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A remark on Berezanskiĭ version of spectral theorem

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We will show in this short note that a spectral theorem of Berezanskiĭ is an immediate corollary of our nuclear theorem.

Preliminaries. Let $\mathscr{A}$ be a commutative C^*-algebra of normal operators in a separable Hilbert space H. If the family $(A_x)_{x\in X}$ generates the algebra $\mathscr{A}$, then, as it is well known (Gelfand theory E), the maximal ideal-space Λ of $\mathscr{A}$ is homeomorphic with $\hat{\Lambda}\subset\prod_{x\in X}\operatorname{sp}(A_x)$, where $\operatorname{sp}(A_x)$ denotes the spectrum of the operator A_x. We shall identify Λ with $\hat{\Lambda}$. More explicitely: the Gelfand isomorphism has now the canonical form

$$\mathscr{A}\ni A_x\leftrightarrow \hat{A}_x(\cdot)\in C(\Lambda),\quad \hat{\Lambda}\subset\prod_{x\in X}\hat{A}_x(\Lambda),\quad \hat{A}_x(\Lambda)=\operatorname{sp}(A_x).$$

But the identity map id_Λ has the form

$$\Lambda\ni\lambda\to\big(\hat{A}_x(\lambda)\big)_{x\in X}\in\Lambda.$$

This means that $\lambda(x)=\hat{A}_x(\lambda)$ for all $x\in X,\lambda\in\Lambda$. We can now reformulate our

NUCLEAR SPECTRAL THEOREM [2]. *Let $(A_x)_{x\in X}$ be a commuting family of normal operators in a separable Hilbert space H. Let $\Phi\subset H$ be a nuclear space such that Φ is dense in H and the imbedding $\Phi\to H$ is continuous. If $A_x(\Phi)\subset\Phi$ for every $x\in X$, then there exists the direct integral decomposition*

$$H\cong\int_\Lambda H(\lambda)\,d\mu(\lambda),\quad \Lambda\subset\prod_{x\in X}\operatorname{sp}A_n\subset \boldsymbol{C}^X,$$

where μ is a Radon measure on the compact space Λ, such that:

1° *there exists a subset $\Lambda_0\subset\Lambda$ of measure 0, $\mu(\Lambda_0)=0$, such tha $H(\lambda)\subset\Phi'$ for all $\lambda\in\Lambda-\Lambda_0$ and $H(\lambda)$ are generalized common eigenspace s of A_x, $x\in X$;*

(1) $$\langle A_x\varphi, e(\lambda)\rangle=\lambda(x)\langle\varphi, e(\lambda)\rangle$$

for each $\varphi\in\Phi$, $e(\lambda)\in H(\lambda)$, $\lambda\in\Lambda-\Lambda_0$;

2° *taking in each* $H(\lambda)$, $\lambda \in \Lambda - \Lambda_0$, *an orthonormal basis* $e_k(\lambda)$, $k = 1, 2, \ldots, \dim H(\lambda)$, *one obtains a complete set of linear continuous functions of* Φ: *to each* $0 \neq \varphi \in \Phi$ *there exists* $e_k(\lambda) \in \Phi'$ *such that*

$$\langle \varphi, e_k(\lambda) \rangle \neq 0, \quad \lambda \in \Lambda - \Lambda_0, \tag{2}$$

and $e_k(\lambda)$ *satisfies equation* (1).

Remark. This version of the complete spectral theorem shows that the elements of the (compact) measure space $\Lambda = (\Lambda, \mu)$ are functions $X \ni x \to \lambda(x) \in \mathrm{Sp}(A_x) \subset \boldsymbol{C}$, and that all $\lambda \in \Lambda - \Lambda_0$ are of the form

$$\lambda(x) = \frac{\langle A_x \varphi, e_k(\lambda) \rangle}{\langle \varphi, e_k(\lambda) \rangle}, \quad x \in X, \ \mu(\Lambda_0) = 0. \tag{3}$$

Identity (3) allows immediately to prove a

SPECTRAL THEOREM OF BEREZANSKIĬ TYPE. *Let the family* $(A_x)_{x \in X}$ *from the preceding theorem satisfy the following regularity condition*:

X *is a topological space (resp. a differentiable manifold). For each* $\varphi \in \Phi$, $\psi' \in \Phi'$, *the function*

$$X \ni x \to \langle A_x \varphi, \psi' \rangle \in \boldsymbol{C} \tag{4}$$

is continuous (resp. differentiable).

Then the measure μ *in the direct integral decompositions*

$$H = \int_\Lambda H(\lambda)\, d\mu, \quad \Lambda \subset \boldsymbol{C}^X,$$

is concentrated on continuous (resp. differentiable) functions, i.e. there exists a subset $\Lambda_0 \subset \Lambda$, $\mu(\Lambda_0) = 0$ *such that for all* $\lambda \in \Lambda - \Lambda_0$, $\lambda \in C(X)$ *(resp.* $\lambda \in C^\infty(X)$*).*

Proof. Obvious, since from (3) and (4) it follows that for $\lambda \notin \Lambda_0$

$$X \ni x \to \lambda(x) = \frac{\langle A_x \varphi, e(\lambda) \rangle}{\langle \varphi, e(\lambda) \rangle} \in \boldsymbol{C}$$

is continuous (differentiable).

Concluding remarks. The theorem suggests that the Wiener measure could be considered as a spectral measure for an adequately chosen family $(A_x)_{x \in X}$ of operators.

I express my gratitude to Professor Berezanskiĭ for giving to my disposition (during the V Winter School of Theoretical Physics at Karpacz 1968) the manuscript of his magnificent paper [1].

References

[1] Yu. M. Berezanskiĭ, Trudy Moskovskogo Matem. Ob. 1969 (to appear).
[2] K. Maurin, *Allgemeine Eigenfunktionsentwicklungen. Spektraldarstellung abstrakter Kerne*, Bull. Acad. Pol. Sc., Série sc. math., astr. et phys., 7 (1959), p. 471–475.
[3] — *Methods of Hilbert spaces*, Monografie Matematyczne 45, Warszawa 1967.
[4] — *General eigenfunction expansions and unitary representation of topological groups*, Monografie Matematyczne 48, Warszawa 1968.

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