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Singular invariant measures on the line

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INTRODUCTION

In this paper we give a method of obtaining a large class of σ-finite measures on the real line which are invariant under translation by every real number. These measures are defined on sub-σ-rings of rings of Borel subsets of R, the real line. Most of these measures are non-atomic and attach positive finite mass to some Lebesgue null set. The object of this paper is to study such measures with special reference to some problems in Harmonic Analysis. We feel that the study of such measures is not only interesting in itself but it has also strong bearing on some unresolved problems raised by Helson and Lowdenslager [7].

In Section 1 we recall the definition of sets of translation and obtain some of their properties. Large part of this section derives from Kahane and Salem [9], Chapter 1. In Section 2 we get a method of obtaining invariant measures as described above and obtain a result regarding the ergodicity of such measures. In Section 3 we define functions called cocycles and coboundaries introduced by Helson and Lowdenslager in [8] and give an example of a cocycle taking values $+1$ or -1 and which is not a coboundary(1). In Section 4 we derive certain elementary consequences of Beurling's well known discription of invariant subspaces of H^2 [1, 8] and in Section 5 we apply these results to obtain the multiplicity of the spectral measure associated with a unitary group of translations on the L^2 of an invariant measure. In Section 6 we give some results about "dual" measures. The results of Sections 5 and 6 are rather special but we feel that they are interesting.

Ergodicity plays a special role in our discussions. The work has connections with the previous work of de Leeuw and Glicksberg [2] on

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(1) First such example was given by Gamelin [5] for compact groups with ordered duals.

analytic measures on compact abelian groups and its generalization by Forelli [4] to arbitrary locally compact spaces where ergodicity was first explicitly considered in connection with analytic measures.

1. SETS OF TRANSLATION

In this section we shall describe the construction and properties of sets of translation (or translation sets). This discription will follow Kahane and Salem [9] with only slight modification which shall result in some technical advantage for our purpose.

1.1. Construction. Let v be an integer $\geqslant 2$ and let $\eta_1, \eta_2, \ldots, \eta_v$ be v distinct numbers satisfying $0 \leqslant \eta_1 < \eta_2 < \ldots < \eta_v < 1$. Let $\xi > 0$ satisfy $\xi < \eta_2 - \eta_1$, $\xi < \eta_3 - \eta_2, \ldots, \xi < 1 - \eta_v$. Let $[a, b]$ denote the closed interval $a \leqslant x \leqslant b$ of length λ. By *dissection* of $[a, b]$ of type $(v, \eta_1, \eta_2, \ldots, \eta_v, \xi)$ we mean the subset of $[a, b]$ consisting of union of v intervals $[a + \lambda\eta_i, a + \lambda\eta_i + \xi\lambda]$, $i = 1, 2, \ldots, v$. Each of these intervals is called a *white interval* and each component interval of the complement (with respect to $[a, b]$) of this union is called a *black interval*. A dissection of type $(v, \eta_1, \eta_2, \ldots, \eta_v, \xi)$ is called *equally spaced* if $\eta_1 = 0$ and $\eta_k = (k-1)\eta_2$, $k = 1, 2, 3, \ldots, v$.

Let I denote the interval $0 \leqslant x \leqslant 1$. Let E_1 be the dissection of I of type $(v_1, \eta_{1,1}, \ldots, \eta_{v_1,1}, \xi_1)$. Let E_2 be obtained from E_1 by performing on each white interval of E_1 a dissection of type $(v_2, \eta_{1,2}, \ldots, \eta_{v_2,2}, \xi_2)$; thus $E_2 \subset E_1$ and consists of $v_1 v_2$ white intervals. Generally, let E_k denote the set obtained from E_{k-1} by performing on each white subinterval of E_{k-1} a dissection of type $(v_k, \eta_{1,k}, \eta_{2,k}, \ldots, \eta_{v_k,k}, \xi_k)$.

Definition 1.1. Let $E = \bigcap_{k=1}^{\infty} E_k$, where $E_1 \supset E_2 \supset \ldots \supset E_k \supset \ldots$ are sets as described above. Then E is called a *set of translation*. A translation set E is called *equally spaced* if each dissection $(v_k, \eta_{1,k}, \ldots, \eta_{v_k,k}\xi_k)$ is equally spaced.

Each left end-point of the white interval in E_k is of the type

(1.1) $$\eta(j_1, 1) + \xi_1\eta(j_2, 2) + \ldots + \xi_1\xi_2 \ldots \xi_{k-1}\eta(j_k, k),$$

where we have written $\eta(j, \mu)$ for $\eta_{j,\mu}$ and where $1 \leqslant j_k \leqslant v_k$. These left end-points and group and semi-group generated by them will play a special role in our discussions. Since every point of E is a limit point of such left end-points, every $x \in E$ has a representation of the type

(1.2) $$x = \eta(j_1, 1) + \xi_1\eta(j_2, 2) + \ldots + \xi_1\xi_2 \ldots \xi_{k-1}\eta(j_k, k) + \ldots$$

1.2. Lebesgue function. Let l_k denote the Lebesgue measure on E_k normalised to 1 and regard l_k as defined on I by making l_k zero for sets outside E_k. Let L_k denote the distribution function of l_k, i.e., $L_k(x) = l_k([0, k])$. It is easy to see that the value of L_k on the left end-point (1.1) is

$$L_k\big(\eta(j_1, 1) + \xi_1\eta(j_2, 2) + \ldots + \xi_1\xi_2 \ldots \xi_k\eta(j_k, k)\big) = \frac{j_1 - 1}{v_1} + \frac{j_2 - 1}{v_1 v_2} + \ldots + \frac{j_k - 1}{v_1 \ldots v_k}.$$

L_k increases linearly on each white interval of E_k and L_k is constant on each subinterval of the complement of E_k. The functions L_k converge uniformly to a continuous function L which has all its points of increase in E and is constant on subintervals of complement of E.

For any $x \in E$ with representation $x = \eta(j_1, 1) + \xi_1\eta(j_2, 2) + \ldots + \xi_1 \ldots \xi_{k-1}\eta(j_k, k) + \ldots$

(1.3) $$L(x) = \frac{j_1 - 1}{v_1} + \frac{j_2 - 1}{v_1 v_2} + \ldots + \frac{j_k - 1}{v_1 v_2 \ldots v_k} + \ldots$$

The restriction of L to E is not one-one but very nearly so. More precisely, if we delete from E all points x with representation of the type

$$x = \eta(j_1, 1) + \xi_1\eta(j_2, 2) + \ldots + \sum_{k=n}^{\infty} \xi_1\xi_2 \ldots \xi_{k-1}\eta(v_k, k)$$

for $n = 1, 2, \ldots$ and call the new set $\bar{E}$, then the restriction of the function L to $\bar{E}$ is one-one and increasing on $\bar{E}$. It maps $\bar{E}$ onto $0 \leqslant x < 1$. Henceforth we shall regard L as a function from $\bar{E}$ onto $\bar{I} = \{x: 0 \leqslant x < 1\}$ and also write E for $\bar{E}$ and I for $\bar{I}$.

1.3. Measure induced by L. Let g denote the measure on E induced by the function L. If W is the λ^{th} white interval of E_k counting from left, then

$$L(W \cap E) = \left(\frac{\lambda - 1}{v_1 \ldots v_k}, \frac{\lambda}{v_1 \ldots v_k}\right) \quad \text{and} \quad g(W \cap E) = \frac{1}{v_1 v_2 \ldots v_k}.$$

Since any function in L^1 or L^2 of I with Lebesgue measure is approximable by linear combinations of indicator functions of the interval

$$\left(\frac{\lambda - 1}{v_1 \ldots v_k}, \frac{\lambda}{v_1 \ldots v_k}\right), \qquad \lambda = 1, 2, \ldots \text{ and } 1 \leqslant k < \infty,$$

we conclude that any function of $L^1(E, g)$ or $L^2(E, g)$ is approximable in the respective norms by linear combinations of the indicator functions of the sets of the type $W \cap E$, where W is a white interval in E_k for some k. Another property of the measure g which we shall use is the following ([6], p. 19):

(*) *If A is a measurable subset of E and t is a real number such that $A + t \subset E$, then $g(A) = g(A + t)$, i.e., measurable subsets of E congruent by translation have the same measure g.*

Let Q be the group of real numbers generated by the left end-points of the intervals in E_k, $k = 1, 2, \ldots$ Let K be the group of real numbers in $0 \leqslant x < 1$ of type

$$k = \frac{j_1-1}{v_1} + \frac{j_2-1}{v_1 v_2} + \ldots + \frac{j_k-1}{v_1 \ldots v_k}, \quad 0 < j_i \leqslant v_i,$$

where the addition in K is defined modulo 1. It is clear that K is a dense subgroup of the group C of real numbers (modulo 1) and hence any measurable subset of C invariant under translation by K has Haar measure zero or one (²). The next lemma transfers this fact to E.

LEMMA 1.1. *Let $A \subset E$ be a measurable set such that $(A \pm q) \cap E \subset A$ for every $q \in Q$. Then either $g(A) = 0$ or $g(E-A) = 0$.*

Proof. It is enough to show that $L(A)$ is invariant under translation by K. Now let $y \in L(A)$ and let it have the representation

$$y = \sum_{i=1}^{\infty} \frac{j_i - 1}{v_1 \ldots v_i}, \quad 1 \leqslant j_i \leqslant v_i.$$

Let

$$x = \sum_{i=1}^{n} \frac{p_i - 1}{v_1 \ldots v_i} \in K, \quad 1 \leqslant p_i \leqslant v_i.$$

We shall show that $y + x \in L(A)$. Now $y+x$ has a representation of type

$$y + x = \sum_{i=1}^{n} \frac{q_i - 1}{v_1 \ldots v_i} + \sum_{i=n+1}^{\infty} \frac{j_i - 1}{v_1 \ldots v_i},$$

so that the terms in the representation of $y+x$ and y agree from $(n+1)^{\text{th}}$ term on. Hence we have

$$L^{-1}(y+x) = \sum_{i=1}^{n} \xi_1 \ldots \xi_i \eta(q_i, i) + \sum_{i=n+1}^{\infty} \xi_1 \ldots \xi_i \eta(j_i, i),$$

$$L^{-1}(y) = \sum_{i=1}^{\infty} \xi_1 \ldots \xi_i \eta(j_i, i) = \sum_{i=1}^{n} \xi_1 \ldots \xi_i \eta(j_i, i) + \sum_{i=n+1}^{\infty} \xi_1 \ldots \xi_i \eta(j_i, i).$$

Consequently, by (1.3),

$$L^{-1}(y+x) - L^{-1}(y) = \Big(\sum_{i=1}^{n} \xi_1 \ldots \xi_i \eta(q_i, i) - \sum_{i=1}^{n} \xi_1 \ldots \xi_i \eta(j_i, i)\Big) \in Q.$$

Let q denote this element in Q; then $L^{-1}(y+x) = L^{-1}(y) + q$ belongs to $(A+q) \cap E \subset A$. Hence $L^{-1}(y+x) \in A$, so that $y + x \in L(A)$. Thus $L(A)$ is invariant under K, q.e.d.

(²) C is the circle group and Haar measure on C is the linear measure on C. This measure is ergodic with respect to translation by members of a dense subgroup.

Remark. Lemma 1.1 permits us to classify the sets of translation in the following manner. Let E_1 and E_2 be two sets of translation with g_1 and g_2 the associated measures supported on E_1 and E_2 respectively. Then either $g_2\big(E_2 \cap (E_1 + t)\big) = 0$ for all t or there exist $t_1, t_2, \ldots$ such that

$$g_2\Big(E_2 - \bigcup_{n=1}^{\infty} (E_1 + t_n)\Big) = 0.$$

2. INVARIANT MEASURES

A σ-finite measure ν defined on the Borel σ-ring $\mathscr{B}$ of R is called *locally invariant* if there exists a support B of ν such that any two measurable subsets of B congruent by translation have the same measure ν. B is then called an *admissible support* of ν. If ν is locally invariant with admissible support B, then ν_t is locally invariant with admissible support $B+t$, where ν_t is defined by $\nu_t(A) = \nu(A - t)$, $A \in \mathscr{B}$. Further ν and ν_t agree on Borel subsets of $B \cap (B+t)$. Let G be a subgroup of R. For each $t \in G$, let S_t be the σ-ring of Borel measurable subsets of $B - t$ and let S be the σ-ring generated by $\bigcup_{t \in G} S_t$. S consists of measurable subsets of R which can be covered by countably many translates of B by members of G.

THEOREM 2.1. *Let ν, B and S be as above. Then there exists a unique measure μ on the σ-ring S such that*

(i) *$\mu(A) = \mu(A+t)$ for all $A \in S$ and $t \in G$;*

(ii) *restriction $\mu|_B$ of μ to B is ν.*

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \emptyset$ it $i \neq j$ and A_i is a Borel subset of $B + t_i$ for some i. Define μ for this A as

$$\mu(A) = \sum_{i=1}^{\infty} \nu_{t_i}(A_i).$$

Since ν and ν_t agree on Borel subsets of $(B+t) \cap B$, μ is unambiguously defined. It is easy to prove that μ is invariant under G and $\mu|_B = \nu$ and that μ is unique, q.e.d.

Now if in Theorem 2.1 we take $G = R$, we get a measure invariant under translation by every real number. We can take ν to be the measure g associated with a translation set E, E being an admissible support of ν. We can also take μ to be any non-atomic finite measure supported on an independent set (³) E and E then becomes an admissible support of μ.

(³) Observe that an independent set and its non-zero translate can intersect in at most one point.

We thus see that there are many measures on R, other than the Lebesgue measure and the cardinality measure which are invariant under translation. A σ-finite measure space (X, S, μ) is called *totally σ-finite* if there exists an $A \in S$ such that $\mu(B) = 0$ for every set $B \in S$ disjoint from A.

2.3. Ergodicity. Henceforth we shall deal with totally σ-finite measures on Borel subsets of R which are invariant under a countable dense subgroup of R. A totally σ-finite measure μ defined on a sub-σ-ring of $\mathscr{B}$ will be regarded as a measure on $\mathscr{B}$ simply by setting $\mu(A) = 0$ for those subsets of $\mathscr{B}$ which do not intersect a measurable support of μ.

Definition 2.1. Let μ be a σ-finite measure on $\mathscr{B}$ which is invariant under translation by a countable dense subgroup Q. We say that μ is *ergodic with respect to* Q if for every measurable set A such that $A+q = A$ for all $q \in Q$, either $\mu(A) = 0$ or $\mu(R-A) = 0$.

Now let E be a set of translation and Q the group generated by the left end-points as in Lemma 1.1. Let g be the locally invariant measure on E given by the Lebesgue function on E. Let μ be the measure obtained by setting in Theorem 2.1 $\nu = g$, $B = E$ and $G = Q$.

THEOREM 2.2. *μ is ergodic under Q.*

Proof. Let A be a measurable set which is invariant under Q, i.e., $A+q = A$ for all $q \in Q$. Let $\Lambda = E \cap A$. Then $(\Lambda+q) \cap E \subset \Lambda$ for all $q \in Q$. Hence by Lemma 1.1 either $\nu(\Lambda) = 0$ or $\nu(E-\Lambda) = 0$. Now μ is invariant under Q and supported on $\bigcup_{q \in Q}(E+q)$, with ν the restriction of μ to the Borel subsets of E. It follows that either $\mu(A) = 0$ or $\mu(R-A) = 0$. This proves the ergodicity of μ, q.e.d.

On the other hand, if a non-atomic σ-finite measure μ is invariant under a subgroup Q and attaches positive mass to a perfect independent set E, then μ can never be ergodic under any subgroup. Thus in terms of their ergodic behaviour perfect independent sets are quite opposite of sets of translation.

3. COCYCLES AND COBOUNDARIES

3.1. Let μ be a measure defined on $\mathscr{B}$ which is invariant under a countable semi-group P of R.

Definition 3.1. A non-vanishing complex-valued function A on $P \times R$ is called a *cocycle* if $A(q, x)$ is $\mathscr{B}$ measurable in x for every q and A satisfies, for all q_1 and q_2,

$$A(q_1+q_2, x) = A(q_1, x)\, A(q_2, x+q_1) \quad \text{a.e. } [\mu]. \tag{3.1}$$

It is called a *unitary cocycle* if $|A(q, x)| = 1$ a.e. $[\mu]$ for all $q \in P$.

Let B be a non-vanishing $\mathscr{B}$-measurable function and write

$$A(q, x) = \frac{B(x+q)}{B(x)}.$$

Then A is easily verified to be a cocycle. A cocycle of this type is called a *coboundary*. A cocycle defined on $P \times R$ can be uniquely extended to $Q \times R$, where Q is the group generated by P, so that (3.1) holds on $Q \times R$. For this we need only write $A(-p, x) = \big(A(p, x+p)\big)^{-1}$, $p \in P$. In view of this we shall assume henceforth that a cocycle is defined with respect to Q.

3.2. We now give an example of a cocycle which is not a coboundary. It will also serve to show that some results of later sections are not vacuous. First example of a cocycle which is not a coboundary was constructed by Helson and Lowdenslager [8] in connection with invariant subspaces in the space of square integrable functions on the Bohr group, where the terminology of cocycles and coboundaries was introduced. Subsequent examples and new results were given by Gamelin [5]. Our example differs from those of above authors in that we are dealing on the real line rather than finite- or infinite-dimensional tori.

Example. Let G be a dense subgroup of R such that $G = \bigcup_{n=0}^{\infty} G_n$, where each G_n is a cyclic group and $G_n \subset G_{n+1}$. Let $\lambda_n > 0$ be the generator of G_n and let $\lambda_n/\lambda_{n+1} = a_n$. Define

$$B_0(x) = (-1)^k, \quad k\lambda_0 \leqslant x < (k+1)\lambda_0,\ -\infty < k < \infty,$$

$$B_1(x) = (-1)^k, \quad k\lambda_1 \leqslant x < (k+1)\lambda_1,\ k = 0, \ldots, a_n-1,$$

and extend B_1 outside $[0, \lambda_0)$ by making it periodic with period λ_0. Generally, define

$$B_n(x) = (-1)^k, \quad k\lambda_n \leqslant x < (k+1)\lambda_n,\ 0 \leqslant k \leqslant a_n-1,$$

and extend B_n outside $[0, \lambda_{n-1})$ by making it periodic with period λ_{n-1}. Let $\Phi_n = B_0 B_1 \ldots B_n$, and define $A(g, x) = \Phi_n(x+g)/\Phi_n(x)$, $g \in G_n$. Suppose $m > n$; then

$$\Phi_m(x+g)\Phi_m^{-1}(x) = \frac{\prod_{k=0}^{m} B_k(x+g)}{\prod_{k=0}^{m} B_k(x)}.$$

Now $B_{n+1}, \ldots, B_m$ are periodic with period λ_n and so if $g \in G_n$, then $\Phi_m(x+g)/\Phi_m(x) = \Phi_n(x+g)/\Phi_n(x)$ which shows that A is unambiguously defined.

Now if $g_1, g_2 \in G$, then $g_1, g_2 \in G_n$ for some n so that

$$A(g_1+g_2, x) = \frac{\Phi_n(x+g_1+g_2)}{\Phi_n(x)} = \frac{\Phi_n(x+g_1+g_2)}{\Phi_n(x+g_1)} \cdot \frac{\Phi_n(x+g_1)}{\Phi_n(x)}$$
$$= A(g_1, x)\, A(g_2, x+g_1),$$

which proves that A is a cocycle. We now prove that A is not a coboundary. Suppose that for some measurable function B we have

$$A(g+x) = \frac{B(x+g)}{B(x)}$$

a.e. with respect to the Lebesgue measure for all g. Then

$$\frac{B(x+g)}{B(x)} = \frac{\Phi_n(x+g)}{\Phi_n(x)}$$

for a.e. x [Lebesgue], $g \in G_n$. Hence $B/\Phi_n = C_n$ is periodic with period λ_n. Now Φ_n is constant on the intervals $k\lambda_n \leqslant x < (k+1)\lambda_n$, the constant value being $+1$ or -1, hence restriction of B to the interval $k\lambda_n \leqslant x < (k+1)\lambda_n$ is equal in absolute value to the restriction of B to the interval $0 \leqslant x < \lambda_n$. Now it is easy to see from the way Φ_n is defined that

$$\int_0^{\lambda_0} B(x)\,dx = \int_0^{\lambda_0} \Phi_n(x) C_n(x)\,dx$$

is either zero or equal to

$$\frac{\lambda_n}{\lambda_0} \int_0^{\lambda_0} B(x)\,dx.$$

Letting $n \to \infty$, it follows that

$$\int_0^{\lambda_0} B(x)\,dx = 0.$$

Same argument shows that for every m,

$$\int_0^{\lambda_m} B(x)\,dx = 0$$

and hence

$$\int_{k\lambda_m}^{(k+r)\lambda_m} B(x)\,dx = 0, \qquad 0 \leqslant k < k+r \leqslant \frac{\lambda_0}{\lambda_m}.$$

This holds for all m so that $B = 0$. But this is impossible. Hence A is not a coboundary, q.e.d.

4. SOME CONSEQUENCES OF BEURLING'S DISCRIPTION OF INVARIANT SUBSPACES OF H^2

4.1. $L_2(R^+, \mathscr{B}_\lambda, \mu)$ and shifts on it. Let R^+ denote the set of non-negative real numbers. Let $0 < \lambda < \infty$, and let $\mathscr{B}_\lambda$ denote the σ-ring of sets generated by the intervals $n\lambda \leqslant x < (n+1)\lambda$, $n = 0, 1, 2, \ldots$ Let μ be a measure defined on $\mathscr{B}_\lambda$ which is invariant under translation by λ. We shall assume that $0 < \mu(I_\lambda) < \infty$, where I_λ is the interval $0 \leqslant x < \lambda$. It is easy to see that $L_2(R^+, \mathscr{B}_\lambda, \mu)$ is not much different from l^2, the space of square summable sequences. Every function in $L_2(R^+, \mathscr{B}_\lambda, \mu)$ is constant on $n\lambda \leqslant x < (n+1)\lambda$ and if C_n be this value, then

$$\sum_{n=0}^{\infty} |C_n|^2 < \infty.$$

By *shift* on $L_2(R^+, \mathscr{B}_\lambda, \mu)$ we mean the operator S defined by

$$(Sf)(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < \lambda, \\ f(x-\lambda) & \text{if } x \geqslant \lambda. \end{cases}$$

S is an isometry from $L_2(R^+, \mathscr{B}_\lambda, \mu)$ into $L_2(R^+, \mathscr{B}_\lambda, \mu)$. Let a be a function which is $\mathscr{B}_\lambda$-measurable and of absolute value 1. Let U be the unitary operator $Uf = af$, where $f \in L_2(R^+, \mathscr{B}_\lambda, \mu)$. Write T for the isometric operator $U^{-1}SU$. Now since U is unitary, the subspace spanned by $f, Tf, T^2 f, \ldots$ is whole of $L_2(R^+, \mathscr{B}_\lambda, \mu)$ if and only if the subspace spanned by $Uf, SUf, S^2 Uf, \ldots$ is the whole of $L_2(R^+, \mathscr{B}_\lambda, \mu)$. Let a_n and f_n stand for values of a and f on the interval $n\lambda \leqslant x < (n+1)\lambda$. The next theorem will be useful in Section 5:

THEOREM 4.1. (i) *$Uf, SUf, S^2 Uf, \ldots$ spans $L_2(R^+, \mathscr{B}_\lambda, \mu)$ if and only if $\sum_{n=0}^{\infty} a_n f_n z^n$ is an outer function in H^2.*

(ii) *Suppose $f_i = 0$ for $i > n$. Then $Uf, S^2 Uf, \ldots$ spans $L_2(R^+, \mathscr{B}_\lambda, \mu)$ if and only if $\sum_{i=0}^{n} a_i f_i z^i$ has no zeros in $|z| < 1$.*

(iii) *Suppose that $a_i f_i = (-1)^i$ for $i = 0, 1, 2, \ldots, n$ and $f_i = 0$ otherwise. Then $Uf, SUf, S^2 Uf, \ldots$ spans $L_2(R^+, \mathscr{B}_\lambda, \mu)$.*

Proof. (i) follows from Beurling's well known discription of invariant subspaces of H^2.

(ii) follows because a polynomial in z is outer if and only if it has no zeros inside the unit disc.

(iii) this follows from (ii) because $\sum_{i=0}^{n} (-1)^i z^i$ has no zeros in $|z| < 1$.

5. UNITARY GROUP OF TRANSLATIONS

5.1. Unitary group of translations [4]. Let $E = \bigcap_{k=1}^{\infty} E_k$ be an equally spaced set of translation, where E_1 is a dissection of $[0, 1]$ of the type $\{v_1+1, 0, \eta_1, 2\eta_1, \ldots, v_1\eta_1, \xi_1\}$ and E_k is obtained from E_{k-1} by performing on each subinterval of E_{k-1} a dissection of the type $\{v_k+1, 0, \eta_k, 2\eta_k, \ldots, \ldots, v_k\eta_k, \xi_1\}$. Let g be the locally invariant measure on E induced by the Lebesgue function L, E being an admissible support of g. Let Q denote the group generated by the left end-points of subintervals of E_k, $k = 1, 2, \ldots$, i.e., by the real numbers of the type

$$\lambda_1\eta_1 + \xi_1\lambda_2\eta_2 + \ldots + \xi_{k-1}\xi_{k-2}\ldots\xi_1\lambda_k\eta_k, \qquad 0 \leqslant \lambda_i \leqslant v_i.$$

Q^+ shall denote the semi-group of non-negative real numbers of Q. We shall denote by $W_{k\lambda}$ the λ^{th} subinterval of E_k counting from left. Let μ be the measure obtained as in theorem 2.1 by taking $G = Q$, $v = g$ and $B = E$. Let A be a unitary cocycle on $Q \times R$ satisfying

$$A(q_1+q_2, x) = A(q_1, x)\,A(q_2, x+q_1)$$

a.e. $[\mu]$. Consider the group of unitary operators on $L_2(R, \mathscr{B}, \mu)$ defined by

$$(U_q f)(x) = A(q, x) f(x+q), \qquad q \in Q. \tag{5.1}$$

It can be verified using the functional equation of the cocycle that U_q, $q \in Q$, is indeed a group of unitary operators.

THEOREM 5.1. *Assume that*

(i) *for all non-negative integers n and k, $A(n\xi_1\xi_2\ldots\xi_{k-1}\eta_k, x)$ is constant on $W_{k_1} \cap E$ and let C_n^k be this constant value;*

(ii) $\sum_{\lambda=1}^{v_k} C_\lambda^k z^{\lambda-1}$ *has no zeros in $|z| < 1$ for all k.*

Then

(a) $U_q 1_E$, $q \in Q^+$, *spans the closed subspace of functions in $L_2(R, \mathscr{B}, \mu)$ which vanish for negative real numbers;*

(b) $U_q 1_E$, $q \in Q$, *spans $L_2(R, \mathscr{B}, \mu)$. Here 1_E denotes the indicator function of E.*

Proof. Let $\mathscr{B}_1$ be the σ-algebra generated by $W_{11} \cap E + k\eta_1$ on $0 \leqslant x < \infty$. Since for all n, $A(n\eta_1, x)$ is constant on $W_{11} \cap E$,

$$A(n\eta_1, x) = \frac{B_1(x+n\eta_1)}{B_1(x)}$$

[4] Actually our unitary groups are translations times a cocycle as given by formula (5.1).

for function B_1 defined by $B_1(x+n\eta_1) = A(n\eta_1, x)$, $0 \leqslant x < \eta_1$. We see now that in $L_2(R^+, \mathscr{B}_1, \mu)$, $U_{n\eta_1}$, $n = 0, 1, 2, \ldots$, is $B_1^{-1} S^n B_1$, where B_1 also stands for the operator consisting of multiplication by B_1. Now $B(x+n\eta_1) = A(n\eta_1, x)$ for $0 \leqslant x < \eta_1$. Because of (ii) therefore

$$\sum_{i=1}^{v_1} C_i^1 z^{i-1} = \sum_{i=1}^{v_1} B(i\eta_1) z^{i-1}$$

has no zeros in $|z| < 1$. By theorem 4.1 (b) it follows that $U_{k\eta_1}1_E$, $k \geqslant 0$, spans $L_2(R^+, \mathscr{B}_1, \mu)$ and hence indicator functions of $W_{11} \cap E + k\eta_1$, $k \geqslant 0$, all belong to the span of $U_q 1_E$, $q \in Q^+$. Again let $\mathscr{B}_2$ be the σ-algebra generated by $W_{21} \cap E + k\eta_2\xi_1$, $k \geqslant 0$. Since $\sum_{\lambda=1}^{v_2} C_\lambda^2 z^{\lambda-1}$ has no zeros in $|z| < 1$, by argument same as before we conclude that $U_{k\eta_2\xi_1}1_{W_{11}\cap E}$, $k \geqslant 0$, spans $L_2(R_1, \mathscr{B}_2, \mu)$ and hence the indicator functions of $W_{21} \cap E + k\eta_2\xi_1$, $k \geqslant 0$, all belong to the span of $U_{k\eta_2\xi_1}1_{W_{11}\cap E}$. If we note that $A(k\eta_1, x)$ is constant on each $W_{1\lambda} \cap E$, $\lambda = 1, 2, \ldots, v_1$, it can be seen that $U_{k\eta_1+\lambda\eta_2\xi_1}1_E$ $(k, \lambda \geqslant 0)$ has a span which contains the indicator functions of each of the set $W_{21} \cap E + k\eta_1 + \lambda\eta_2\xi_1$ $(k, \lambda \geqslant 0)$. Proceeding thus we see that the closed linear subspace spanned by $\{U_q 1_E\}_{q\geqslant 0}$, $q \in Q$, contains the indicator functions of each of the sets

$$W_{n1} \cap E + k_1\eta_1 + k_2\eta_2\xi_1 + \ldots + \xi_1\ldots\xi_{n-1}k_n\eta_n,$$
$$0 \leqslant k_i < \infty, \qquad i = 1, 2, \ldots, n, \qquad n = 1, 2, \ldots$$

But every function in $L_2(R^+, \mathscr{B}, \mu)$ is approximable by linear combinations of indicator functions of such sets. Consequently, $U_q 1_E$, $q \in Q^+$, spans the subspace of functions in $L_2(R, \mathscr{B}, \mu)$ which vanish for negative real numbers.

(b) this is now obvious, q.e.d.

COROLLARY. *Translates of 1_E by members of Q^+ span $L^2(R^+, \mathscr{B}, \mu)$.*

Proof. In this case each $C_n^k = 1$, so that $\sum_{\lambda=0}^{v_k} z^{\lambda-1}$ has no zero in $|z| < 1$. Hence theorem 5.1 is applicable, q.e.d.

The cocycle constructed in 3.2 together with theorem 4.1 (iii) can be used to show that our theorem is non-vacuous in a non-trivial way, i.e., A can really be chosen to be a cocycle which is not a coboundary and for which the hypothesis of theorem 5.1 are satisfied. We leave the details of this verification to the reader.

Remark. In contrast with theorem 5.1 there exist measures μ invariant under a group Q such that for every group U_q, $q \in Q$, of unitary operators on $L_2(R, \mathscr{B}, \mu)$ of type (5.1) no function f exists in $L_2(R, \mathscr{B}, \mu)$ so that $U_q f$, $q \in Q$, spans $L_2(R, \mathscr{B}, \mu)$ [3].

6. DUAL MEASURES AND THEIR PROPERTIES

6.1. Dual measures. Let μ be a σ-finite measure on $\mathscr{B}$ which is invariant under a countable dense subgroup Q and let $A(\cdot, \cdot)$ be a unitary cocycle on $Q\times R$, which satisfies functional equation (3.1) a.e. with respect to μ. Let U_q $(q\in Q)$ and V_t $(t\in R)$ be groups of unitary operators on $L_2(R, \mathscr{B}, \mu)$ defined by

$$(U_q f)(x) = A(q, x) f(x+q),$$
$$(V_t f)(x) = e^{itx} f(x).$$

It is easy to check that U_q and V_t together satisfy the following important equation:

$$V_t U_q = e^{itq} U_q V_t. \tag{6.1}$$

Now consider Q as an abelian group with discrete topology and let B denote its compact dual. There is a continuous isomorphism Φ of the real line with usual topology into B given by $\big(\Phi(t), q\big) = \Phi(t)(q) = e^{itq}$. It can further be shown that $\Phi(R)$ is dense in B. Now by Stone's theorem for groups we can write

$$U_q = \int_B (q, r)\,\beta(dr),$$

where β is a spectral measure on the Borel subsets of B, whose values are projections in $L_2(R, \mathscr{B}, \mu)$. It follows as a consequence of (6.1) that

$$V_t\beta(\Delta)\,V_{-t} = \beta\big(\Delta+\Phi(t)\big), \tag{6.2}$$

where Δ is a measurable subset of B. See [9].

We assume now that $L_2(R, \mathscr{B}, \mu)$ has a single generator f with respect to U_q, $q\in Q$. Then β is of multiplicity 1 and $T: U_q f\to (q, \cdot)$ is an invertible isometry from $L_2(R, \mathscr{B}, \mu)$ onto $L_2(B, v)$, where v is the measure $v(\Delta) = \big(\beta(\Delta)f, f\big)$. Because of (6.2) the measure v is quasi-invariant under $\Phi(R)$, i.e., $v(\Delta) = 0$ if and only if $v\big(\Delta+\Phi(t)\big) = 0$, $t\in R$. Further one can prove

THEOREM 6.1. *If μ is ergodic under Q, then v is ergodic under $\Phi(R)$.*

Remark. Our method shows a way of getting on the dual B of a subgroup of R a measure which is quasi-invariant and ergodic under $\Phi(R)$ but which is neither equivalent to the Haar measure on B nor equivalent to the linear measure on a coset of $\Phi(R)$. Here *equivalent measures* means measures mutually absolutely continuous.

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