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On the zeroes of some random functions

by

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Let $F(t)$ be a Fourier series with random coefficients and phases,

$$F(t) = \sum_{n=1}^{\infty} a_n X_n \cos(nt + \Phi_n).$$

Here $(X_n)_1^\infty$ is a sequence of mutually independent Gaussian variables of type $N(0,1)$; $(\Phi(n))_1^\infty$ is a sequence of mutually independent variables, uniformly distributed upon $[0, 2\pi]$; and the X's and Φ's are mutually independent. (The basic probability space will be denoted (Ω, P).) About the numbers a_n we suppose

$$a_n > 0, \quad \log a_n = -\beta \log n + o(\log n), \quad \text{with } \tfrac{1}{2} < \beta \leqslant 1.$$

Our goal is an estimation of the zero-set of F, $Z(\omega) = \{t\colon F(t, \omega) = 0\}$.

THEOREM. *Let B be a closed set in $[0, 2\pi]$ of Hausdorff dimension d. Then*

$$P\{\dim(Z \cap B) \leqslant d - \beta + \tfrac{1}{2}\} = 1, \quad d \geqslant \beta - \tfrac{1}{2},$$

$$P\{Z \cap B = \emptyset\} = 1, \quad d < \beta - \tfrac{1}{2},$$

$$P\{\dim(Z \cap B) \geqslant d_1\} > 0, \quad 0 < d_1 < d - \beta + \tfrac{1}{2}.$$

In § 1 we prove a general principle for the lower bound, whose application is dependent upon specific estimates, derived later about F. In § 2 we review some conclusions from [2] about the uniform convergence of F and its modulus of continuity, and we also obtain a technical lemma about the local character of the trajectories of F. In § 3 we obtain an upper bound for the dimension, and in § 4 a lower bound is obtained by combining the work of §§ 1 and 2.

§ 1. Let B be a compact set of real numbers and μ a probability measure in B such that:

(i) $\mu([a, a+h]) \leqslant C_1 h^d$ for constants $C_1, d > 0$ and all intervals $[a, a+h]$.

Let $Y(t, \omega)$ be a stochastic process, for $t \in B$, and suppose that:
(ii) Each path $t \to Y(t, \omega)$ is continuous.
(iii) $P\{\omega\colon |Y(t, \omega)| < y\} \geqslant C_2 y$, for $t \in B$, $0 < y \leqslant 1$, and $C_2 > 0$.
(iv) For some constant α in $(0, d)$ and $C_3 > 0$

$$P\{\omega\colon |Y(t, \omega)| < y, |Y(s, \omega)| < y\} \leqslant C_3 y^2 |t-s|^{-\alpha}$$

for $0 < y \leqslant 1$, $t \in B$, $s \in B$.

THEOREM. *$P\{\dim(Z \cap B) \geqslant d-\alpha\} > 0$, where $Z(\omega)$ is the zero-set of $Y(\cdot, \omega)$.*

Proof. Let $f(u)$ be defined for $u > 0$:

$$f(u) = \begin{cases} u^{\alpha-d}\log^{-2}(u^{-1}), & 0 < u \leqslant e^{-1}. \\ f(e^{-1}), & e^{-1} < u. \end{cases}$$

Define a (random) measure σ_y in B for $0 < y \leqslant 1$:

$$\sigma_y\{|Y| \geqslant y\} = 0, \qquad \sigma_y(G) = y^{-1}\mu(G) \quad \text{if} \quad G \subseteq \{|Y| < y\}.$$

Then

$$E(\sigma_y(B)) = y^{-1}\int P\{Y(t) < y\}\mu(dt) \geqslant C_2 \quad \text{for all } y.$$

$$E\left(\iint f(|t-s|)\,\sigma_y(dt)\,\sigma_y(ds)\right)$$
$$= y^{-2}\iint f(|t-s|) P\{|Y(t)| < y, |Y(s)| < y\}\mu(dt)\mu(ds)$$
$$\leqslant C_3 \iint f(|t-s|)|t-s|^{-\alpha}\mu(ds)\mu(dt) \leqslant C_4 < \infty$$

by (i), (iv) and the definition of f. Because f is bounded away from 0, $E\big((\sigma_y(B))^2\big) \leqslant C_5$ and so ([2], p. 8) there is a constant K such that

$$P\{\sigma_y(B) \geqslant K\} \geqslant K > 0,$$

and then

$$P\left\{\iint f(|t-s|)\,\sigma_y(dt)\,\sigma_y(ds) \leqslant 2K^{-1}C_4\right\} \geqslant 1-\tfrac{1}{2}K.$$

Thus if we set, for $0 < y \leqslant 1$,

$$A_y = \left\{\omega\colon \sigma_y(B) \geqslant K, \iint f(|t-s|)\,\sigma_y(dt)\,\sigma_y(ds) \leqslant 2K^{-1}C_4\right\},$$

then $P(A_y) \geqslant \frac{1}{2}K$, and so $P(\limsup A_{1/m}) \neq 0$. Now let $\omega \in \limsup A_{1/m}$, so that $\omega \in A_{1/m_j}$ for a sequence $m_j \to \infty$. Let σ be a weak* limit-point of the measures σ_{1/m_j}. We have $\|\sigma\| = \sigma(Z \cap B) \geqslant K > 0$,

$$\iint f(|t-s|)\,\sigma(dt)\,\sigma(ds) \leqslant 2K^{-1}C_4$$

and so ([3], III) $\dim(Z \cap B) \geqslant d-\alpha$.

§ 2. Let $1 \leqslant \mu \leqslant k \leqslant \nu$ be integers and

$$S_{\mu,k}(t) = \sum_{\mu}^{k} a_n X_n \cos(nt+\Phi_n).$$

Let E_k denote conditional expectation for the Borel field of $(X_1, \Phi_1, \ldots, X_k, \Phi_k)$. Then

$$E^{k-1}(S_{\mu,k}(t)) = S_{\mu,k-1}(t), \qquad u < k \leqslant \nu.$$

By Jensen's inequality, for any real y

$$E^{k-1}(\exp y|S_{\mu,k}(t)|) \geqslant \exp y|S_{\mu,k-1}(t)|,$$

$$E^{k-1}\Big(\int_0^{2\pi} \exp y|S_{\mu,k}(t)|\,dt\Big) \geqslant \int_0^{2\pi} \exp y|S_{\mu,k-1}(t)|\,dt.$$

Hence the integrals, say $I_k(y)$, form a submartingale for $\mu \leqslant k \leqslant \nu$; therefore ([1], p. 302) for $\lambda > 0$

$$P\{\max_k I_k(y) \geqslant \lambda\} \leqslant \lambda^{-1}E(I_\nu(y)).$$

To estimate $E(I_\nu(y))$ we fix t and $(\Phi_1, \ldots, \Phi_\nu)$ and find an integral $\leqslant \exp[\frac{1}{2}y^2 \sum_\mu^\nu a_n^2]$, whence

$$E(I_\nu(y)) \leqslant 2\pi\exp\Big[\tfrac{1}{2}y^2\sum a_n^2\Big].$$

Using S. Bernstein's inequality as in [2] we find that for any $z > 0$, $y > 0$

$$P\{\max_t\max_k |S_{\mu,k}(t)| \geqslant z\} \leqslant P\{\max_k I_k(y) \geqslant \nu^{-1}e^{1/2zy}\}$$
$$\leqslant 2\pi\nu\exp\Big[\tfrac{1}{2}y^2\sum a_n^2 - \tfrac{1}{2}zy\Big]$$
$$= 2\pi\nu\exp\Big[-\tfrac{1}{8}z^2\Big(\sum a_n^2\Big)^{-1}\Big],$$

for an optimal choice of $y > 0$. Finally, if $z = 4\log^{1/2}\nu(\sum a_n^2)^{1/2}$

(1) $$P\{\max_t\max_k |S_{\mu,k}(t)| \geqslant z\} \leqslant 2\pi\nu^{-2}.$$

This is substantially inequality (9) of [2], p. 7. Two useful consequences of (1) are these:

(a) F is almost surely uniformly convergent ([2], p. 8).
(b) F belongs almost surely to each class $\mathrm{Lip}(\alpha)$, $\alpha < \beta-\frac{1}{2}$ ([2], p. 13).

The remainder of this paragraph is concerned with the covariance

$$\sum_{n=1}^{\infty} a_n^2\cos(nt+\Phi_n)\cos(ns+\Phi_n) = g(t, s).$$

LEMMA. (i) *For almost all* $\Phi = (\Phi_n)_1^\infty$, $\inf_t g(t,t) > 0$, *and* $g(t,t) + g(s,s) > 2g(s,t)$ *for* $0 \leqslant s < t < 2\pi$.

(ii) *For each* $\varepsilon > 0$ *and almost all* Φ,

$$|g(t,t) - g(s,t)| = O(|t-s|^{2\beta-1-\varepsilon}) \quad \text{as} \quad |t-s| \to 0,$$

$$g(t,t) + g(s,s) - 2g(s,t) \geqslant |t-s|^{2\beta-1+\varepsilon} \quad \text{for small } |t-s|.$$

Proof. (i) Because $g(t,t) \geqslant a_1^2\cos^2(t+\Phi_1) + a_2^2\cos^2(2t+\Phi_2)$, $\inf g(t,t) > 0$ if $\Phi_2 \not\equiv 2\Phi_1 (\text{modulo } \frac{1}{2}\pi)$, and so $\inf g(t,t) > 0$ almost surely. A slightly more elaborate argument yields the second assertion of (i).

To prove (ii) we begin with the formula

$$\cos(ns+\Phi_n)\cos(nt+\Phi_n) = \tfrac{1}{2}\cos(ns-nt) + \tfrac{1}{2}\cos(ns+nt+2\Phi_n).$$

In the inequalities

$$\sum_{n=1}^{k} a_n^2 |1-\cos(ns-nt)| \leqslant |t-s|^2 \sum_{n=1}^{k} n^2 a_n^2 = |t-s|^2 O(k^{3-2\beta+\varepsilon}),$$

$$\sum_{n=k+1}^{\infty} a_n^2 = O(k^{1-2\beta+\varepsilon})$$

we choose k so that $1 \leqslant k|t-s| \leqslant 2$ and obtain

$$\sum_{n=1}^{\infty} a_n^2 |1-\cos(ns-nt)| = O(|t-s|^{2\beta-1-\varepsilon}).$$

We have also the inequality

$$\sum_{n=1}^{k} a_n^2 |\cos(ns+nt+2\Phi_n) - \cos(2nt+2\Phi_n)| \leqslant |t-s| \sum_{n=1}^{k} n a_n^2 = |t-s| O(k^{2-2\beta+\varepsilon}).$$

Exactly as we proved (1), we can prove that

$$\max \Big| \sum_{n=k+1}^{2k} a_n^2 \cos(ns+nt+2\Phi_n) \Big| \leqslant C_1 \log^{1/2} k \Big(\sum_{n=k+1}^{\infty} a_n^4 \Big)^{1/2}$$

except on a set of probability $P \leqslant C' k^{-2}$. Hence

$$\max \Big| \sum_{n=k+1}^{2k} a_n^2 \cos(ns+nt+2\Phi_n) \Big| = O(k^{1/2-2\beta+\varepsilon})$$

almost surely.

As $\frac{1}{2} - 2\beta + \varepsilon < 0$ for small ε, $\sum_{n=k+1}^{\infty} a_n^2 \cos(ns+nt+2\Phi_n)$ can be estimated by a geometric series, and then

$$\max \Big| \sum_{n=k+1}^{\infty} a_n^2 \cos(ns+nt+2\Phi_n) \Big| = O(k^{1/2-2\beta+\varepsilon}).$$

In the inequalities for $1 \leqslant n \leqslant k$ and $k < n < \infty$ we choose $k \sim |t-s|^{-2/3}$ and find

$$\sum_{n=1}^{\infty} a_n^2 \big(\cos(ns+nt+2\Phi_n) - \cos(2nt+2\Phi_n)\big) = O(|t-s|^{-\frac{1}{3}+\frac{4}{3}\beta-\varepsilon}).$$

But $\beta \leqslant 1$ implies $2\beta - 1 \leqslant \frac{4}{3}\beta - \frac{1}{3}$, and the first inequality in (ii) follows.

To prove the second inequality we use the fact that all the addends in the series $g(t,t) + g(s,s) - 2g(s,t)$ are positive. We estimate the sum upon the interval $k_1 \leqslant n \leqslant k_2$, where $|k_1|t-s|-1| < |t-s|$, $|k_2|t-s|-2| < |t-s|$. We have already proved that

$$\sum_{k_1}^{k_2} a_n^2 (\cos 2nt + 2\Phi_n) + \cos(2ns+2\Phi_n) - 2\cos(ns+nt+2\Phi_n) = O(k_1^{1/2-2\beta+\varepsilon}).$$

But

$$\sum_{k_1}^{k_2} a_n^2 (1-\cos(ns-nt)) \geqslant C|t-s|^2 \sum_{k_1}^{k_2} n^2 a_n^2 \geqslant C|t-s|^2 k_2^{3-2\beta-\varepsilon} \geqslant C|t-s|^{2\beta-1+\varepsilon}.$$

The last two inequalities complete the proof of the lemma.

§ 3. Here we give an upper bound to the stochastic closed set $Z(\omega) \cap B$. By a small effort we obtain a variant of the bound proposed in the introduction. Let G be a function in every class $\mathrm{Lip}(\alpha)$, $\alpha < \beta - \frac{1}{2}$, and let $Z^*(\omega)$ be the zero-set of

$$H(t,\omega) = F(t,\omega) + G(t).$$

THEOREM.

$$P\{\dim(Z^* \cap B) \leqslant d - \beta + \tfrac{1}{2}\} = 1 \quad \textit{provided} \quad \dim B = d > \beta - \tfrac{1}{2}.$$

$$P\{Z^* \cap B = \emptyset\} = 1 \quad \textit{provided} \quad d < \beta - \tfrac{1}{2}.$$

Proof. For each Φ, $H(t,\omega)$ is conditionally Gaussian, with variance

$$\sum_{n=1}^{\infty} a_n^2 \cos^2(nt+\Phi_n) \geqslant \varrho^2 > 0$$

for almost all Φ. Thus for numbers $\eta < \xi$ and any t

$$P^{\Phi}\{\eta < H(t) < \xi\} = P^{\Phi}\{\eta - G(t) < F(t) < \xi - G(t)\} \leqslant \varrho(\xi-\eta) = O(\xi-\eta).$$

Here the symbol O depends upon Φ but not upon t, ξ, or η.

Let us choose numbers s_1, s_2 so that $s_1 > d-\beta+\frac{1}{2}$, $s_2 < \beta-\frac{1}{2}$, $s_1+s_2 > d$. For each $r > 0$ we can find intervals $(a_1, b_1), \ldots, (a_N, b_N)$ so that

$$B \subseteq \bigcup_1^N (a_j, b_j), \quad \sum (b_j-a_j)^{s_1+s_2} < r, \quad \text{and} \quad 0 < b_j - a_j < r$$

for each j. Define

$$T_r(\omega) = \sum (b_j - a_j)^{s_1}$$

summed for all j such that $|H(b_j, \omega)| \leqslant (b_j-a_j)^{s_2}$.

Because H is almost surely in some class $\mathrm{Lip}(\alpha)$ with $\alpha > s_2$, there is a number $r_0(\omega) > 0$ so that, if $b_j - a_j < r < r_0(\omega)$ and H has a zero in $[a_j, b_j]$, then

$$|H(b_j, \omega)| \leqslant (b_j-a_j)^{s_1}.$$

Hence (for these values of r) the set $Z^*(\omega) \cap B$ can be enclosed in intervals (a_j', b_j') for which

$$0 < b_j' - a_j' < r, \quad \sum (b_j'-a_j')^{s_2} \leqslant T_r(\omega).$$

Therefore we can establish that

$$P\{\dim(Z^* \cap B) \leqslant s_1\} = 1$$

by showing that $T_r \to 0$ in P-measure as $r \to 0$. The latter follows from the fact that for almost all Φ $E^{\Phi}(T_r) = o(1)$ as $r \to 0$. In fact,

$$E^{\Phi}(T_r) = \sum_{j=1}^{N} (b_j - a_j)^{s_1} P^{\Phi}\{|H(b_j)| < (b_j-a_j)^{s_2}\}$$
$$= \sum_{j=1}^{N} O(b_j-a_j)^{s_1+s_2} = o(1),$$

as required.

By choosing a sequence of pairs (s_1, s_2) with $s_1 \to d-\beta+\frac{1}{2}$, we obtain the first part of the lemma.

In the case that $d < \beta - \frac{1}{2}$ we can always choose $s_1 = 0$, so that $T_r(\omega) < 1$ implies $Z^*(\omega) \cap B = \emptyset$. This completes the proof.

§ 4. In this paragraph we assemble results already obtained, to prove the lower bound asserted in the introduction. First, if $0 < d_1 < d$, B carries a probability measure μ such that

$$\mu([a, a+h]) \leqslant Ch^{d_1}$$

([3], II).

We apply the theorem to the process $F(t)$ with P^{Φ} evaluated at Φ. Because $\sum a_n^2 < \infty$ we have

$$P^{\Phi}\{|F(t)| < y\} \geqslant C_1 y$$

for a constant C_1 and $0 < y \leqslant 1$. The joint distribution of $F(t)$ and $F(s)$ can be determined as follows. We have $F(s) = A(s, t)F(t) + V$, where V is Gaussian, V has variance $g(s, s) - g^2(t, s)/g(t, t)$, and V is independent of $F(t)$.

Let us show that

$$g(t, t)g(s, s) - g^2(s, t) \geqslant C_2|t-s|^{2\beta-1+\varepsilon} \quad \text{for} \quad |t-s| \text{ small.}$$

In fact,

$$g(t, t)g(s, s) = g^2(s; t) + g(s, t)\big(g(t, t) + g(s, s) - 2g(s, t)\big) + $$
$$+ \big(g(t, t) - g(s, t)\big)\big(g(s, s) - g(s, t)\big),$$

or

$$g(t, t)g(s, s) - g^2(s, t) \geqslant O\big(|t-s|^{4\beta-2-\varepsilon}\big) + g(s, t)\,|t-s|^{2\beta-1+\varepsilon}$$

for $|t-s|$ small.

From the last estimation it easily follows that

$$P^{\Phi}\{|F(t)| < y, |F(s)| < y\} = O(|y| \cdot |y|\, |t-s|^{1/2-\beta-\varepsilon}).$$

Thus for almost all Φ we can apply the Theorem of § 2 with $\alpha = \beta - \frac{1}{2} + \varepsilon$, and find

$$P^{\Phi}\{\dim(Z \cap B) \geqslant d' + \tfrac{1}{2} - \beta - \varepsilon\} > 0.$$

This completes the proof.

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