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On Mikusiński operators *

by

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I. Introduction. We shall use the following standard notation. $\mathscr{C}$ is the space of complex-valued continuous functions on the half-line $0 \leqslant t < \infty$. F, the space of Mikusiński operators, is the field of quotients of the integral domain $\mathscr{C}$ where the product operation is convolution. Thus if $f \in \mathscr{C}$ and $g \in \mathscr{C}$, then

$$(fg)(t) = \int_0^t f(t-u) g(u) du$$

for all $t \geqslant 0$, and if $x \in F$ then $x = f/g$ for some $f \in \mathscr{C}, g \in \mathscr{C}, g \neq 0$. The shift operator (the measure with unit mass concentrated at $t = a \geqslant 0$) is denoted by e^{-as} and $e^{as} = 1/e^{-as}$. The symbol h stands for the constant function $h(t) = 1$ for $t \geqslant 0$. Thus $h^2(t) = \int_0^t du = t$ for $t \geqslant 0$, $h^3(t) = t^2/2$, etc. The reciprocal of h in F is $s = 1/h$ and this is a differentiation operator. $\mathscr{C}_0$ is the set $\{f \mid f \in \mathscr{C}, 0 \in \text{Support } f\}$.

The space $\mathscr{C}$ has a topology defined by the semi-norms

$$\|f\|_N = \operatorname*{Max}_{[0,N]} |f(t)|.$$

$\mathscr{L}$ is the space of (equivalence classes of) functions which are integrable on each finite interval $[0, T]$, $T > 0$. $\mathscr{L}$ has the topology defined by the semi-norms

$$\|g\|_T^1 = \int_0^T |g| \quad \text{for } T > 0.$$

If a sequence $f_n \in \mathscr{C}$ (or $\mathscr{L}$) converges to f in $\mathscr{C}$ (or in $\mathscr{L}$) we will write $f_n \xrightarrow{\mathscr{C}} f$ (or $f_n \xrightarrow{\mathscr{L}} f$).

Some other terminology has not become standard and we will make some conventions. In the first place there are several types of topological

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spaces which are otherwise unrelated but go under the name of Fréchet spaces. A locally convex, metrisable, complete, topological vector space is commonly called a Fréchet space. Also, any topological space with the property that the sequential closure of every set is closed is usually called a Fréchet space. Unfortunately both of these types of spaces occur in this paper; thus on the hypothesis that the sequential closure of every set being closed means that sequential closure is a Kuratowski closure operator we will call this last type of space a Kuratowski space. We will reserve the name Fréchet space to mean a locally convex, metrizable, complete, topological vector space. Both $\mathscr{C}$ and $\mathscr{L}$ are Fréchet spaces.

The field of Mikusiński operators does not have a topology defined for it. Mikusiński initially defined a sequential convergence for F, which we will call *type* I *convergence*, as follows: a sequence $x_n \in F$ converges to $x \in F$ if and only if there is an $f \in \mathscr{C}$, $f \neq 0$, such that $fx_n \in \mathscr{C}$ for each n and $fx_n \overset{\mathscr{C}}{\to} fx$. If x_n converges to x type I we will write $x_n \overset{\mathrm{I}}{\to} x$. This is the convergence most commonly used in F. Another type of convergence which we will call *type* II *convergence* is as follows: if there exist $f_n \in \mathscr{C}$, $f_n \overset{\mathscr{C}}{\to} f \neq 0$ such that $f_n x_n \in \mathscr{C}$ for each n and $f_n x_n \overset{\mathscr{C}}{\to} fx$, then x_n converges to x type II and we write $x_n \overset{\mathrm{II}}{\to} x$.

Let $\mathscr{U}$ be a class of sequences on a space X (with or without a topology). We say $x_n \overset{\mathscr{U}}{\to} x$ if $(x, x_1, x_2, \ldots)$ is in $\mathscr{U}$. The sequential closure with respect to $\mathscr{U}$, of a subset $S \subset X$, is the set

$$\bar{S} = \{x \mid x_n \in S,\ x_n \overset{\mathscr{U}}{\to} x\}.$$

Urbanik [1] calls a convergence class "topological in the sense of Kuratowski" if sequential closure is a Kuratowski closure operator, i.e., if for every $S \subset X$, $\bar{S} = \bar{\bar{S}}$. Since, even if $\mathscr{U}$ is the class of convergence sequences for a topology on X it may fail to have this property we will use the term "topological" differently. Namely, we call $\mathscr{U}$ *topological* if and only if there exists a topology $\mathscr{T}$ for X such that $x_n \overset{\mathscr{U}}{\to} x \Leftrightarrow \Leftrightarrow x_n \overset{\mathscr{T}}{\to} x$.

This paper is a discussion of the connection between type I convergence and a topology for F. In [9] it is shown that sequential closure is not a Kuratowski closure operator for either type I or type II convergence. Thus if type I or type II convergence is topological and $\mathscr{T}$ is a topology such that $x_n \overset{\mathscr{T}}{\to} x \Leftrightarrow x_n \overset{\mathrm{I}}{\to} x$ (or $x_n \overset{\mathrm{II}}{\to} x$), then $\langle F, \mathscr{T}\rangle$ is not a Kuratowski space. In [7] Norris imposes a topology on a subspace of F; however, the topology is strictly stronger than type I convergence in that there are sequences of continuous functions $\varphi_n \in \mathscr{C}$ such that $\varphi_n \overset{\mathrm{I}}{\to} 0$ but they fail to converge to zero in his topology.

In section II we review some of the facts about sequential convergence and topologies. All of this material can be found in [4] or [5]. It is placed here because it is the basis for what follows.

In section III a theorem is proved which answers a question raised by [1]. In [1] it is shown that if $f_n \in \mathscr{C}$ and $0 \in \operatorname{Support} f_n$ for each n then there are $\varrho_n \in \mathscr{C}$ and a non-zero function $k \in \mathscr{C}$ such that $f_n \varrho_n = k$ for all n. A natural question is "Suppose $f_n \in \mathscr{C}$, $0 \in \operatorname{Support} f_n$ each, and $f_n \overset{\mathscr{C}}{\to} f \neq 0$. Can we pick $\varrho_n \in \mathscr{C}$, $\varrho_n \overset{\mathscr{C}}{\to} \varrho \neq 0$ such that $f_n \varrho_n = f\varrho$ for all n?" This question is answered by Theorem 4.

In section IV it is shown that type I convergence is not topological, and it is demonstrated what is the smallest topological convergence class which contains the type I convergence class. F is endowed with a topology whose convergent sequences are exactly the convergent sequences of this last convergence class. It is shown that F with this topology is a sort of inductive limit of Fréchet spaces (but it is *not* locally convex). Theorem 9 shows exactly what sequences convergent to x in F have the property that $1/x_n \to 1/x$.

In section V the space F is exhibited as the quotient of a metric space.

II. Sequential Convergence. A convergence class $\mathscr{U}$ on a set X is any collection of sequences $(x, x_1, x_2, \ldots)$ from the set X. If $(x, x_1, x_2, \ldots)$ lies in $\mathscr{U}$ we say $x_n \overset{\mathscr{U}}{\to} x$. $\mathscr{U}$ is said to be an *L-convergence class* if it satisfies the two conditions:

(i) $x_n \overset{\mathscr{U}}{\to} x \Rightarrow$ each subsequence of (x_n) converges $\overset{\mathscr{U}}{\to}$ to x.

(ii) each constant sequence $(x, x, \ldots)$ converges $\overset{\mathscr{U}}{\to}$ to x.

$\mathscr{U}$ is said to be an *L^*-convergence class* if in addition to (i) and (ii) it satisfies the criterion for convergence

(iii) if a sequence (x_n) is such that each subsequence has a subsequence which converges $\overset{\mathscr{U}}{\to}$ to x, then in fact $x_n \overset{\mathscr{U}}{\to} x$.

Of course, every topology on X gives rise in a natural way to a convergence class; namely, the class of all $(x, x_1, x_2, \ldots)$ where $x_n \to x$ as $n \to \infty$ with convergence in the given topology. This convergence class will be called the *convergence class of the topology*.

The convergence class of a topology is always an L^*-convergence class.

Conversely, we would like if possible to associate a topology with a convergence class $\mathscr{U}$. We say $\mathscr{U}$ is *topological* if there exists a topology for X such that $\mathscr{U}$ is exactly the convergence class of that topology.

A topological convergence class is an L^*-convergence class. Also, the convergence classes in which we are interested have unique sequential limits (i.e., $x_n \overset{\mathcal{U}}{\to} x$ and $x_n \overset{\mathcal{U}}{\to} y \Leftrightarrow x = y$). In this case Kisyński [6] has shown

THEOREM 1. *A convergence class which has unique sequential limits is topological if and only if it is an L^*-convergence class.*

If $\mathcal{U}$ is topological, there are ordinarily many topologies with the property that $x_n \to x \Leftrightarrow x_n \overset{\mathcal{U}}{\to} x$. We will pick one of these topologies–namely, the largest one. In fact, start with any $\mathcal{U}$ which is an L-convergence class. We define $\mathcal{T}$ by

(1) $O \in \mathcal{T}$ if and only if each sequence $x_n \overset{\mathcal{U}}{\to} x \in O$ is eventually in O (i.e., there exists an N such that $n > N \Rightarrow x_n \in O$).

Or equivalently

(1′) C is closed in $\mathcal{T}$ if and only if C is sequentially closed with respect to $\mathcal{U}$ (i.e., $\bar{C} = \bar{\bar{C}}$).

$\mathcal{T}$ will be called the *topology of the convergence class* $\mathcal{U}$. $\mathcal{T}$ has the following properties:

(a) $\mathcal{T}$ is weaker than $\mathcal{U}$ in the sense that $x_n \overset{\mathcal{U}}{\to} x \Rightarrow x_n \overset{\mathcal{T}}{\to} x$.

(b) $\mathcal{T}$ is the largest topology weaker than $\mathcal{U}$.

If $\mathcal{U}$ has unique sequential limits, then

(c) a sequence $x_n \in X$ is convergent to x in the topology $\mathcal{T}$ if and only if every subsequence contains a subsequence which converges $\overset{\mathcal{U}}{\to}$ to x.

If $\mathcal{U}$ is also an L^*-convergence class, then

(d) $\mathcal{T}$ is the strongest topology on X which has $\mathcal{U}$ for its convergence class.

In this last case $\mathcal{T}$ is a topology which can be used to prove Theorem 1.

Now start with a topology $\mathcal{T}_1$ on X. Consider the convergence class of $\mathcal{T}_1$ and let $\mathcal{T}$ be the largest topology which has this same convergence class for its convergence class. We have $\mathcal{T} \supset \mathcal{T}_1$ and if $\mathcal{T} = \mathcal{T}_1$ we say that $\mathcal{T}_1$ is a *sequential topology* for X. Thus, a topology with unique sequential limits is a sequential topology if and only if every sequentially closed set is closed.

Franklin [5] has characterized the sequential topologies by

THEOREM 2. *A necessary and sufficient condition that $\mathcal{T}$ be a sequential topology for X is that $\langle X, \mathcal{T}\rangle$ be the quotient of a metric space.*

III. The main theorem in [1] states:

THEOREM 3. *If there exists a $T > 0$ such that none of the functions $f_n \in \mathcal{C}$, $n = 1, 2, \ldots$, vanish identically on $[0, T]$, then there exist functions $\varrho_n \in \mathcal{C}$ and a $k \neq 0$ in $\mathcal{C}$ such that $f_n \varrho_n = k$ for all $n = 1, 2, \ldots$*

In view of this theorem if $f_n \overset{\mathcal{C}}{\to} f \neq 0$, there is a $\varrho \neq 0$ and a sequence $\varrho_n \in \mathcal{C}$ such that $f_n \varrho_n = f\varrho$ for all n sufficiently large. One can now ask, "Can the functions ϱ_n and ϱ be chosen so that $\varrho_n \overset{\mathcal{C}}{\to} \varrho$"? That the answer to this question can be negative is seen by taking a sequence $f_n \in \mathcal{C}_0$ converging in $\mathcal{C}$ to an f such that $f(t) \equiv 0$ for $t \in [0, 1]$. Basically the answer to the question seems to be

THEOREM 4. *If $f_n \overset{\mathcal{C}}{\to} f \in \mathcal{C}_0$ as $n \to \infty$, then each sufficiently rapidly increasing sequence of integers has a subsequence (n_k) such that there exist $\varrho_k \in \mathcal{C}$, $\varrho_k \overset{\mathcal{C}}{\to} \varrho \neq 0$ and*

$$f_{n_k} \varrho_k = f\varrho$$

for all $k = 1, 2, \ldots$

In order to prove the theorem we will use the theory of summable series as exposited in Schwartz [8], Chapter 1. Theorem 4 then results from a series of six lemmas.

Some of the basic facts concerning summable series are given below before we begin the proof of the lemmas.

Definition 1. Let S be an index set for a collection of complex numbers $\{a_\alpha \mid \alpha \in S\}$. Then $A = \sum_S a_\alpha$ is *summable to the number* A if for every $\varepsilon > 0$ there is a finite set $F \subset S$ with the property that for each finite subset U, $F \subset U \subset S$, we have $|A - \sum_U a_\alpha| < \varepsilon$.

Definition 2. Let $\{f_\alpha \mid \alpha \in S\}$ be a collection of continuous functions on an interval I. The series $\sum f_\alpha$ is *uniformly summable to* f if for each $\varepsilon > 0$ there is a finite set $F \subset S$ with the property that for each finite subset U, $F \subset U \subset S$, we have $|f(t) - \sum_U f_\alpha(t)| < \varepsilon$ for all $t \in I$.

Definition 3. Let $\{f_\alpha \mid \alpha \in S\}$ be a collection of continuous functions on I. The series $\sum f_\alpha$ is said to be *norm summable* if $\sum_S \|f_\alpha\|$ is summable, where $\|f_\alpha\| = \operatorname{Sup}_I |f_\alpha(t)|$.

A norm summable series of continuous functions is uniformly summable to a continuous function.

If $A = \sum_S a_\alpha$ is summable and $S' \subset S$, then $A' = \sum_{S'} a_\alpha$ is summable, and is called a *partial sum* of $\sum a_\alpha$. A sequence A_n of partial sums of A is called *cofinal* if $A_n = \sum_{S_n} a_\alpha$ and for every finite set F there is an n_0 such that $F \subset S_n$ for all $n > n_0$.

Any cofinal sequence of partial sums A_n of a summable series $A = \sum a_\alpha$ is convergent to A. A series of non-negative terms is summable if the partial sums over finite subsets of S are bounded. Thus a series of non-negative terms is summable if any cofinal sequence of partial sums is bounded.

If $\sum f_a$ is norm summable, every partial sum is norm summable and thus uniformly summable, and a cofinal sequence of partial sums, say $f_n = \sum_{S_n} f_a$, converges uniformly on I to $f = \sum_S f_a$.

We have for norm summable series

$$\|f\| \leqslant \sum_S \|f_a\|.$$

In the remainder of this section we will use the notation

$$S = \{F \mid F \text{ is a finite, non-empty, set of positive integers}\}$$

and

$$\beta = \prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right) = \frac{\sinh \pi}{\pi}.$$

LEMMA 1. *We have*

(1) $$\beta = 1+\sum_S \Big(\prod_F k^{-2}\Big).$$

Proof. The partial products

$$\beta_N = \prod_{k=1}^{N}\left(1+\frac{1}{k^2}\right)$$

form a cofinal collection of partial sums of the non-negative series on the right (1). Since the partial sums β_N converge to β, equation (1) is established.

Recall that for $f \in C[0, k]$

$$\|f\|_k = \operatorname*{Max}_{[0,k]} |f(t)|.$$

LEMMA 2. *Let ε_k and r_k be in $\mathscr{C}$, for each $k = 1, 2, \ldots$ Suppose that*

(i) $$\|\varepsilon_k\|_k \leqslant \frac{1}{k^2}\,\frac{1}{1+\sum_1^k \|r_i\|_i}$$

for each $k = 1, 2, \ldots$ Define, for T any positive integer,

(ii) $$A(T) = \operatorname*{Max}_{k \leqslant T} k^2 \|\varepsilon_k\|_T,$$

(iii) $$B(T) = \operatorname{Max}[A(T), 1],$$

(iv) $$C(T) \geqslant \operatorname{Max}\{\|r_i\|_T \mid i \leqslant T\},$$

(*we suppose $C(T) \geqslant 1$ for all $T > 0$*),

(v) $$|F| = \text{number of elements in } F,$$

(vi) $$u_F = r_{|F|}\Big(\prod_F \varepsilon_k\Big).$$

Then

$$u = \sum_S u_F$$

is norm summable and

$$\|u\|_T \leqslant e^T B^T(T)\, C(T)(\beta-1).$$

Proof. For any finite sequence of functions in $\mathscr{C}$, say $a_1, a_2, \ldots, a_N$ we have the inequalities

$$\Big\|\prod_1^N a_i\Big\|_T \leqslant \frac{T^{(N-1)}}{(N-1)!}\prod_1^N \|a_i\|_T \leqslant e^T \prod_1^N \|a_i\|_T.$$

Thus

$$\|u_F\|_T \leqslant e^T \|r_{|F|}\|_T \prod_F \|\varepsilon_k\|_T.$$

We will show that in fact

(2) $$\|u_F\|_T \leqslant e^T B^T(T)\, C(T) \prod_F \frac{1}{k^2}$$

for every $F \in S$, $T > 0$. The proof of (2) will be divided into two parts. First, the case $|F| \geqslant T$ and second $|F| < T$.

$|F| \geqslant T$. Since the $|F|$ elements of F are all distinct at most T of the functions ε_k, $k \in F$, have subscripts $k \leqslant T$. These satisfy the inequality

(3) $$\|\varepsilon_k\|_T \leqslant \frac{B(T)}{k^2}$$

by (ii) and (iii). At least one $k \in F$ is such that $k \geqslant |F| \geqslant T$ and for this k

$$\|\varepsilon_k\|_T \|r_{|F|}\|_T \leqslant \|\varepsilon_k\|_k \|r_{|F|}\|_{|F|} \leqslant \frac{1}{k^2}$$

by (i). Also by (i) the remaining $k \in F$ are such that

(4) $$\|\varepsilon_k\|_T \leqslant \frac{1}{k^2}.$$

Thus when $|F| \geqslant T$ we have

$$\|u_F\|_T \leqslant e^T B^T(T) \prod_F \frac{1}{k^2}$$

and since $C(T) \geqslant 1$, inequality (2) is established for $|F| \geqslant T$.

$|F| < T$. Since $|F| < T$, we have from (iv) the inequality $\|r_{|F|}\|_T \leqslant C(T)$ and the elements of F are divided again into two groups. For those $k \leq T$ we have inequality (3) again by (ii) and (iii) and for $k > T$ inequality (4). Thus

$$\|u_F\|_T \leq e^T C(T) B^T(T) \prod_F \frac{1}{k^2}$$

hold also when $|F| < T$, and (2) is established for all $F \in S$ and $T > 0$.

The statement of the Lemma now immediately follows from

$$\sum_S \|u_F\|_T \leq e^T B^T(T) C(T) \sum_S \prod_F \frac{1}{k^2} = e^T B^T(T) C(T)(\beta - 1).$$

We will now pick the functions r_k once and for all. Let $f_n \xrightarrow{\mathscr{C}} f \in \mathscr{C}_0$. By Theorem 3 there is a sequence $r_n \in \mathscr{C}_0$ and an $r \in \mathscr{C}_0$ such that $r = f^k r_k$ $(k = 1, 2, \ldots)$. Then

LEMMA 3. *If the functions $\varepsilon_k = f_{n_k} - f$ satisfy condition* (i) *of Lemma* 2, *the partials products*

$$R_N = r \prod_1^N \left(1 + \frac{\varepsilon_k}{f}\right)$$

converge in $\mathscr{C}$ to a function R which we denote by

$$R = r \prod_1^\infty \left(1 + \frac{\varepsilon_k}{f}\right).$$

We have

$$R = r + \sum_S u_F$$

(*using the notation of Lemma* 2.) *Thus*

$$\|R\|_T \leq \|r\|_T + \sum_S \|u_F\|_T \leq \|r\|_T + e^T B^T(T) C(T)(\beta - 1).$$

Proof. The finite products R_N form a cofinal sequence of partial sums for the series $r + u = r + \sum_S u_F$, where u_F is given in Lemma 2. By Lemma 2 the sequence R_N must converge uniformly on $[0, T]$ to $R = r + u$ and $\|R\|_T \leq \|r\|_T + e^T B^T(T) C(T)(\beta - 1)$ for each T.

LEMMA 4. *With the notation and hypothesis of Lemma* 3

$$\frac{R_N}{1 + \varepsilon_k / f} \xrightarrow{\mathscr{C}} \frac{R}{1 + \varepsilon_k / f} \quad (\textit{each } k = 1, 2, \ldots \textit{ as } N \to \infty)$$

and

$$\left\| \frac{R}{1 + \varepsilon_k / f} \right\|_T \leq \|r\|_T + e^T B^T(T) C(T)(\beta - 1)$$

for each k.

Proof. The products

$$\frac{R_N}{1 + \varepsilon_k / f}$$

are in $\mathscr{C}$ for $N \geq k$ and form a cofinal sequence of partial sums for series obtained from $r + \sum_S u_F$ by striking out those $F \in S$ which contain k. Let $S_k = \{F \mid F \in S, k \notin F\}$. Then $\sum_{S_k} u_F$ is a partial sum of $\sum_S u_F$ and is thus norm convergent, and

$$\frac{R_N}{1 + \varepsilon_k / f} \xrightarrow{\mathscr{C}} r + \sum_{S_k} u_F, \quad \textit{as } N \to \infty.$$

Since $R_N \xrightarrow{\mathscr{C}} R$, we have

$$\frac{R}{1 + \varepsilon_k / f} = r + \sum_{S_k} u_F,$$

which is the first statement in the lemma. Since

$$\left\| \frac{R}{1 + \varepsilon_k / f} \right\|_T \leq \|r\|_T + \sum_{S_k} \|u_F\|_T \leq \|r\|_T + \sum_S \|u_F\|,$$

the proof of the lemma is complete.

LEMMA 5. *If $f_n \xrightarrow{\mathscr{C}} f \in \mathscr{C}_0$, there is a subsequence f_{n_k} such that, for $\varepsilon_k = f_{n_k} - f$,*

(i) $$R = r \prod_1^\infty \left(1 + \frac{\varepsilon_k}{f}\right) \in \mathscr{C},$$

(ii) $$\frac{R}{1 + \varepsilon_k / f} \textit{ is a bounded sequence in } \mathscr{C}, \quad k = 1, 2, \ldots,$$

(iii) $$R \neq 0.$$

Proof. (i) and (ii) are satisfied by taking n_k increasing sufficiently rapidly so that (i) of Lemma 2 is satisfied. To see that, we can assume $R \neq 0$; we note that since $r \in \mathscr{C}_0$, $\|r\|_1 > 0$. In fact, we can assume $\|r\|_1 > 1/(\beta - 1)$, since $r = f^k r_k$ implies $(ar) = f^k(ar_k)$ for any real number a. By discarding the first few terms of the sequence f_{n_k} a new sequence can be obtained with $eB(1)C(1) < 1$. Then

$$\|R\|_1 \geq \|r\|_1 - eB(1)C(1)(\beta - 1) > 0.$$

LEMMA 6. *If $g \in \mathscr{C}$ and m_k is a bounded sequence in $\mathscr{C}$, then there is a subsequence of gm_k which is convergent in $\mathscr{C}$.*

Proof. This is a well-known result. The proof consists of noting that if the sequence m_k is uniformly bounded on each finite interval, then the sequence gm_k is uniformly bounded and uniformly equicontinuous on each finite interval. By Arzela's theorem gm_k has compact closure in $\mathscr{C}$. Since $\mathscr{C}$ is metrizable gm_k must have a convergent subsequence.

Proof of Theorem 4. By Theorem 3 there exist $g_n \in \mathscr{C}$, $g \neq 0$, such that $fg = f_n g_n$ for all n sufficiently large. Pick a subsequence f_{n_k} such that the conditions of Lemma 5 are satisfied. Now

$$g_{n_k} = \frac{fg}{f_{n_k}} = \frac{fg}{f+\varepsilon_k} = \frac{g}{1+\varepsilon_k/f}.$$

Thus

$$Rg_{n_k} = g\frac{R}{1+\varepsilon_k/f}$$

has, by Lemma 6, a subsequence convergent in $\mathscr{C}$. In order to simplify the notation we will suppose the entire sequence Rg_{n_k} is convergent in $\mathscr{C}$. Then since $f_{n_k}(Rg_{n_k}) = f(Rg)$ and $f_n \overset{\mathscr{C}}{\to} f$ and Rg_{n_k} convergent in $\mathscr{C}$, it follows that $Rg_{n_k} \overset{\mathscr{C}}{\to} Rg$. Thus we can take $\varrho_k = Rg_{n_k}$, $\varrho = Rg \neq 0$ and the theorem is proved.

IV. A topology for F. Type I convergence is an L-convergence and it provides unique sequential limits. Thus by Theorem 1 it is topological if and only if it satisfies condition (iii) of section II. We shall show that it does not satisfy this condition.

THEOREM 5. *Type* I *convergence is not topological.*

Proof. It was shown by Mikusiński that the sequence

$$x_n = 1/(s-n) = \{e^{nt}\} \qquad (n = 1, 2, \ldots)$$

fails to converge type I. We will show that every subsequence possesses a subsequence which converges type I to zero and this will prove the theorem.

Let f_n and f be given by

$$f_n = \frac{1}{n}\frac{h^2}{x_n} = \frac{h}{n} - h^2 \qquad (n = 1, 2, \ldots),$$

$$f = -h^2.$$

Then $f_n \to f$ in $\mathscr{C}$ and by Theorem 4 given any subsequence of integers it has a subsequence such that there exist functions ϱ_{n_k}, $\varrho \neq 0$,

$$f_{n_k}\varrho_{n_k} = f\varrho \qquad (k = 1, 2, \ldots)$$

and $\varrho_{n_k} \to \varrho$ in $\mathscr{C}$. Thus

$$f\varrho x_{n_k} = \frac{h^2}{n_k}\varrho_{n_k} \to 0$$

in $\mathscr{C}$ as $k \to \infty$ and the theorem is proved.

We take for the topology on F the sequential topology for type I convergence. *In the remainder of the paper when we speak of convergence in F we mean convergence in this topology.* Since type I convergence is an L-convergence, we have

(i) O is open in $F \Leftrightarrow$ for each sequence $x_n \overset{\mathrm{I}}{\to} x \in O$, implies x_n is eventually in O.

(ii) C is closed in $F \Leftrightarrow C$ is sequentially closed under type I convergence.

Since type I convergence is not topological, there are sequences $x_n \to x$ in F but x_n does not converge to x type I. The new sequences, however, are found from the type I convergent sequences in a simple manner:

(iii) $x_n \to x$ in $F \Leftrightarrow$ every subsequence of x_n has a subsequence which converges to x type I.

The space F is thus a sequential space in the terminology of section II, and by Theorem 2 it is the quotient of a metric space. The metric space utilized by Franklin in the proof of Theorem 2 is the disjoint union of a large number of spaces each of which consists of one convergent sequence. We can do somewhat better for our particular space. First, however, we shall see that F is in the nature of an inductive limit of Fréchet spaces (of course, however, it is not locally convex). To obtain this view of F, we express F as the union of a partially ordered collection of Fréchet spaces.

For each $f \neq 0$, $f \in \mathscr{L}$ define $B_f = \{x \mid x \in F, xf \in \mathscr{L}\}$ and give B_f the semi-norms $\|x\|_{f,N} = \|xf\|_N^1$ for $N = 1, 2, \ldots$ The collection $\{B_f \mid f \in \mathscr{L}, f \neq 0\}$ has a natural ordering, $B_f \geqslant B_g$ if and only if $B_f \supset B_g$. We use this partial ordering to partially order $\mathscr{L} - \{0\}$ by $f \geqslant g$ if and only if $B_f \supset B_g$.

THEOREM 6. *$f \geqslant g$ if and only if there is a finite measure μ on $[0, \infty]$ such that $f = g\mu$. $f \sim g$ (i.e., $B_f = B_g$) if and only if μ assigns non-zero mass to the origin.*

Proof. This is essentially Theorem 3.2 in [2].

COROLLARY. *If $f \geqslant g$, the natural injection of B_g into B_f is continuous.*

Since $F = \bigcup_{f \neq 0} B_f$, the topology on F can now be described in terms of the spaces B_f.

THEOREM 7. *F has the strongest topology so that all the cononical injections $B_f \to f$ are continuous.*

Proof. All the injections are continuous for a topology on F if and only if every type I convergent sequence in F is convergent in this topology. The sequential topology for type I convergence is the strongest topology with this property.

COROLLARY. *A set O is open in F if and only if $O \cap B_f$ is open in B_f for each $f \neq 0$.*

Some properties of the topology on F follow directly from the definition. Thus

THEOREM 8. (i) *The topology on F is translation invariant.*

(ii) *The mapping $(a, x) \to ax$ on $C \times F \to F$ is continuous.*

Proof. Property (i) follows from the above corollary.

Property (ii) follows from the fact (proved in [3]) that the product topology on the product of a locally compact sequential space with a sequential space is sequential. Thus the topology on $C \times F$ is sequential and since the map $(a, x) \to ax$ is sequentially continuous on $C \times F \to F$, it is continuous.

On the other hand, it is difficult to tell if the topology on $F \times F$ is a sequential topology. In particular, it is not known if F is a topological vector space.

The map $x \to 1/x$ is not continuous at any point $x \in F$. However, in the case of reciprocals we can find out exactly what happens by the use of support numbers. We will follow Norris by defining a finite number a for each $x \neq 0$ in F, as that unique real number a such that

$$x = \frac{f}{g} e^{-as},$$

where f and g are in $\mathscr{C}_0$. This number is independent of the particular representation of x which is chosen, and if $x = y/z$, where $y, z \in F$, then $a(x) = a(y) - a(z)$. The number $a(x)$ is called the *support number* of x. Then

LEMMA. *If $x_n \to x \in F$, $x \neq 0$, then*

$$\overline{\lim}\, a(x_n) \leqslant a(x).$$

Proof. If $f_n \in \mathscr{C}$, $f \neq 0$ is in $\mathscr{C}$, and $f_n \to f$ in $\mathscr{C}$, then $\overline{\lim}\, a(f_n) \leqslant a(f)$. The lemma follows.

There is a slight generalization of Theorem 4 which can be stated utilizing support numbers.

THEOREM 4'. *Let $f_n \in \mathscr{C}$ converge in $\mathscr{C}$ to a non-zero function f. If $a(f_n) \to a(f)$, then each subsequence of the integers has a subsequence (n_k) and there is a corresponding sequence $\varrho_k \in \mathscr{C}$ such that $\varrho_k \to \varrho \neq 0$ in $\mathscr{C}$ and $f_{n_k} \varrho_k = f \varrho$ for every $k = 1, 2, \ldots$*

Proof. Define functions $\hat{f}_n$ and $\hat{f}$ by $f_n = \hat{f}_n e^{-a(f_n)s}$ and $f = \hat{f} e^{-a(f)s}$. Then $\hat{f}_n \to \hat{f}$ in $\mathscr{C}$, $\hat{f} \in \mathscr{C}_0$ and by Theorem 4 there exist $\hat{\varrho}_k \to \hat{\varrho}$ in $\mathscr{C}$, $\hat{\varrho} \neq 0$ such that $\hat{f}_{n_k} \hat{\varrho}_k = \hat{f} \hat{\varrho}$. Let $A = \operatorname{Sup} a(f_n)$; then the sequence

$$\varrho_k = \hat{\varrho}_k e^{-[A - a(f_n)]s}, \quad \varrho = \hat{\varrho} e^{-[A - a(f)]s}$$

satisfies the conditions of the theorem.

We can now clarify the relationship between convergent sequences and their reciprocals.

THEOREM 9. *Suppose that $x_n \to x \neq 0$ in F. Then $1/x_n \to 1/x$ if and only if $a(x_n) \to a(x)$.*

Proof. The only if part follows from the above lemma and the fact that $a(1/x) = -a(x)$.

Now for the if part. Suppose that $x_n \to x \neq 0$ and $a(x_n) \to a(x)$. By Theorem 4' every subsequence of x_n has a subsequence such that $1/x_n \xrightarrow{\text{I}} 1/x$ and this proves the theorem.

V. F as a quotient of a metric space. Let $X = \{(x, f) \mid x \in B_f, f \in \mathscr{L}, f \neq 0\}$. Since each B_f is a metric space, we can make X metric with the metric

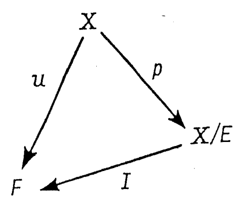
$$\varrho[(x, f), (y, f)] = \frac{\varrho_f(x, y)}{1 + \varrho_f(x, y)},$$

$$\varrho[(x, f), (y, g)] = 1, \quad f \neq g.$$

Since $(x, f) \in X$ implies $x \in B_f \subset F$, the relation $(x, f) \sim (y, g) \Leftrightarrow x = y$ is an equivalence relation on X. Let E be this equivalence relation. Then

THEOREM 10. $X/E = F$.

Proof. Each element of X/E is identified in a natural way with a unique element in F. We claim the identity map on X/E onto F is a homeomorphism. Let I be this map, p the natural projection, $p[(x, f)]$



$= x'$ of X to X/E and u the map $u\big((x, f)\big) = x$ on X to F. First to see that I is continuous. The map u is continuous and if $A \subset F$ is open, then $u^{-1}(A)$ is an open collection of cosets in X; since p is the quotient map, $I^{-1}(A) = p\big(u^{-1}(A)\big)$ is open in X/E.

To see that I^{-1} is continuous, let B be open X/E. Then $I(B) = u(p^{-1}(B))$. Now $p^{-1}(B) = [B]$ is a collection of cosets in X and is open; we want to show $I(B) = u([B])$ is open. Suppose $x_n \in F$, $x_n \xrightarrow{\mathrm{I}} x \in u([B])$. Because $x_n \to x$ type I, there is an $f \neq 0$ in $\mathscr{L}$ such that $\varrho_f[(x_n, f), (x, f)] \to 0$ and since $x \in u([B])$, $(x, f) \in [B]$, thus (x_n, f) is eventually in $[B]$ and x_n is eventually in $u([B])$. Thus $u([B]) = I(B)$ is open in F.

Conclusion. Some of the unresolved questions with respect to the sequential topology for type I convergence are as follows. First what is the connection between convergence in the topology and type II convergence. If $x_n \xrightarrow{\mathrm{II}} x$ and the regularizing sequence f_n, such that $f_n \xrightarrow{\mathscr{C}} f \neq 0$ and $f_n x_n \xrightarrow{\mathscr{C}} fx$, can be chosen so that $f \in \mathscr{C}_0$, then Theorem 4 shows that in fact $x_n \to x$ in the topology. If a regularizing sequence with $f \in \mathscr{C}_0$ can be chosen, then, in particular, $\overline{\lim}\, a(x_n) \leqslant a(x)$. A reasonable conjecture is that if $x_n \xrightarrow{\mathrm{II}} x \neq 0$, then $x_n \to x$ if and only if $\overline{\lim}\, a(x_n) \leqslant a(x)$.

If O is such that $O \cap B_f$ is open in B_f for each $f \in \mathscr{L} - \{0\}$, then O is open in F. An unresolved question is: if V is such that $V \cap B_f$ contains an open neighborhood of the origin in B_f for each $f \in \mathscr{L} - \{0\}$ does V necessarily contain an open neighborhood of the origin in F?

Is F Hausdorff?

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Reflexivity and summability: the Nakano $l(p_i)$ spaces

by

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1. A classical theorem of Banach and Saks asserts that every bounded sequence in L_p, $p > 1$, has a subsequence whose $(C, 1)$ means converge. Nishiura and Waterman [4] showed that a Banach space is reflexive if and only if, for every bounded sequence, there is a summability method T of a particular kind and a subsequence whose T-means converge (either weakly or strongly). This has been discussed further by Singer [7], Pełczyński [5], and Waterman [8].

In his review [6] of the paper of Nishiura and Waterman, Sakai raised the following question: Is there a reflexive space for which $(C, 1)$ is not the suitable method? Klee [1] attempted to answer this and showed that certain $l(p_i)$-spaces of Nakano contained bounded sequences with no $(C, 1)$ summable subsequences. In section 2 we will show that these spaces exhibit a more striking property, namely that, for any regular method T or any regular* method T^* of Zygmund [10], p. 202-205, there exists a bounded sequence without T (T^*)-summable subsequences. However, as we will show in section 3, it is precisely these $l(p_i)$-spaces which are *not* reflexive. Thus the question of Sakai remains unanswered. The result in section 3 was stated in our review [9] of [1].

2. Let $\{p_i\}$ be a sequence of real numbers, $1 \leqslant p_i \leqslant \infty$. Then $l(p_i)$ denotes the set of all real sequences $x = \{t_i\}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{p_i} |at_i|^{p_i} < \infty$$

for some $a > 0$ depending on x. We adopt the convention that, for a function f of a finite real variable, the value at ∞ is given by

$$f(\infty) = \lim_{u \to \infty} f(u).$$

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