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On a theorem of H. Goldstine

by

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The purpose of this paper is to generalize to locally convex spaces the theorem of H. Goldstine [2] which states that a Banach space with a separable dual is reflexive iff it is weakly sequentially complete. Our locally convex spaces may be metrizable or not. In the non-metrizable case nets will play the role which sequences play in the metrizable case. We shall work with special nets which we call Mackey nets. In terms of these we obtain a characterization of weak compactness and from this our generalization of Goldstine's result.

1. We begin with a short discussion of the terminology and notations which we will use. Throughout this paper the letter E will denote a locally convex, topological vector space over the field of real numbers. The dual of E will be denoted by E', the weak topology on E by $\sigma(E, E')$ and the weak* topology on E' by $\sigma(E', E)$. A subset U of E is said to be *circled* if $aU \subset U$ for all real numbers a such that $|a| \leqslant 1$. If U is circled we define U^0 to be: $\{f \text{ in } E' \mid |f(x)| \leqslant 1 \text{ for all } x \text{ in } U\}$. If we want to call attention to a specific locally convex topology t on E, we will write $E[t]$. A compact subset of $E[t]$ will be called *t-compact*. Similarly, we will speak of t-continuous linear functions, t-convergent nets, etc. A subset of E' is said to be *total* if its linear hull is $\sigma(E', E)$-dense in E'. Recall that the *Mackey topology* on E is the topology of uniform convergence on the convex $\sigma(E', E)$-compact subsets of E'. This topology will be denoted by $\tau(E, E')$ or, when there is no possibility of misunderstanding, by τ. Unless explicitly stated otherwise we shall assume that τ is non-metrizable, i.e. $E[\tau]$ does not have a countable, fundamental system of neighborhoods of zero.

Definition 1. Let $\mathscr{U}$ be a fixed fundamental system of circled, closed, convex, τ-neighborhoods of zero in E. A net $\{x_a \mid a \text{ in } A\}$ of points of E (see [6], p. 65, for a discussion of our notation and terminology for nets) will be called a *Mackey net* provided: (i) Cardinality of $A \equiv \operatorname{card} A = \operatorname{card} \mathscr{U}$; (ii) No countable subset of A is cofinal, i.e. no subnet of $\{x_a \mid a \text{ in } A\}$ is a sequence.

Our characterization of weak compactness is motivated by a simple observation of R. C. James ([4]; Theorem 2, (15), p. 107). He noticed that a bounded, weakly closed subset K of a Banach space B is weakly compact if for every sequence $\{x_n\} \subset K$ and any sequence $\{f_n\} \subset B'$ satisfying $\lim_n f_k(x_n)$ exists for every k, there is a point x_0 in K such that: $\lim_n f_k(x_n) = f_k(x_0)$ for every k. This condition is very weak, for there is little connection between the sequence $\{x_n\}$ and the point x_0. If, for instance, K is countably weakly compact, then x_0 is in the weak closure of $\{x_n\}$. However, if K satisfies condition (3) of James' theorem 1 ([4], p. 103), then we can only say that x_0 is in the closed, convex hull of $\{x_n\}$.

Definition 2. A bounded subset K of E is said to be *m-compact* if for every Mackey net $\{x_\alpha |\ \alpha \text{ in } A\}$ of points of K and any subset $\Gamma \subset E'$ with card Γ = card $\mathscr{U}$, there is a point x_0 in $K \cap$ [closed, linear hull of $\{x_\alpha |\ \alpha \text{ in } A\}$] such that $f(x_0)$ is an adherent point of $\{f(x_\alpha)|\ \alpha \text{ in } A\}$ for every f in Γ.

The following theorems remain valid in the event that τ is metrizable. However, Mackey nets must be replaced by sequences and some slight modifications must be made in the arguments.

Theorem 1. *An m-compact subset of a locally convex space is weakly closed.*

Proof. Let K be an m-compact subset of E and let x_0 be a point in the weak closure of K. Assume, for the moment, that card $K \leqslant$ card $\mathscr{U}$. Let H be the closed, linear subspace of E generated by K. Let $\mathscr{V} = \{U \cap H|\ U \text{ in } \mathscr{U}\}$ and, for V in $\mathscr{V}$, let $N_v = \{x \text{ in } H|\ x \text{ is in } \lambda V \text{ for all } \lambda > 0\}$. It is well known that the gauge function of V ([3], p. 94) is a norm for the space H/N_v. The natural map, n_v, from H onto H/N_v is continuous when H has $\tau(E, E')|H$ and H/N_v has its norm topology. Thus, $n_v(K)$ (i.e. $\{n_v(x)|\ x \text{ in } K\}$) generates the space H/N_v. It is easy to construct a subset Γ_v of $(H/N_v)'$ which is total and has cardinality less than or equal to that of $\mathscr{U}$. Since n_v is onto, its transpose, n_v^*, is a one-to-one map from $(H/N_v)'$ into H' ([3], p. 254). This map is continuous when both spaces have their weak* topologies ([3], Corollary to Proposition 3, p. 256] and so $n_v^*(\Gamma_v)$ is total in $n_v^*[(H/N_v)']$. But $H' = \bigcup \{n_v^*[H/N_v)']|\ V \text{ in } \mathscr{V}\}$, hence $\Gamma = \bigcup \{n_v^*(\Gamma_v)|\ V \text{ in } \mathscr{V}\}$ is total in H'. We may assume that this last union, which has cardinality less than or equal to that of $\mathscr{U}$, has cardinality equal to that of $\mathscr{U}$.

Now x_0 is in the $\sigma(E, E')$-closure of K and hence is in H. Recall that $\sigma(H, H') = \sigma(E, E')|H$ ([3], Proposition 1, p. 261). Let $\{x_\lambda | \lambda \text{ in } A\}$ be a net in K which is weakly convergent to x_0. Since K is bounded, the set $S(f) = \{f(x)|\ x \text{ in } K\}$ is a bounded set of real numbers for every f in Γ. If $\mathrm{cl}\, S(f)$ denotes the closure of $S(f)$ for the usual topology of the reals, then $\Pi\{\mathrm{cl}\, S(f)|\ f \text{ in } \Gamma\}$ is a compact space. For each λ in A let p_λ be the point $\{f(x_\lambda)|\ f \text{ in } \Gamma\}$ and, for λ_0 in A, let F_{λ_0} be the set $\{p_\lambda |\ \lambda \geqslant \lambda_0\}$. Since A is a directed set, any finite number of the sets F_λ has a point in common. If $\mathrm{cl}\, F_\lambda$ denotes the closure of F_λ for the product space topology, then there is a point p in $\bigcap \{\mathrm{cl}\, F_\lambda | \lambda \text{ in } A\}$. Let $\Sigma = \{\lambda; f_1, f_2, \ldots, f_n)|\ \lambda$ in $A; f_1, f_2, \ldots, f_n$ in Γ; n finite$\}$. Clearly card Σ = card $\mathscr{U}$. We partially order Σ by defining $(\lambda_0; f_1, \ldots, f_k) \leqslant (\lambda_1; g_1, \ldots, g_m)$ to mean: $\lambda_0 \leqslant \lambda_1$ and $\{f_i|\ 1 \leqslant i \leqslant k\} \subset \{g_j|\ 1 \leqslant j \leqslant m\}$. Note that Σ becomes a directed set and if it had a countable cofinal set, then so would $\{(f_1, f_2, \ldots, f_n)|\ f_1, f_2, \ldots, f_n \text{ in } \Gamma; n \text{ finite}\}$ which is impossible because Γ is uncountable. Define a map φ from Σ into A as follows: If $\sigma = (\lambda_0; f_1, f_2, \ldots, f_k)$ choose p_λ in F_{λ_0} such that $|p_\lambda(f_i) - p(f_i)| < 1/k$ for $1 \leqslant i \leqslant k$, and then let $\varphi(\sigma)$ be this λ. Clearly $\{x_{\varphi(\sigma)}|\ \sigma \text{ in } \Sigma\}$ is a Mackey net which is a subnet of $\{x_\lambda | \lambda \text{ in } A\}$. Since the latter net is weakly convergent to x_0, so is the former. Now K is m-compact so there is a point y in K such that $f(y)$ is an adherent point of $\{f(x_{\varphi(\sigma)}|\ \sigma \text{ in } \Sigma\}$ for every f in Γ. It follows that $f(x_0) = f(y)$ for every f in Γ. But x_0 and y are in H and Γ is total in H', hence $x_0 = y$ and K is weakly closed.

We began with an m-compact set K and a point x_0 in its weak closure. To complete the proof of our theorem we need only show that K has a subset, N say, such that: x_0 is in the weak closure of N and card $N \leqslant$ card $\mathscr{U}$. Let $\mathscr{F} = \{U^0|\ U \text{ in } \mathscr{U}\}$. The family $\mathscr{F}$ is a covering of E' by sets which are circled, convex and weak* compact ([3], Proposition 1 (e), p. 190, Proposition 6, p. 200, and Theorem 1, p. 201). Recall that each x in E can be regarded as a continuous function $\hat{x}$ on $E'[\sigma(E', E)]$; here $\hat{x}(f) = f(x)$ for all f in E'. If X is in $\mathscr{F}$, let $C(X)$ be the family of all real-valued, continuous functions on X, and let φ_X be the map which takes each x in E to $\hat{x}|X$. It is clear that φ_X is linear and, if E has $\sigma(E, E')$ and $C(X)$ the topology of pointwise convergence, continuous. Hence $\varphi_X(x_0)$ is in the closure of $\varphi_X(K)$ for the topology of pointwise convergence.

We shall prove that there is a countable set $N'_X \subset \varphi_X(K)$ whose closure, for the topology of pointwise convergence, contains $\varphi_X(x_0)$. Choose two positive integers m and n. Let P denote the product space constructed from m copies of the space X. We shall find a finite set $M(m, n) \subset \varphi_X(K)$ such that: Given $y = (y_1, y_2, \ldots, y_m)$ in P we can find f in $M(m, n)$ for which $|f(y_i) - \varphi_X(x_0) y_i| < 1/n$ for $1 \leqslant i \leqslant m$. Once we have done this it is easy to see that N'_X can be taken to be $\bigcup\{M(m, n)|\ m, n \text{ positive integers}\}$. Since $\varphi_X(x_0)$ is in the pointwise closure of $\varphi_X(K)$, we can, given $y = (y_1, y_2, \ldots, y_m)$ in P, find f in $\varphi_X(K)$ such that $|f(y_i) - \varphi_X(x_0) y_i| < 1/n$ for $1 \leqslant i \leqslant m$. By the continuity of f and $\varphi_X(x_0)$ there is a neighborhood V_i of y_i, $1 \leqslant i \leqslant m$, such that $|f(x) - \varphi_X(x_0) x| < 1/n$ for all x in V_i and for $1 \leqslant i \leqslant m$. Clearly $V_1 \times V_2 \times \ldots \times V_m$ is a neighborhood of y in P. By varying the point y we obtain an open covering of P which, since P is compact, has a finite subcovering. Let this finite subcovering be $\{V_1^j \times V_2^j \times \ldots \times V_m^j \mid 1 \leqslant j \leqslant k\}$. To each set in this covering there is associated a function f^j in $\varphi_X(K)$ and it is clear that we may take $M(m, n)$ to be $\{f^j \mid 1 \leqslant j \leqslant k\}$.

We have found a countable set $N'_X \subset \varphi_X(K)$ whose closure contains $\varphi_X(x_0)$. For each X in $\mathscr{F}$ let N_X be a countable subset of K such that $\varphi_X(N_X) = N'_X$, and let $N = \bigcup\{N_X \mid X \text{ in } \mathscr{F}\}$. Clearly $N \subset K$ and card $N \leqslant$ card $\mathscr{U}$. We shall now show that x_0 is in the $\sigma(E, E')$-closure of N. Let $f_1, f_2, \ldots, f_n$ in E' and $\varepsilon > 0$ be given. Choose U in $\mathscr{U}$ such that f_i is in U^0 for $1 \leqslant i \leqslant n$. For notational convenience let U^0, which is in $\mathscr{F}$, be denoted by X. If $V = \{x \text{ in } E \mid |f_i x - f_i x_0| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\}$, then $\varphi_X(V) = \{g \text{ in } C(X) \mid |g(f_i) - \varphi_X(x_0) f_i| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \cap \varphi_X(E)$. Since $\varphi_X(N_X) \subset \varphi_X(E)$, $\varphi_X(V)$ must meet $\varphi_X(N_X)$. Thus, there is a point y in V and a point z in N_X such that $\varphi_X(y) = \varphi_X(z)$. It follows that $f_i y = f_i z$ for $1 \leqslant i \leqslant n$. But y is in V so $|f_i y - f_i x_0| < \varepsilon$ for $1 \leqslant i \leqslant n$. We conclude that z is in V also, and hence that z is in $V \cap N_X \subset V \cap N$.

Remark 1. In the second paragraph of the proof of Theorem 1 we have proved the following: If $\{x_\beta \mid \beta \text{ in } B\}$ is any bounded net in E and Γ is any subset of E' with card Γ = card $\mathscr{U}$, then there is a subnet of $\{x_\beta \mid \beta \text{ in } B\}$, say $\{x_\sigma \mid \sigma \text{ in } \Sigma\}$, which satisfies: (a) $\{x_\sigma \mid \sigma \text{ in } \Sigma\}$ is a Mackey net; (b) $\lim f(x_\sigma)$ exists for every f in Γ.

Remark 2. The proof of Theorem 1 can be modified to prove the following: If τ is metrizable, then a countably weakly compact subset of E is weakly closed iff it is weakly sequentially closed.

2. The second dual of E, E'' is the space of all linear functionals on E' which are $\beta(E', E)$-continuous. Recall that $\beta(E', E)$ is the topology of uniform convergence on the convex, $\sigma(E, E')$-bounded subsets of E. There is a natural algebraic isomorphism from E into E''. One maps each x in E to the linear functional $\hat{x}$ on E' defined by: $\hat{x}(f) = f(x)$ for all f in E'. We shall often identify E with its image in E'' under this map.

THEOREM 2. *An m-compact subset of a complete locally convex space is weakly compact.*

Proof. Let K be an m-compact subset of the complete space $E[\tau(E, E')]$, and let Γ be any subset of E' having cardinality equal to that of $\mathscr{U}$. By Theorem 1, K is weakly closed. Hence, by a deep result of R. C. James ([5], Theorem 6, p. 139), it suffices to show that every element of E' attains its supremum over K at some point of K. Let f_0 be in E'. Since the closure of K for $E''[\sigma(E'', E')]$ is compact ([3], Theorem 1, p. 201), there is a point z in this closure at which f_0 attains its supremum over K. Let $\{x_\lambda \mid \lambda \text{ in } \Lambda\}$ be a net in K which is $\sigma(E'', E')$-convergent to z. Since $\Gamma \cup \{f_0\}$ has cardinality equal to that of $\mathscr{U}$, we can, by Remark 1, construct a Mackey net $\{x_\sigma \mid \sigma \text{ in } \Sigma\}$ which is a subnet of $\{x_\lambda \mid \lambda \text{ in } \Lambda\}$ and for which $\lim f(x_\sigma)$ exists for all f in $\Gamma \cup \{f_0\}$. Since K is m-compact, it contains a point y such that $\lim f(x_\sigma) = f(y)$ for every f in $\Gamma \cup \{f_0\}$. In particular, $f_0(y) = \lim f_0(x_\sigma) = f_0(z)$. But this says that f_0 attains its supremum over K at the point y of K.

We are now prepared to prove our generalization of Goldstine's theorem:

THEOREM 3. *Let E be a complete locally convex space. Assume that E' contains a $\beta(E', E)$-dense subset Γ with* card Γ = card $\mathscr{U}$. *Then E is semi-reflexive iff every bounded, weakly Cauchy, Mackey net in E is weakly convergent to a point of E.*

Proof. Recall: E is said to be semi-reflexive iff $E = E''$. It is well known that E is semi-reflexive iff every bounded, weakly closed subset of E is weakly compact ([3], Proposition 1, p. 227). Thus, the necessity of our condition is obvious.

Let K be any bounded, weakly closed subset of E and let $\{x_\alpha \mid \alpha \text{ in } A\}$ be any Mackey net in K. By Remark 1 we can find a Mackey net $\{x_\sigma \mid \sigma \text{ in } \Sigma\}$ which is a subnet of $\{x_\alpha \mid \alpha \text{ in } A\}$ and for which $\lim f(x_\sigma)$ exists for every f in Γ. Let f in E' but not in Γ, and $\varepsilon > 0$ be given. Since $\{x_\sigma \mid \sigma \text{ in } \Sigma\}$ is bounded and Γ is $\beta(E', E)$-dense in E', we can find f_0 in Γ such that $|(f - f_0) x_\sigma| < \varepsilon$ for every σ in Σ. Combining this with the fact that $\lim f_0(x_\sigma)$ exists, we see that $\lim f(x_\sigma)$ exists; i.e. we see that $\{x_\sigma \mid \sigma \text{ in } \Sigma\}$ is weakly Cauchy. Thus, by hypothesis, this net converges weakly to a point of K. Clearly, K is m-compact. Hence by Theorem 2 and [3], Proposition 1, p. 227, E is semi-reflexive.

THEOREM 4. *Let K be an m-compact subset of E. A subset of E' having cardinality less than or equal to that of $\mathscr{U}$ is uniformly bounded on K iff it is bounded at each point of K.*

Proof. This follows easily from Remark 1 and a recent result of Brace and Nielsen ([1], Theorem 1, p. 625).

References

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