Soc. 121 (1966), p. 408-423.
[16] L. Y. H. Yap, *On the impossibility of representing certain functions by convolutions*, Math. Scand. 24 (1969).
[17] A. Zygmund, *Trigonometric series*, Vol. I, 2nd ed., Cambridge 1959.

Reçu par la Rédaction le 23. 4. 1969