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A remark on p-absolutely summing operators in l_r-spaces

by

S. KWAPIEŃ (Warszawa)

Pietsch [6] and Pełczyński [3] have proved that in a Hilbert space an operator is p-absolutely summing if and only if it is 1-absolutely summing. The aim of the present note is to generalize this result on the case of l_r-spaces.

The method employed here is that used by Persson and Pietsch [4] in the case of Hilbert spaces.

In the sequel we shall need the following notations:

$B(X, Y)$ — the class of all linear, continuous operators from a Banach space X into a Banach space Y;

$\Pi_p(X, Y)$ — the class of all p-absolutely summing operators from X into Y.

(A is *p-absolutely summing* if there exists a constant C such that for each $x_1, x_2, \ldots, x_n \in X$

$$\sum_{i=1}^{n} \|Ax_i\|^p \leqslant C \sup_{\|x^*\|\leqslant 1} \sum_{i=1}^{n} |x^*(x_i)|^p).$$

$N_p(X, Y)$ — the class of all p-nuclear operators from X into Y (A is *p-nuclear* if it admits a factorization $A\colon X \xrightarrow{U} l_\infty \xrightarrow{\Delta} l_p \xrightarrow{V} Y$, where U, V are continuous operators and Δ is diagonal, i.e. $\Delta((a_n)) = (\lambda_n a_n)$ and (λ_n) is a fixed sequence from l_p).

THEOREM. *If* $1 \leqslant r \leqslant 2$, $1 \leqslant p \leqslant 2$, *and* $2 \leqslant q < \infty$, *then for each Banach space* X

(a) $$\Pi_p(l_r, X) = \Pi_1(l_r, X),$$

(b) $$\Pi_q(X, l_r) = \Pi_2(X, l_r).$$

Proof. (a) Since (cf. Pietsch [6])

(1) $$\Pi_s(X, Y) \subset \Pi_{s'}(X, Y) \quad \text{for } s \leqslant s',$$

it is enough to prove the inclusion $\Pi_2(l_r, X) \subset \Pi_1(l_r, X)$. This is a result of the following three facts:

(2) $A \in \Pi_1(Y, X)$ iff $AB \in \Pi_1(l_\infty, X)$ for each $B \in B(l_\infty, Y)$;

(3) $\Pi_2(l_\infty, l_r) = B(l_\infty, l_r)$ for $1 \leqslant r \leqslant 2$ (cf. [2]);

(4) If $B \in \Pi_2(X_1, X_2)$, $A \in \Pi_2(X_2, X_3)$, then $AB \in \Pi_1(X_1 X_3)$ (cf. Pietsch [6]).

(b) By (1) it is sufficient to prove only the inclusion $\Pi_q(X, l_r) \subset \Pi_2(X, l_r)$.

According to the results of Persson and Pietsch (cf. [4], Satz 5.2 and 5.3) and Saphar [8] the space $\Pi_s(X, l_r)$ is the dual of the space $N_{s^*}(l_r, X)$ (where $s^* = s/(s-1)$). Hence the above inclusion is equivalent to $N_2(l_r, X) \subset N_q{}^*(l_r, X)$.

Let $A \in N_2(l_r, X)$ and let $A: l_r \xrightarrow{U} l_\infty \xrightarrow{\Delta} l_2 \xrightarrow{V} X$ be its factorization. By (3) and (1) ΔU is r^*-absolutely summing. Thus

$$\sum_{k=1}^{\infty} \|\Delta U e_k\|^{r^*} < +\infty, \tag{5}$$

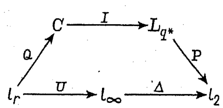
where (e_k) is the unit-vector basis of l_r.

Making use of the Rademacher system, an operator $P: L_{q^*} \to l_2$ may be constructed such that the operator $B: C \xrightarrow{I} L_{q^*} \xrightarrow{P} l_2$ is surjective (cf. [1] [7.1.3] and [7.3.6]) (I is the inclusion operator of C into L_{q^*}). So by the Banach "open map" theorem a sequence $(x_k)_{k=1}^{\infty}$ in C may be found such that

$$B(x_k) = \Delta U e_k, \ \|x_k\| \leqslant K \|\Delta U e_k\|, \quad k = 1, 2, \ldots \tag{6}$$

(K is a constant independent of k).

(5) together with (6) imply that the assignment $e_k \to x_k$ for $k = 1, 2, \ldots$ may be extended to a bounded linear operator $Q: l_r \to C$. Now, it is seen that the following diagram is commutative:



This means that ΔU is q^*-integral.

If $r > 1$, then l_r is reflexive. Hence ΔU is q^*-nuclear (cf. Persson [5]) and so $A = V\Delta U$ is q^*-nuclear as well.

Let $r = 1$. Since ΔU is compact, ΔU may be factorized into

$$\Delta U: l_1 \xrightarrow{D} l_1 \xrightarrow{E} l_2,$$

where D is compact and E is a continuous operator. Indeed, let E be any operator from l_1 onto l_2. Since E is open, there exists a sequence (x_k) in l_1 relatively compact such that $Ex_k = \Delta U e_k$ for $k = 1, 2, \ldots$ The operator $D: l_1 \to l_1$ which maps e_k into x_k for each k is compact and $ED = \Delta U$. Using again the fact that there exists a surjection $B: C \to l_2$ which is factorized by the natural embedding $J: C \to L_{q^*}$ and the fact that the space l_1 has the lifting property, we infer that every bounded linear operator from l_1 into l_2 is q^*-integral.

But D is compact, so ED is q^*-nuclear (cf. Persson [5]). Thus $A = VED$ is also q^*-nuclear, which completes the proof.

As an immediate consequence, we obtain

COROLLARY. *If* $1 \leqslant r, s \leqslant 2$ *and* $1 \leqslant p < \infty$, *then*

$$\Pi_p(l_r, l_s) = \Pi_1(l_r, l_s).$$

Remark 1. In the case of $r = s = 2$, Corollary coincides with Pietsch [6] and Pełczyński [3] theorem.

Remark 2. In the statements of the Theorem and of the Corollary the spaces l_r and l_s may be replaced by general $\mathscr{L}_r$ and $\mathscr{L}_s$ spaces of Lindenstrauss and Pełczyński [2] respectively.

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