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Some curious bases for c_0 and $C[0,1]$

by

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§ 1. Introduction. In the first part of this note we exhibit two interesting bases for c_0, the Banach space of sequences tending to 0 with the sup norm.

The first example unifies some recent results [28], [24], [16], [17], concerning bases in this space. Our second example is actually an analysis of a previous example of J. Lindenstrauss [15]. This second example is used in the second part of the note to provide a particular basis for $C[0,1]$.

Indeed, if (x_i) is Schauder basis for a Banach space X with coefficient functionals (f_i), then $[f_i]$ is called the *coefficient space* of the basis (x_i). We show in part 2 that $C[0,1]$, the continuous functions on $[0,1]$ with the sup norm, has a basis (x_i) with coefficient space $[f_i]$ isomorphic to neither l^1 nor $L^1[0,1]$. After presenting the example we make some remarks concerning the intrinsic difficulties in constructing such an example. In the final section we raise some questions which appear to be related to a classical problem of Banach.

§ 2. Notation and terminology. Let X be a Banach space. A sequence (x_i) in X is a (*Schauder*) *basis* for X provided for each $x \in X$ there is a unique sequence of scalars (a_i), such that

(2.1) $$x = \sum_{i=1}^{\infty} a_i x_i,$$

with convergence in the norm topology of X. The linear forms given by

(2.2) $$f_i(x) = a_i$$

are continuous and $f_i(x_{ij}) = \delta_{ij}$, i.e. (x_i, f_i) is a *biorthogonal pair*. A basis (x_i) is *unconditional* if each expansion (2.1) is unconditionally convergent, i.e. each rearrangement of (2.1) converges to x. A basis (x_i) is *shrinking* provided (f_i) is a basis for X^*, where f_i is given by (2.2) (there are many equivalent ways to define shrinking; see, e.g. [23]); *boundedly complete* if $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < +\infty$ implies $\sum_{i=1}^{\infty} a_i x_i$ converges; *type P* provided

$\inf_n \|x_n\| > 0$ and $\sup_n \|\sum_{i=1}^n x_i\| < +\infty$; and *type* P^* provided $\sup_n \|x_n\| < +\infty$ and $\sup_n \|\sum_{i=1}^n f_i\| < +\infty$.

The notions of shrinking and boundedly complete are due to R. C. James [13] (the terminology to Day [7]) and the notions of type P and type P^* are due to I. Singer [28].

EXAMPLE 1. *A conditional, shrinking, type P basis for* c_0.

Before proceeding to the example we list some known facts to show the pertinence of this example to the literature.

I. ([13], Lemma 2, p. 520). *Every unconditional basis for* c_0 *is shrinking.*

II. ([28], p. 358; see also [24]). *Every unconditional basis of type* P *is equivalent to the unit vector basis* (e_n) *of* c_0 $(e_n = (\delta_{n_i}))$.

III. ([16], Theorem 6.1, p. 295). *Every unconditional basis* (x_n) *for* c_0 *satisfying* $0 < \inf_n \|x_n\| \leqslant \sup_n \|x_n\| < +\infty$ *is of type* P.

Of course, c_0 has conditional bases, e.g. (x_n) where $x_n = \sum_{i=1}^n e_i$. The basis (x_n) is non-shrinking ($[f_i]$, the closed linear span of (f_i) in $c_0^* = l^1$, has co-dimension 1) and is of type P^*.

We construct our example in two steps.

Let $x_1 = e_1$ and $x_n = -x_{n-1} + (n-1)e_n$ for $n \geqslant 2$, and $f_1 = e_1 + e_2$, $f_n = \frac{1}{n-1} e_n + \frac{1}{n} e_{n+1}$ for $n \geqslant 2$.

1° (x_n) *is a basis for* c_0.

If $a = (a_i) \in c_0$, then

$$a = \sum_{i=1}^{\infty} a_i e_i = a_1 e_1 + \sum_{i=2}^{\infty} a_i \left(\frac{x_i + x_{i-1}}{i-1}\right)$$

$$= (a_1 + a_2) e_1 + \sum_{i=2}^{\infty} \left(\frac{a_i}{i-1} + \frac{a_{i+1}}{i}\right) x_i = \sum_{i=1}^{\infty} f_i(a) x_i.$$

This expansion is unique since (x_n, f_n) is a biorthogonal pair.

2° (x_n) *is shrinking.*

It suffices ([23], Theorem 3.1, p. 843) to show that $[f_m] = l^1$. Now

$$\left\| e_1 - \sum_{j=1}^{N} (-1)^{j+1} f_j \right\| = \frac{1}{N}$$

and it follows that $e_1 \in [f_m]$. From the definition of f_n it now follows that $e_n \in [f_m]$ for all n and hence $[f_m] = l^1$.

3° (x_n) *is a conditional basis.*

If (x_n) were unconditional, then by [13] (Theorem 3, p. 522) (f_n) would be an unconditional basis for l^1.

In the proof of 2° we showed that $\sum_{j=1}^{\infty} (-1)^j f_j$ converges. Thus if (f_i) were unconditional, $\sum_{j=1}^{\infty} f_j$ would converge and so

$$\sum_{n=2}^{\infty} \left(\frac{1}{n-1} e_n + \frac{1}{n} e_{n+1}\right).$$

would converge. Of course, in l^1 this is impossible.

Now (x_n) is not of type P. To remedy this we do the following: Let $y_1 = x_1$, $y_2 = x_2$ and $y_n = \frac{1}{n-1} x_n$ for $n \geqslant 3$. Then (y_n) still obviously satisfies 1°, 2° and 3°. Moreover,

4° (y_n) *is of type* P.

Since $\|x_1\| = \|x_2\| = 1$ and $\|x_p\| = p-1$ for $p \geqslant 3$ it follows that $\|y_N\| = 1$ for all N. Also

$$\left\| \sum_{j=1}^{N} y_j \right\| = \left\| x_1 + x_2 - \left(\sum_{i=2}^{N-1} \frac{x_i}{i}\right) + \sum_{j=3}^{N} e_j \right\|$$

$$= \left\| e_1 + (e_2 - e_1) - \left[\frac{e_2 - e_1}{2} + \frac{e_1 - e_2 + 2e_3}{3} + \ldots + (-1)^{N+1}\left(\frac{e_1 - \sum_{i=2}^{N-1} (-1)^i (i-1) e_i}{N-1}\right)\right] + \sum_{j=3}^{N} e_j \right\|$$

$$= \left\| \sum_{j=1}^{N} e_j - \left\{\left[1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{(-1)^{N-1}}{N-1}\right] e_1 + \left[\frac{1}{2} - \frac{1}{3} + \ldots + \frac{(-1)^{N-1}}{N-1}\right] e_2 + 2\left[\frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{N-1}}{N-1}\right] e_3 + \ldots + (N-2)\left[\frac{1}{N-2} - \frac{1}{N-1}\right] e_{N-2}\right\} + e_{N-1} \right\| \leqslant 2 \left\| \sum_{j=1}^{N} e_j \right\| = 2$$

since the coefficient of each e_j in the bracketed expression is less than 1.

This completes the first example.

We remark that it now follows immediately from the duality results of [28] that l^1 *has a conditional, boundedly complete type* P^* *basis.*

EXAMPLE 2. *A Lindenstrauss basis for* c_0.

A sequence (x_n) in a Banach space X is a *basic sequence* if (x_n) is a basis for its closed linear span $[x_n]$.

In a remarkable paper [15] Lindenstrauss constructed a basic sequence (f_n) in l^1 with a number of unusual properties:

(a) $[f_n]$ has no unconditional basis;
(b) $[f_n]$ is not isomorphic to a conjugate space;
(c) $[f_n]$ is complemented in no conjugate space;
(d) $[f_n]^*$ is isomorphic to l^∞.

If (x_i) is a basis for a Banach space X with coefficient functionals (f_i), then we will call (x_i) a *Lindenstrauss basis* if $[f_n]$ satisfies (a), (b), and (c) above.

We now show that the sequence (f_n) constructed by Lindenstrauss is the sequence of coefficient functionals for a basis (x_n) for c_0, i.e. c_0 has a Lindenstrauss basis.

For a real number λ, $[\lambda]$ denotes the greatest integer $\leqslant \lambda$. For each positive integer n let $\gamma_0(n) = n$ and $\gamma_{j+1}(n) = \left[\frac{\gamma_j(n)-1}{2}\right]$.

Let e_i be the ith unit vector in c_0 (resp. l^1) for $i > 0$ and $e_i = 0$ for $i \leqslant 0$. Define x_n and f_n by

$$x_n = \sum_{j=0}^{n} \frac{1}{2^j} e_{\gamma_j(n)} \text{ in } c_0 \quad \text{and} \quad f_n = e_n - \frac{1}{2}(e_{2n+1}+e_{2n+2}) \text{ in } l^1.$$

The sequence (f_n) is precisely that constructed in [15]. We first show that (x_n, f_n) is a biorthogonal pair. To see this we consider three cases.

1° $m < n$: For $j = 0, 1, \ldots, m$, $\gamma_j(m) \leqslant \gamma_0(m) = m < n$ and thus

$$e_k\Big(\sum_{j=0}^{m} \frac{1}{2^j} e_{\gamma_j(m)}\Big) = 0$$

for $k = n, 2n+1$ and $2n+2$ and it follows that $f_n(x_m) = 0$.

2° $m = n$: Now $\gamma_0(m) = m = n$ and $\gamma_j(m) < m = n$ for $j = 1, 2, \ldots, m$. Hence

$$e_n\Big(\sum_{j=0}^{m} \frac{1}{2^j} e_{\gamma_j(m)}\Big) = 1 \quad \text{and} \quad e_{2n+i}\Big(\sum_{j=0}^{m} \frac{1}{2^j} e_{\gamma_j(m)}\Big) = 0 \quad \text{for } i = 1, 2.$$

It follows that $f_n(x_n) = 1$.

3° $n < m$: If $n < m < 2n+1$, then $n < \gamma_0(m) < 2n+1$ and $\gamma_j(m) < n$ for $j = 1, \ldots, m$, by the definition of $\gamma_j(m)$. Hence $f_n(x_m) = 0$. Now

suppose $m \geqslant 2n+1$. In this case $\gamma_0(m) \neq n$ and by definition $\gamma_m(m) \leqslant 0$, $\gamma_j(m) = n$ if and only if $\gamma_{j-1}(m) = 2n+1$ or $\gamma_{j-1}(m) = 2n+2$. With these observations, it follows from the definition of f_n and x_m that $f_n(x_m) = 0$. Thus (x_n, f_n) is a biorthogonal pair. We now claim that (x_n) is a basis for c_0.

For, if $a = (a_i) \in c_0$, then

$$\sum_{i=1}^{n} f_i(a) x_i = \sum_{i=1}^{n} \Big(a_i - \frac{1}{2} a_{2i+1} - \frac{1}{2} a_{2i+2}\Big)\Big(\sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big)$$

$$= \sum_{i=1}^{n} a_i e_i + \Big\{\sum_{i=1}^{n} a_i \Big(\sum_{j=1}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big) - \frac{1}{2}\sum_{i=1}^{n} (a_{2i+1} + a_{2i+2}) \Big(\sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big)\Big\}$$

$$= \sum_{i=1}^{n} a_i e_i + P(a, n),$$

where $P(a, n)$ denotes the bracketed expression. To show that (x_n) is a basis it is enough to show that $P(a, n) \to 0$ as $n \to \infty$ for each $a \in c_0$.

Let $n \geqslant 3$. Then $n = 2k+1$ or $2k+2$. We claim that in either case

$$\|P(a, n)\| \leqslant 7 \sup_{i \geqslant 2k+1} |a_i|.$$

To see this, first observe that

$$P(a, 2k+1) = \sum_{i=1}^{2k+1} a_i \Big(\sum_{j=1}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big) - \frac{1}{2}\sum_{i=1}^{2k+1} (a_{2i+1} + a_{2i+2}) \Big(\sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big)$$

$$= -\frac{1}{2} a_{2k+2} \sum_{j=0}^{k} \frac{1}{2^j} e_{\gamma_j(i)} - \frac{1}{2}\sum_{i=k+1}^{2k+1} (a_{2i+1} + a_{2i+2}) \Big(\sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big).$$

To see this, last equality we argue as follows. To prove the equality it is clearly enough to show that

$$\sum_{i=1}^{2k+1} a_i \Big(\sum_{j=1}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big) - \frac{1}{2}\sum_{i=1}^{k-1} (a_{2i+1} + a_{2i+2}) \Big(\sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)}\Big) -$$

$$- \frac{1}{2} a_{2k+1} \Big(\sum_{j=0}^{k} \frac{1}{2^j} e_{\gamma_j(k)}\Big) = 0.$$

This is readily seen from the following lemma:

LEMMA. *For all* $n = 1, 2, \ldots$,

$$\sum_{j=1}^{2n+1} \frac{1}{2^j} e_{\gamma_j(2n+1)} = \frac{1}{2} \sum_{j=0}^{n} \frac{1}{2^j} e_{\gamma_j(n)} \quad \textit{and} \quad \sum_{j=1}^{2n+2} \frac{1}{2^j} e_{\gamma_j(2n+2)} = \frac{1}{2} \sum_{j=0}^{n} \frac{1}{2^j} e_{\gamma_j(n)}.$$

Proof. Clearly $\gamma_1(2n+1) = \gamma_0(n)$. Suppose $\gamma_i(2n+1) = \gamma_{i-1}(n)$ for $i = 1, \ldots, k$. Then

$$\gamma_{k+1}(2n+1) = \left[\frac{\gamma_k(2n+1)-1}{2}\right] = \left[\frac{\gamma_{k-1}(n)-1}{2}\right] = \gamma_k(n).$$

Thus, by induction, $\gamma_i(2n+1) = \gamma_{i-1}(n)$ for all $i \geqslant 1$. In exactly the same way, $\gamma_i(2n+2) = \gamma_{i-1}(n)$ for all $i \geqslant 1$. Thus

$$\sum_{j=1}^{2n+1} \frac{1}{2^j} e_{\gamma_j(2n+1)} = \sum_{j=1}^{2n+1} \frac{1}{2^j} e_{\gamma_{j-1}(n)} = \frac{1}{2} \sum_{j=0}^{n} \frac{1}{2^j} e_{\gamma_j(n)},$$

since $\gamma_j(k) \leqslant 0$ for $j \geqslant k$. Similarly for $2n+2$. This proves the lemma.

Now

$$\sum_{j=1}^{i} \frac{1}{2^j} e_{\gamma_j(i)} = 0 \quad \text{for} \quad i = 1, 2.$$

For $i \geqslant 3$, $i = 2n+1$ or $2n+2$ for some $n \geqslant 1$. Suppose $i = 2n+1$. Then by the lemma,

$$a_{2n+1} \sum_{j=1}^{2n+1} \frac{1}{2^j} e_{\gamma_j(2n+1)} - \frac{1}{2} a_{2n+1} \left(\sum_{j=0}^{n} \frac{1}{2^j} e_{\gamma_j(n)} \right) = 0.$$

Similarly if $i = 2n+2$, and thus

$$\sum_{i=3}^{2k} a_i \left(\sum_{j=1}^{i} \frac{1}{2^j} e_{\gamma_j(i)} \right) - \frac{1}{2} \sum_{i=1}^{k-1} (a_{2i+1} + a_{2i+2}) \left(\sum_{j=0}^{i} \frac{1}{2^j} \gamma_j(i) \right) = 0.$$

Since

$$a_{2k+1} \left(\sum_{j=1}^{2k+1} \frac{1}{2^j} e_{\gamma_j(2k+1)} \right) = \frac{1}{2} a_{2k+1} \sum_{j=0}^{k} \frac{1}{2^j} e_{\gamma_j(k)}$$

by the lemma, we have the desired equality.

From the definition of $\gamma_j(i)$ it follows that

$$\left\| \sum_{i=k+1}^{2k+1} \left(\sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)} \right) \right\| < 2 \quad \text{for} \quad k = 1, 2, 3, \ldots,$$

and

$$\left\| \sum_{j=0}^{i} \frac{1}{2^j} e_{\gamma_j(i)} \right\| < 2 \quad \text{for any } i.$$

Thus,

$$\|P(a, 2k+1)\| \leqslant \frac{1}{2} \cdot 2 |a_{2k+2}| + \frac{1}{2} \cdot 2 \max_{k<i\leqslant 2k+1} |a_{2i+1} + a_{2i+2}| \leqslant 3 \sup_{i \geqslant 2k+1} |a_i|.$$

Also

$$P(a, 2k+2) = P(a, 2k+1) + a_{2k+2} \sum_{j=1}^{2k+2} \frac{1}{2^j} e_{\gamma_j(2k+2)} - \frac{1}{2}(a_{4k+5} + a_{4k+6}) \sum_{j=0}^{2k+2} \frac{1}{2^j} e_{\gamma_j(2k+2)},$$

whence

$$\|P(a, 2k+2)\| \leqslant 3 \sup_{i \geqslant 2k+1} |a_i| + 2|a_{2k+2}| + |a_{4k+5}| + |a_{4k+6}| \leqslant 7 \sup_{i \geqslant 2k+1} |a_i|.$$

Thus $P(a, n) \to 0$ as $n \to \infty$.

By [1] (Theorem 1, p. 68) (f_n) is an ω^*-Schauder basis for l^1. Thus we have the following interesting corollary:

l^1 has an ω^*-Schauder basis (f_n) such that $l^1/[f_n]$ is isomorphic to $L^1[0, 1]$.

This result is immediate from our construction and the result of Lindenstrauss [15].

§ 3. We now use the Lindenstrauss basis for c_0 to construct a basis for $C[0, 1]$ whose coefficient space is isomorphic to neither l^1 nor $L^1[0, 1]$.

Let us recall that the classical Schauder basis (s_n) for $C[0, 1]$ is given by $s_0(t) = 1$, $s_1(t) = t$,

$$s_{2^k+j}(t) = \begin{cases} 0 & \text{for } t \in \left(\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}} \right), \\ 1 & \text{for } t = \frac{2j-1}{2^{k+1}}, \\ \text{linear for the other } t \end{cases}$$

$$(j = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots).$$

If (l_n) is the sequence of coefficient functionals associated with (s_n), we let $F = [l_n] \subset (C[0, 1])^* = \mathrm{NBV}[0, 1]$, the normalized functions of bounded variation (see, e.g., [9]).

I. *F is isometrically isomorphic to l^1.*

Now I has been shown by Foiaş and Singer ([10], Theorem 3, p. 942). We reproduce the proof here, for we need the explicit form of the isomorphism.

Proof. Z. Ciesielski [3] has observed that the sequence (l_n) is given by $l_0(x) = x(0)$, $l_1(x) = x(1) - x(0)$,

$$(*) \qquad l_{2^k+j}(x) = x\left(\frac{2j-1}{2^{k+1}}\right) - \frac{1}{2} x\left(\frac{2j-2}{2^{k+1}}\right) - \frac{1}{2} x\left(\frac{2j}{2^{k+1}}\right)$$

$$(x \in C[0, 1];\ j = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots).$$

Let $t_0=0, t_1=t$ and $t_{2^k+j}=(2j-1)/2^{k+2}$ $(j=1,2,\ldots,2^k; k=0,1,\ldots)$, and $\xi_{t_n}(x)=x(t_n)$ for $x\in C[0,1]$. It is clear from (*) that $[\xi_{t_n}]=[l_n]=F$. If $a_0, a_1, \ldots, a_n$ are scalars, then

$$(**)\qquad \Big\|\sum_{i=0}^{n} a_i\xi_{t_i}\Big\| = \sup_{\substack{x\in C[0,1]\\ \|x\|\leqslant 1}}\Big|\sum_{i=0}^{n} a_i\xi_{t_i}(x)\Big| = \sum_{i=0}^{n}|a_i|$$

($\leqslant$ is obvious; for $\geqslant$ choose $x\in C[0,1]$ such that $\|x\|=1$ and $x(t_i)=\operatorname{sgn} a_i$ $(i=0,1,\ldots,n)$). Thus (ξ_{t_n}) is equivalent to unit vector basis (e_i) of l^1 and the map $\xi_{t_i}\to e_i$ is an isometry by (**).

Now let (φ_i) be any orthogonal basis for $C[0,1]$ (e.g. the Franklin basis obtained by applying the Gram-Schmidt process to (s_n), see [4]) with coefficient functionals (ψ_i). Let $G=[\psi_i]$. The following is surely well-known but does not seem to appear in the literature.

II. *G is isometrically isomorphic to $L^1[0,1]$.*

Proof. Since (φ_i) is orthogonal, i.e. $\int_0^1 \varphi_i(t)\varphi_j(t)\,dt=\delta_{ij}$, it is clear that each ψ_i is given by

$$\psi_i(t)=\int_0^t \varphi_i(s)\,ds.$$

For any scalars $a_1,\ldots,a_n$,

$$\Big\|\sum_{i=1}^{n} a_i\psi_i\Big\|_V=\int_0^1\Big|\sum_{i=1}^{n} a_i\varphi_i(t)\Big|\,dt,$$

where V denotes the variation norm. Thus the mapping $\psi_i\to\varphi_i$ is the desired isometry, since it is well known that (φ_i) is a basis for $L^1[0,1]$ (see e.g. [9], p. 358). (For another example of a basis with $[\psi_i]$ isomorphic to $L^1[0,1]$, see [10].)

Let us recall that, for Banach spaces X and Y, $X\hat{\hat{\otimes}}Y$ denotes the completion of the algebraic tensor product $X\otimes Y$ in the norm given by

$$\Big\|\sum_{i=1}^{n} x_i\otimes y_i\Big\|^{\wedge\!\!\wedge}=\sup_{\substack{\|f\|\leqslant 1, f\in X^*\\ \|g\|\leqslant 1, g\in Y^*}}\Big|\sum_{i=1}^{n} f(x_i)g(y_i)\Big|$$

and $X\hat{\otimes}Y$ denotes the completion of $X\otimes Y$ in the norm

$$\Big\|\sum_{i=1}^{n} x_i\otimes y_i\Big\|^{\wedge}=\inf\Big\{\sum_{i=1}^{k}\|x_i'\|\,\|y_i'\|:\ \sum_{i=1}^{k} x_i'\otimes y_i'=\sum_{i=1}^{n} x_i\otimes y_i\Big\}.$$

Our next result is a trivial consequence of the work of Grothendieck [12].

III. *If X and Y are Banach spaces and X^* has a basis, then*

$$(X\hat{\hat{\otimes}}Y)^*=X^*\hat{\otimes}Y^*.$$

Proof. Since X^* has a basis, it certainly satisfies Grothendieck's condition of approximation (via the Banach-Steinhaus theorem) and hence ([12], Equivalence (B_1), p. 165) the canonical map from $X^{**}\hat{\otimes}X^*\to\mathscr{L}(X^*,X^*)$ is one-to-one. Thus by [12], p. 123, the canonical map from $X^*\hat{\otimes}Y^*\to B(X,Y)$ is one-to-one ($\mathscr{L}(X^*,X^*)$ denotes the continuous linear operators from X^* into X^* and $B(X,Y)$ denotes the continuous bilinear forms on $X\times Y$). Now III follows from [12], Theorem 8, p. 122.

The final two results we need are well known. Before stating them we make the following definition: Let X and Y be Banach spaces with Schauder bases (x_i) and (y_i) respectively. By the *tensor product*, $(x_i)\otimes(y_i)$, of (x_i) and (y_i) we mean the set $\{x_i\otimes y_j\}$ ordered as a sequence in the following fashion:

$$\begin{array}{l} \underline{x_1\otimes y_1|},\ x_2\otimes y_1,|\ \ldots\\ \underline{x_1\otimes y_2,\ x_2\otimes y_2,|}\ \ldots\\ \ldots\ldots\ldots\ldots\ldots \end{array}$$

IV. (Theorem of Gelbaum-DeLamadrid [11]) *Let X and Y be Banach spaces with Schauder bases (x_i) and (y_i) respectively. Then $(x_i)\otimes(y_i)$ is a Schauder basis for $X\tilde{\otimes}Y$, where $\sim$ denotes $\wedge$ or $\wedge\!\!\wedge$. Moreover, the set of coefficient functionals for $(x_i)\otimes(y_i)$ is precisely the tensor product of the coefficient functionals of (x_i) and (y_i).*

Finally, we need the profound theorem of Milutin.

V. (Theorem of Milutin [18]) *Let S and T be uncountable, compact metric spaces. Then $C(S)$ is linearly homeomorphic to $C(T)$.*

For a penetrating study of the work of Milutin see [20] and [8].

§ 4. The Example. If K denotes the one-point compactification of the positive integers, then $C(K)$ is isometrically isomorphic to c, the space of convergent sequences. By [1], p. 181, c is isomorphic to c_0 and so

$$C[0,1]\hat{\hat{\otimes}}c_0=C[0,1]\hat{\hat{\otimes}}c=C[0,1]\hat{\hat{\otimes}}C(K)=C([0,1]\times K)=C[0,1].$$

Here we are using "=" to mean "is isomorphic to". The third "equality" is the result of [12], p. 90, and the last is from V.

Thus, to obtain the desired example we need only find a basis for $c_0\hat{\hat{\otimes}}C[0,1]$ whose coefficient functionals in $(c_0\otimes C[0,1])^*$ span a space isomorphic to neither l^1 nor $L^1[0,1]$.

Let (s_i, l_i) denote the classical Schauder basis for $C[0,1]$ and let (x_n, f_n) be the Lindenstrauss basis for c_0, constructed in § 2. We let $\mathscr{L} = [f_n]$. By IV, $(x_i) \otimes (s_i)$ is a basis for $c_0 \hat{\hat{\otimes}} C[0,1]$ with coefficient functionals $(f_i) \otimes (l_i)$. We wish to compute the space spanned by $(f_i) \otimes (l_i)$ in

$$(c_0 \hat{\hat{\otimes}} C[0,1])^* = l^1 \hat{\otimes} \mathrm{NBV}[0,1],$$

the above equality coming from III.

(a) *The closed linear span of* $(f_i) \otimes (l_i)$ *in* $l^1 \hat{\otimes} \mathrm{NBV}(0,1]$ *is isometrically isomorphic to* $\mathscr{L} \hat{\otimes} l^1$.

Since, by I, $[l_i] = F \subset \mathrm{NBV}[0,1]$ is isometrically isomorphic to l^1, we have $\mathscr{L} \hat{\otimes} l^1$ is isometrically isomorphic to $\mathscr{L} \hat{\otimes} F$. We claim there is an isometric isomorphism Q from $\mathscr{L} \hat{\otimes} F$ into $l^1 \hat{\otimes} \mathrm{NBV}[0,1]$ such that $Q(f_i \otimes l_j) = f_i \otimes l_j$ for each i and j. Since $(f_i) \otimes (l_i)$ is a basic sequence in $l^1 \hat{\otimes} \mathrm{NBV}[0,1]$ and, by IV, a basis for $\mathscr{L} \hat{\otimes} F$, we will have the desired result.

Consider the following diagram:

$$\begin{array}{ccccc} \mathscr{L} \hat{\otimes} F & \xrightarrow{J \otimes T} & \mathscr{L} \hat{\otimes} l^1 & \xrightarrow{i} & l^1 \hat{\otimes} l^1 \\ & & {\scriptstyle S_1} \updownarrow & & \updownarrow {\scriptstyle S_2} \\ & & A(\mathscr{L}) & \xrightarrow{\hat{i}} & A(l^1) \end{array}$$

Here $A(\mathscr{L})$ and $A(l^1)$ denote the absolutely summable sequences in $\mathscr{L}$ and l^1 respectively, J the identity map $\mathscr{L} \to \mathscr{L}$. T is the isometry of I, $\hat{i}$ the injection map and i the indicated composition. S_1 and S_2 denote the well-known isometries which extend the mappings

$$\sum_{i=1}^{n} y_i \otimes e_i \to (y_1, \ldots, y_n, 0, 0, \ldots),$$

where $(y_i)_{i=1}^n$ are in $\mathscr{L}$ or l^1 and e_i denotes the ith unit vector in l^1. Since the proof of I shows that $T(l_i)$ is a finite linear combination of the unit vectors in l^1, say $Tl_j = \sum_{k=1}^{n(j)} a_k^{(j)} e_k$, we have

$$i(f_i \otimes Tl_j) = S_2 \circ \hat{i} \circ S_1 (f_i \otimes Tl_j) = f_i \otimes Tl_j.$$

Now clearly $l^1 \hat{\otimes} l^1$ is isometric to $l^1 \hat{\otimes} F$ under the mapping $J_1 \otimes T^{-1}$, where J_1 is the identity map $l^1 \to l^1$. Moreover

$$J_1 \otimes T^{-1} \circ i \circ J \otimes T (f_i \otimes l_i) = J_1 \otimes T^{-1} (f_1 \otimes Tl_i) = f_i \otimes l_i.$$

Now consider the following diagram:

$$\begin{array}{ccccc} l^1 \hat{\otimes} l^1 & \xrightarrow{J_1 \otimes T^{-1}} & l^1 \hat{\otimes} F & \xrightarrow{j} & l^1 \hat{\otimes} \mathrm{NBV}[0,1] \\ & & {\scriptstyle R_1} \updownarrow & & \updownarrow {\scriptstyle R_2} \\ & & A(F) & \xrightarrow{\hat{j}} & A(\mathrm{NBV}[0,1]) \end{array}$$

where R_1 and R_2 are defined as for S_1 and S_2, $\hat{j}$ is injection and j is the indicated composition. Again, since each f_i is a finite linear combination of the unit vectors in l^1, we obtain $j(f_i \otimes l_i) = f_i \otimes l_i$ and j is an isometry of $l^1 \hat{\otimes} F$ into $l^1 \hat{\otimes} \mathrm{NBV}[0,1]$.

Letting $Q = j \circ J_1 \otimes T^{-1} \circ i \circ J \otimes T$ we obtain (a).

(b) $\mathscr{L} \hat{\otimes} l^1$ *is not isomorphic to* $L^1[0,1]$.

If $\mathscr{L} \hat{\otimes} l^1$ were isomorphic to $L^1[0,1]$, then since $\mathscr{L} \hat{\otimes} l^1$ can be isometrically embedded in $l^1 \hat{\otimes} l^1$ (shown in (a)) and, as is well known, $l^1 \hat{\otimes} l^1$ is isometrically isomorphic to l^1, we would have $L^1[0,1]$ embedded in a space with an unconditional basis, contradicting [21].

(c) $\mathscr{L} \otimes l^1$ *is not isomorphic to* l^1.

Representing $\mathscr{L} \hat{\otimes} l^1$ as $A(\mathscr{L})$, we see that there is a continuous linear projection onto a space isomorphic to $\mathscr{L}$ (e.g. $\{(x_i) \in A(\mathscr{L}) \colon x_1 \in \mathscr{L}, x_i = 0, i \geq 2\}$).

Thus if $\mathscr{L} \otimes l^1 = A(\mathscr{L})$ were isomorphic to l^1, we could infer that $\mathscr{L}$ is complemented in a conjugate space, contradicting [15], p. 540.

Thus we have the desired example.

§ 5. Remarks and unsolved problems.

Remark 1. If E and F are closed subspaces of Banach spaces X and Y respectively, it is not true in general that $E \hat{\otimes} F$ is a closed subspace of $X \hat{\otimes} Y$. For a discussion of this phenomenon see [25].

It is obvious that we have relied heavily on the structure of both $\mathscr{L}$ and l^1 to achieve our result.

In conclusion we raise the following problems.

Problem 1. *How many Lindenstrauss bases with mutually non-isomorphic (i.e. not linearly homeomorphic) coefficient spaces does* c_0 *admit?*

Problem 2. *Let* $W = \{(f_n) \subset l^1 \colon (f_n)$ *be an* ω^**-Schauder basis for* l^1 *and* $l^1/[f_i]$ *be isomorphic to* $L^1[0,1]\}$. *Define an equivalence relation* $\sim$ *on* W *by* $(f_n) \sim (g_n)$ *if and only if* $[f_n]$ *is isomorphic to* $[g_n]$. *Into how many equivalence classes does* $\sim$ *partition* W?

From the remarks above if $(f_n) \in W$ and has coefficient functionals (x_n), then (x_n) is a Lindenstrauss basis for c_0. We conjecture that the answer to Problem 2 and hence to Problem 1 is $\mathfrak{c}$, the power of continuum.

PROBLEM 3. *How many non-isomorphic coefficient spaces does $C[0,1]$ admit?*

Problem 3 is closely related to Problem 1. We conjecture that the answer to Problem 3 is $\mathfrak{c}$. Of course, our example shows the intrinsic difficulties in constructing such examples. Moreover (and roughly speaking), one usually constructs examples of bases with certain desired properties by perturbing in some manner a "standard" basis. However, bases for $C[0,1]$ can be extremely pathological and still have coefficient spaces isomorphic to l^1. In fact,

Remark 2. Pełczyński's [22] universal basis for $C[0,1]$ has coefficient space isomorphic to l^1. This is easily seen from I and the proof of the Zippin extension lemma [29].

A problem related to Problem 3 is the following:

Let $\mathscr{B}$ denote the class of all Banach spaces with Schauder bases and let $\mathscr{C}$ be the subset of $\mathscr{B}$ consisting of those spaces $X \in \mathscr{B}$ admitting non-isomorphic coefficient spaces.

PROBLEM 4. *Classify the elements of $\mathscr{C}$.*

We conjecture that $\mathscr{C}$ consists precisely of the non-quasi-reflexive Banach spaces [5] with bases. Now by a theorem of R. C. James [13] no reflexive space with a basis can be in $\mathscr{C}$ (all coefficient spaces coincide with the conjugate space). Moreover,

Remark 3. For each positive integer n, there is a Banach space X_n with basis such that X_n is quasi-reflexive of order n and $X_n \notin \mathscr{C}$.

To see this, let J denote the space of James [13]. Since J has codimension 1 in J^{**} and is also isometrically isomorphic to J^{**} [13], it follows from a remark of Bessaga and Pełczyński [2] that J is isomorphic to each of its subspaces of co-dimension 1. (This fact has been observed in [26], p. 345.) By a result of Y. Cuttle [6] and I. Singer [26] (Theorem 3, p. 205) if (x_n) is a basis for J with coefficient functionals (f_i), then either $[f_i] = J^*$ or codim $[f_i] = 1$. Thus $J \in \mathscr{B}$ but $J \notin \mathscr{C}$. (Since by [26] ((C), p. 343) J^* is isomorphic to each of its subspaces of codimension 1.) Now let $X_n = J \times J \times \ldots \times J$ be the n-fold product of J. Then [5] X_n is quasi-reflexive of order n and the result of Singer above together with a simple induction argument shows that $X_n \notin \mathscr{C}$ for any n.

The classical problem of Banach as to whether an infinite-dimensional Banach space is isomorphic to each of its subspaces of codimension 1 is still unsolved. If the answer to Banach's problem is affirmative, then we see from [26] (Theorem 3, p. 205) that no quasi-reflexive space in $\mathscr{B}$ can be in $\mathscr{C}$.

We conclude by remarking that it is easy to show that c_0 and l^1 are in $\mathscr{C}$. Indeed, the Lindenstrauss basis (x_n) for c_0 constructed in § 2 and the unit vector basis for c_0 show that $c_0 \in \mathscr{C}$. Also the basis (l_n) for $[l_n]$ described in I is easily seen to have its coefficient space isomorphic to $C[0,1]$. Hence the unit vector basis for l^1 and the basis for l^1 determined by (l_n) under the isometry described in I show that $l^1 \in \mathscr{C}$.

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Симметризуемые операторы, не удовлетворяющие условиям положительной определенности и их приложения

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1. Введение. В работе получают дальнейшее развитие результаты, установленные в статьях [1], [2]. Пусть X — гильбертово пространство над полем комплексных (или вещественных) чисел, $R \subseteq X$ — линейное множество, плотное в X, A и H линейные (вообще говоря, неограниченные) операторы, определенные на R, H симметричен в R и симметризует A.

В работах [1], [2] изучались симметризуемые операторы A с дискретным спектром, для которых выполнялось одно из следующих двух условий: $(Hx, x) \geqslant 0$ или $(HAx, x) \geqslant 0$ для любого $x \in R$.

В этой работе симметризуемые операторы A исследуются в предположении, что ни одно из указанных условий положительной определенности не выполняется. Полученные результаты прилагаются к изучению задач о собственных значениях обыкновенных дифференциальных операторов широкого класса, содержащих, как частные случаи, классы, изучавшиеся ранее Э. Камке [3] и Л. Коллатцем [4].

2. Симетризуемые операторы, не удовлетворяющие условиям определенности. Пусть X — гильбертово пространство над полем комплексных чисел, R — линейное множество, плотное в X, $R \subseteq X$. Рассмотрим аддитивные и однородные (вообще говоря, неограниченные) операторы A и H, определенные на R $(A(R) \subseteq R; H(R) \subseteq X)$, удовлетворяющие следующим условиям:

1) уравнение $Hx = 0$ имеет только нулевое решение $x = 0$;

2) H — симметрический оператор на R, симметризующий оператор A:

$$(Hx, y) = (x, Hy), \quad (HAx, y) = (x, HAy), \quad x, y \in R; \tag{1}$$

3) спектр оператора A может содержать только собственные значения конечной кратности уравнения

$$x - \lambda Ax = 0, \quad x \in R, \tag{2}$$