and in view of (36), it is easy to see that (35) is satisfied. Now we shall prove that (ii) is also satisfied. To do so, let us remark that from the proof of part (a) it follows that there exist two real finite numbers a, $\bar{a}$ such that $\mu(A) \leqslant a\mu_1(A)$ and $\mu_1(A) \leqslant \bar{a}\mu(A)$ for each Borel set A. Therefore it suffices to show that (ii) is satisfied when we replace the measure μ by μ_1. For this purpose let us put $L_1 = \beta$, where β is from (16). Further, in view of Lemma 5, it follows that there exists a positive integer s_0 such that $\varphi^{s_0}(A_{i_0}) = M$ for $i_0 = 1, 2, \ldots, k$, where A_{i_0} is from the proof of Lemma 4. Since φ^{s_0} is a local diffeomorphism and in view of the remarks preceding Lemma 4, it easily follows that there exists a finite real number L_2 such that $\mu_1(\varphi^{s_0}(Z)) \leqslant L_2\mu_1(Z)$ for each Borel set Z. Now let us put for each set $A_{i_0,\ldots,i_n} (n \geqslant 1)$ $n_1 = n$ and $n_2 = n+s_0$. Then (ii$_b$) and (ii$_c$) are satisfied; to prove (ii$_a$) it suffices to apply (21) and act as in the proofs of Lemma 4 and (31) (see the proof of the theorem on p. 525 in [4]). Thus the proof of part (b) is completed.

It remains to prove (c). In fact, in view of (a) and (b), the part (c) follows from the well-known theorem which states that any invariant normalized measure absolutely continuous with respect to a normalized invariant ergodic measure is equal to this measure.

Added in proof. After the paper was submitted for publication, the paper [7] came to our attention. As we understand, there is announced the following result: for each expanding mapping of the compact manifold M into itself there exists an invariant regular Borel measure μ positive on each open set in M.

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Reçu par la Rédaction le 28. 5. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

Existence of differentiable structure in the set of submanifolds

An attempt of geometrization of calculus of variations

by

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1. Introduction. Vigorous development of calculus of variations we witness recently shows only too clearly the effectiveness of the functional analysis approach towards this branch of mathematics. Thanks to the systematic studies started by Eells and his school (see e.g. [4] or [5]) we know that many important families of mappings, encountered in the calculus of variations, can be naturally equipped with the structure of an infinitely dimensional differentiable Banach manifold. E.g. the set $C^r(K, X)$ of r times continuously differentiable mappings of a compact manifold K into a finite-dimensional manifold X, is a manifold modelled on some vector space $C^r(K, Y)$.

Such approach enabled Palais and Smale (see [11], [12] or [13]) to found the unified general Morse theory, the method of steepest descent (cf. also [2]) and to prove many beautiful global theorems about functionals of the calculus of variations. The excellent report by Eells [6], containing also wide bibliography on the subject shows how considerable is the bulk of work done in this field.

As it appears, however, for many purposes like

a) classic field theory

b) differential geometry

the Banach manifolds fail to suffice. E.g. let us consider the classic electrodynamics. In the relativistic Lagrange formulation of the theory the essential role is played by states of the field in bounded domains of the space-time, limited within the dimensions of the laboratory and the duration of an experiment. Such a state is therefore a section over compact set of the bundle $\overset{2}{\wedge} T^*(R^4)$, where R^4 is the space-time. (As we know, the electromagnetic field is an exterior differential 2-form). The range of such a section is a compact 4-dimensional submanifold with boundary of $\overset{2}{\wedge} T^*(R^4)$. Among all such submanifolds with common boundary only the one which is the critical point of action is of interest for physicists.

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As we see it is worth considering the family $\mathscr{P}^r_m(X)$ — of all m-dimensional compact submanifolds with boundary of the class C^r in the finite-dimensional manifold X. But such family can not be naturally equipped with the Banach structure. It is connected with the fact that $\mathscr{P}^r_m(X)$ is the result of division of the manifold $C^r(\Omega, X)$ (where Ω is some fixed model compact manifold with boundary) by the equivalence relation which identifies mappings having the same range. The received quotient-manifold is not even of the class C^1 (although it is a topological manifold). If to calculate in this quotient-manifold the mappings $(\varkappa_1 \circ \varkappa_2^{-1})$ giving the change of a coordinate chart, we see that their derivatives contain differential operators, which are not continuous in spaces $C^r(\Omega, Y)$.

The remedy for this is as follows: we must deal from the beginning with the set $\mathscr{P}_m(X)$ which contains only elements of the class C^∞. In this case the model-spaces will be the spaces $C^\infty(\Omega, Y)$, in which differential operators are continuous.

Although the satisfactory theory of differentiation in general locally convex topological vector spaces does not exists (which predetermines character of the attempts at studies on non-Banach manifolds as merely tentative and having not much to do with concrete applications and models — see e.g. [1]) still for wide class of spaces — which are at the same time Fréchet and Schwartz spaces — such theory has been put forward in [9].

Using this theory we prove in present paper that $\mathscr{P}_m(X)$ is a differentiable manifold of the C^∞-class, the tangent spaces of which are Fréchet and Schwartz spaces. It comes out that these tangent spaces have the simple representation: its vectors are geometrical objects in the finite-dimensional manifold X.

We give the formulation of variational problems in X in the terms of differential calculus and local differential geometry of $\mathscr{P}_m(X)$. This language is especially useful for the problems with free boundary. As we know, in this case and in the case of calculus of variations in non-linear spaces as well, the variational problem can be defined by means of homotopies (see e.g. [3]).

The present paper investigates homotopies, i.e. mappings

$$\Omega \times]-\varepsilon, \varepsilon[\ni (p, t) \to h(p, t) \in X,$$

where $\Omega \in \mathscr{P}_m(X)$, and $h(p, 0) \equiv p$.

Since the sets $\Omega^h_t = \{h(p, t) : p \in \Omega\}$ are elements of $\mathscr{P}_m(X)$, so a homotopy is a representation of a curve in $\mathscr{P}_m(X)$. We calculate the tangent vector to the curve given by a homotopy.

Instead of the definition of the critical point of the functional f, based on homotopies:

"For every homotopy h such that $\Omega^h_0 = \Omega_0$ the following is true:

$$\frac{d}{dt} f(\Omega^h_t)\Big|_{t=0} = 0\text{"}$$

we propose the differential definition: $f'(\Omega_0) = 0$.

The present paper is a continuation of [8], where the differential structure in the set of all compact sections of the topological bundle was constructed.

My grateful acknowledgments are due to dr. W. Tulczyjew who encouraged me to write this paper and who devoted a lot of time for very fruitful discussions.

I am also very much indebted to prof. K. Maurin for his lively interest in the paper, his valuable suggestions and constructive criticism.

2. Preliminaries. All constructions will be carried out in some fixed n-dimensional C^∞-differentiable manifold X.

We shall consider m-dimensional, compact C^∞-submanifolds (possibly with the boundary) imbedded in X. (For the notion of imbedded submanifold see e.g. [14].) For the sake of language simplicity we shall mean the above-mentioned objects saying "compact m-submanifolds", or simply "c. m-s.".

The set of all compact m-submanifolds in X will be denoted by $\mathscr{P}_m(X)$ or simply $\mathscr{P}_m$ if it does not lead to any confusion.

Let $\Omega \in \mathscr{P}_m$. It follows from the definition, that for every $x_0 \in \operatorname{int} \Omega$ there exists such a coordinate chart (μ, O):

$$O \ni x \to \mu(x) = (x^1, \ldots, x^n) \in R^n,$$

where O is a neighbourhood of x_0 in X, that

(1) $$(x \in \Omega \cap O) \Leftrightarrow (x^j \equiv 0 \text{ for } j > m)$$

and $\tilde{\mu}(x) := (x^1, \ldots, x^m) \in R^m$ gives a coordinate chart $(\tilde{\mu}, \Omega \cap O)$ compatible with the differentiable structure of the manifold Ω.

Moreover, for every $x_0 \in \partial\Omega$ (the boundary of Ω) there exists its neighbourhood $O \subset X$ and the coordinate chart (μ, O)

$$O \ni x \to \mu(x) = (x^1, \ldots, x^n) \in R^n$$

that

(2) $$(x \in \Omega \cap O) \Leftrightarrow (x^j \equiv 0 \text{ for } j > m;\ x^m \leqslant 0).$$

It follows from (2) that in this coordinate chart

$$(x \in \partial\Omega \cap O) \Leftrightarrow (x^j \equiv 0 \text{ for } j \geqslant m)$$

and that the coordinate charts $(\tilde{\tilde{\mu}}, \partial\Omega \cap O)$, where $\tilde{\tilde{\mu}}(x) = (x^1, \ldots, x^{m-1})$ and μ satisfies (2), define in $\partial\Omega$ the structure of an $(m-1)$-dimensional C^∞-differentiable submanifold of X.

3. Topology in the space of compact submanifolds. We shall introduce in $\mathscr{P}_m$ an inductive topology, defined by the family of mappings.

Let us settle the following notation:

By E_Ω (where $\Omega \in \mathscr{P}_m$) we denote the set of all C^∞-diffeomorphisms transforming Ω in X (a differentiable mapping of the manifold with the boundary Ω is a mapping which is differentiable in the interior of Ω (int Ω), and in every coordinate chart of a type (2) it admits a differentiable extension beyond the boundary).

We shall treat the E_Ω as a topological space with the topology of uniform convergence of all derivatives (the derivatives can be counted in the fixed covering of the manifold Ω and its image by the domains of coordinate charts. Those derivatives, however, depend on this covering, but the topology of their uniform convergence does not depend on it).

Let us define the mappings I_Ω,

$$E_\Omega \ni \eta \to I_\Omega(\eta) \in \mathscr{P}_m,$$

where $I_\Omega(\eta) := \eta(\Omega)$ (of course $\eta(\Omega)$ is a compact m-submanifold as a diffeomorphic image of Ω).

Now the space $\mathscr{P}_m$ can be equipped with the inductive topology with respect to all pairs (E_Ω, I_Ω). This topology will be denoted by $\mathscr{T}$ (it is the finest topology in which all I_Ω are continuous). It is characterized by the following

THEOREM 1. *The space $(\mathscr{P}_m, \mathscr{T})$ satisfies the first axiom of countability. The base of neighbourhoods $\{\mathscr{V}_i\}_{i=1}^\infty$ of the point $\Omega \in \mathscr{P}_m$ can be given as*

$$\mathscr{V}_i = I_\Omega(V_i),$$

where $\{V_i\}_{i=1}^\infty$ is a base of neighbourhoods of the identical map in E_Ω.

This theorem immediately follows from the given below compatibility condition for mappings I_Ω.

LEMMA 1. *If $V_1 \subset E_{\Omega_1}$ and $V_2 \subset E_{\Omega_2}$ are open sets, then the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$, given in the set*

$$\mathscr{V} = I_{\Omega_1}(V_1) \cap I_{\Omega_2}(V_2)$$

by I_{Ω_1} and I_{Ω_2} respectively, are identical.

Proof. Let $\mathscr{U} \in \mathscr{V}$ be an arbitrary $\mathscr{T}_1$-open set. This means that

$$U = I_{\Omega_1}^{-1}(\mathscr{U}) \subset E_{\Omega_1}$$

is open. We shall show that $I_{\Omega_2}^{-1}(\mathscr{U}) \subset E_{\Omega_2}$ is also open, i.e. $\mathscr{U}$ is $\mathscr{T}_2$-open. This means that $\mathscr{T}_1 \subset \mathscr{T}_2$. In the analogical way we receive $\mathscr{T}_2 \subset \mathscr{T}_1$, whence $\mathscr{T}_1 = \mathscr{T}_2$.

So if $\mathscr{U}$ is non-empty set let us take an arbitrary $\eta_2 \in I_{\Omega_2}^{-1}(\mathscr{U})$. Let $\eta_1 \in U$ be such that $\eta_2(\Omega_2) = \eta_1(\Omega_1)$. Writing $\eta_1^{-1} \circ \eta_2 = \eta: \Omega_2 \to \Omega_1$ we have $\eta_2 = \eta_1 \circ \eta$.

On the contrary, for every diffeomorphism $\eta: \Omega_2 \to \Omega_1$ and every $\eta_1 \in U$ we have $\eta_1 \circ \eta \in I_{\Omega_2}^{-1}(\mathscr{U})$.

Denoting by $Z \in E_{\Omega_2}$ the set of all diffeomorphisms Ω_2 on Ω_1, we can write the result of the above discussions as follows:

$$I_{\Omega_2}^{-1}(\mathscr{U}) = \bigcup_{\eta \in Z} \bigcup_{\eta_1 \in U} \eta_1 \circ \eta.$$

But for fixed η the set $V_\eta = \bigcup_{\eta_1 \in U} \eta_1 \circ \eta \subset E_{\Omega_2}$ is open when U is open. It follows from the fact that the mapping

$$E_{\Omega_2} \in \eta_1 \circ \eta \to \eta_1 \circ \eta \circ \eta^{-1} = \eta_1 \in E_{\Omega_1}$$

is continuous, and V_η is the inverse image of U.

So $I_{\Omega_2}^{-1}(\mathscr{U})$ is open as a sum of open sets.

Remark. The Eells' standard procedure (see [4]) enables us to introduce the structure of infinitely dimensional manifold into the spaces E_Ω. Unfortunately $\mathscr{P}_m$ is not locally homeomorphic to E_Ω, so this method is not useful in defining the differentiable structure in $\mathscr{P}_m$.

$\mathscr{P}_m$ is locally homeomorphic to the spaces E_Ω/R, where R is the equivalence relation:

$$(\eta_1, \eta_2) \in R \Leftrightarrow \eta_1(\Omega) = \eta_2(\Omega).$$

4. Differentiable ($\mathscr{F}-S$)-manifolds. In this paper we shall consider such differentiable manifolds, tangent spaces of which are not Banach-spaces, however, they are topological vector Fréchet spaces of the S-type (Schwartz-spaces). Such spaces will be called *($\mathscr{F}-S$)-spaces.* The definition of Schwartz space was given in [7] (it is also rewritten in [9]).

The differentiability in ($\mathscr{F}-S$)-spaces will be understood in the sence of [9].

Definition. *A differentiable ($\mathscr{F}-S$)-manifold* (resp. *Banach-manifold*) *of the class C^k* is the triplet $(\mathscr{P}, T, K)$, where

1° $\mathscr{P}$ is a topological Hausdorff space;

2° $T = T(\mathscr{P}) = \bigcup_{p=\mathscr{P}} T_p(\mathscr{P})$ and all $T_p(\mathscr{P})$ are vector ($\mathscr{F}-S$)-spaces (Banach-spaces resp.);

3° $K = \bigcup_{p=\mathscr{P}} K_p$, where every K_p is a non-empty set of homeomorphisms mapping neighbourhoods of $p \in \mathscr{P}$ on neighbourhoods of zero in $T_p(\mathscr{P})$;

4° the following axioms are satisfied:

a) If $\varkappa \in K_p$, then $\varkappa(p) = 0$.

b) If $\varkappa_1, \varkappa_2 \in K$, then the mapping $(\varkappa_2 \circ \varkappa_1^{-1})$ is a C^k-diffeomorphism (on the set where it is well defined).

c) If $\varkappa_1, \varkappa_2 \in K_p$, then $(\varkappa_2 \circ \varkappa_1^{-1})'(0) = I$ (I denotes here the identical operator in T_p).

d) The set K is complete in the sense that every larger set does not satisfy the axioms (a), (b) and (c).

The space $T_p(\mathscr{P})$ is called the *tangent space* at the point p (it will be also denoted T_p when no confusion is to be afraid of).

The above definition is a little more general then the standard one. It admits the fact that the spaces T_p and T_q $(p \neq q)$ can be non-isomorphic, however, they are isomorphic if p and q lie in the same connected component of $\mathscr{P}$. So these two definitions coincide in every connected component of $\mathscr{P}$.

5. Differentiable structure in $\mathscr{P}_m(X)$. Now we are going to define the tangent spaces $T_\Omega(\mathscr{P}_m)$ and projections $\varkappa \in K$, which give the differentiable structure in $\mathscr{P}_m(X)$.

Let us denote by $\Gamma(\Omega, T(X))$ the set of all C^∞-sections of the tangent bundle of X the domain of which is $\Omega \subset X$ (such a section is a C^∞-vector field tangent to X, defined on Ω). Since Ω is not, in general, an open set, we mean the differentiability of a mapping defined on Ω in usual sense: it can be differentiably extended on some neighbourhood of $\Omega \subset X$.

Let us denote by $\Gamma(\Omega, T(\Omega)) \subset \Gamma(\Omega, T(X))$ such vector fields which are infinitesimal transformations of Ω (which are generators of one-parameter groups of transformations of the manifold Ω). The set $\Gamma(\Omega, T(\Omega))$ consists of such vector fields which are tangent to Ω at every point $x \in \operatorname{int}\Omega$, and are tangent to $\partial\Omega$ at every point $x \in \partial\Omega$.

The vector space $\Gamma(\Omega, T(X))$, equipped with the topology of uniform convergence of all derivatives is an $(\mathscr{F}-S)$-space. The set $\Gamma(\Omega, T(\Omega))$ is its closed subspace. So the quotient space $\Gamma(\Omega, T(X))/\Gamma(\Omega, T(\Omega))$ is also an $(\mathscr{F}-S)$-space.

Definition. $T_\Omega(\mathscr{P}_m) := \Gamma(\Omega, T(X))/\Gamma(\Omega, T(\Omega))$.

The elements of the space $T_\Omega(\mathscr{P}_m)$ can be treated as fields, defined on Ω, the values of which lie in the spaces $T_x(X)/T_x(\Omega)$ for $x \in \operatorname{int}\Omega$ and in $T_x(X)/T_x(\partial\Omega)$ for $x \in \partial\Omega$.

Now it is obvious why, while defining the differentiable manifold, we have admitted the possibility of non-isomorphic tangent spaces: if Ω_1 and Ω_2 are not diffeomorphic, then T_{Ω_1} and T_{Ω_2} can be non-isomorphic. Thus $\mathscr{P}_m(X)$ is not a differentiable manifold in the sense of the standard definition, although its connected components, consisting of diffeomorphic compact m-submanifolds, are.

In order to construct the mappings $\varkappa \in K$, let us introduce the following notion:

Definition. Every m-dimensional C^∞ imbedded submanifold $\tilde{\Omega}$ of X for which $\Omega \subset \operatorname{int}\tilde{\Omega}$ will be called the *extension* of Ω.

Lemma 2. *For every $\Omega \in \mathscr{P}_m(X)$ and $\tilde{\Omega}$ — its extension, there exist: (1) the neighbourhood $O \subset X$ of Ω; (2) the vector bundle N of the base $\tilde{\Omega}$, the fiber of which over $x \in \tilde{\Omega}$ is an $(n-m)$-dimensional subspace of $T_x(X)$, transversal to $T_x(\tilde{\Omega})$; (3) C^∞-diffeomorphism $\Phi\colon O \to N$ such that, for $x \in \tilde{\Omega}$, $\Phi(x) = O_x$ (zero in the fiber over x), and $\Phi'(x)$ is the identical operator in $T_x(X)$ (it is easily seen that for $x \in \tilde{\Omega}$ both spaces $T_x(N)$ and $T_x(X)$ can be naturally identified).*

The proof of this fact will be given in the last section of the paper.

Let us apply Lemma 2 to the $(m-1)$-submanifold $\partial\Omega \in \mathscr{P}_{m-1}(\tilde{\Omega})$. We shall obtain the vector bundle H with the base $\partial\Omega$ ($\partial\Omega$ having no boundary is the extension of $\partial\Omega$ itself). The bundle H can be obtained explicitly as in [8], that means constructing a vector field transversal to $\partial\Omega$. The fibers of this bundle will be 1-dimensional integral curves of this field. The section of this bundle can be treated as C^∞-function on the manifold $\partial\Omega$, so it can be trivialized: $H = \partial\Omega \times R^1$. To each point $(p, t) \in H$ corresponds the point of the manifold $\tilde{\Omega}$, obtained by translating of the point $p \in \partial\Omega$ for the t-distance along the integral curve of the field.

So let the bundles N, H and their isomorphisms Φ, Ψ with some neighbourhoods of the sets $\Omega \subset X$, $\partial\Omega \subset \tilde{\Omega}$ be fixed.

Let us take any $u \in T_\Omega(\mathscr{P}_m)$. For each element $u(x) \in T_x(X)/T_x(\Omega)$, $x \in \operatorname{int}\Omega$, there exists unique representative $u_N(x) \in T_x(X)$ being the vertical vector in N.

For each element $u(x) \in T_x(X)/T_x(\partial\Omega)$, $x \in \partial\Omega$, there exists unique representative of the form $u_N(x) + u_H(x)$, where $u_N(x)$ is vertical in N and $u_H(x)$ is vertical in H.

So to every $u \in T_\Omega(\mathscr{P}_m)$ corresponds the pair (u_N, u_H), where u_N is the section of N over Ω and u_H is the section of H over $\partial\Omega$.

Proposition 1. *For fixed (N, Φ, H, Ψ) the mapping*

$$T_\Omega(\mathscr{P}_m) \ni u \to (u_N, u_H) \in \Gamma(\Omega, N) \times \Gamma(\partial\Omega, H) \tag{3}$$

is an isomorphism of topological vector spaces provided the spaces $\Gamma(\Omega, N)$ and $\Gamma(\partial\Omega, H)$ are equipped with the topology of uniform convergence of all derivatives.

It is easy to see that there exists a neighbourhood $\mathscr{V}$ of $\Omega \in \mathscr{P}_m$ such that all elements of the set $\mathscr{V}$ correspond by Φ to sections of the bundle N. So we can define in $\mathscr{V}$ the mappings $\varkappa$ in the same way as it was done in [8].

We are going to describe shortly this construction.

If to trivialize the bundle H, one can treat points in some neighbourhood of $\partial\Omega$ in $\tilde{\Omega}$ as pairs: $(p,t)\in\partial\Omega\times]-\varepsilon,\varepsilon[$.

Let us write $A_p := \{(p,t): p = \text{constant}, t\in]-\varepsilon,\varepsilon[\}$. So A_p is the fiber of H.

If the sets $\pi_N^{-1}(A_p)$ (where π_N is the projection in the bundle N) are treated as bundles over A_p, then the choice of an arbitrary linear connection C in N defines in these bundles the absolute parallelism.

So points in some neighbourhood of the set $\partial\Omega$ in X can be treated as elements $(p,t,a)\in\partial\Omega\times]-\varepsilon,\varepsilon[\times\pi_N^{-1}(p)$.

Now let us take an arbitrary C^∞-function $\xi(t)$ such that

1° $\operatorname{supp}\xi\in]-\varepsilon/2,\varepsilon/2[$;

2° $\xi(0)=1$;

3° $\xi(t)\geqslant 0$.

For every $\varphi\in C^\infty(\partial\Omega)$, satisfying the condition

$$|\varphi(p)| < \left(\sup\left|\frac{d\xi(t)}{dt}\right|\right)^{-1}$$

the following diffeomorphism of N on N can be defined:

$$N\ni x=(p,t,a)\to\tau_\varphi(x) := \big(p, t+\varphi(p)\cdot\xi(t), a\big)$$

for x's which can be represented as a triplet (p,t,a), and $\tau_\varphi(x)=x$ for the remaining x's.

Let us notice that the mapping

$$\Gamma(\partial\Omega, H)\cong C^\infty(\partial\Omega)\ni\varphi\to\tau_\varphi\in C^\infty(N,N)$$

is continuous provided both spaces are equipped with the topology of uniform convergence of all derivatives.

Definition. For such $u\in T_\Omega(\mathscr{P}_m)$ for which it is well defined we take

$$\varkappa^{-1}(u) = \Phi\circ\tau_{u_H}(\operatorname{range} u_N).$$

As it is shown in [8] the above mapping is an injection. So for every fixed (N,Φ,H,Ψ,C,ξ) the mapping $\varkappa$ is well defined.

THEOREM 2. *The set K^0 of all above-constructed mappings $\varkappa$ satisfies the axioms* (a), (b), (c). *So* $(\mathscr{P}_m, T, K)$ (where $K\supset K^0$ is the maximal set of mappings satisfying (a), (b), (c)) is an $(\mathscr{F}-\mathscr{S})$-differentiable manifold of the class C^∞.

6. Elements of differential calculus in $\mathscr{P}_m(X)$ and its connection with the calculus of variations.

(a) *Differentiable curves.* For the purpose of the calculus of variations we shall consider curves in the space $\mathscr{P}_m(X)$, i.e. the one-parameter families of c. m-s. The most useful representations of such curves are differentiable homotopies

$$\Omega\times]-r,r[\ni(p,t)\to h(p,t)\in X,$$

where $h(p,0)=p$ and $h(\cdot,t)\in E_\Omega$ for every $t\in]-r,r[$.

If the function h is a C^∞-function then it defines the C^∞-curve in $\mathscr{P}_m$. We shall show that the reverse is also true: every curve in $\mathscr{P}_m(X)$ can be given by a homotopy h. Of course, this correspondence is not one-to-one. The different homotopies can define the same curve.

For the curve $\Omega(t)$, $\Omega(0)=\Omega$, the corresponding homotopy can be taken as follows:

Take some $\varkappa=\varkappa(N,\Phi,H,\Psi,C,\xi)$, defined in the last section. Define $u(t) := \varkappa(\Omega(t))$. Then $h(p,t) := \Phi\circ\tau_{u_H(t)}\circ u_N(p,t)$ for $(p,t)\in\Omega\times]-r,r[$. The following is true:

THEOREM 3. *If $\Omega(t)$ is the curve corresponding to homotopy h, then*

$$(*)\qquad \frac{d}{dt}\Omega(t)\bigg|_{t=t_0} = \left[\frac{\partial h(\cdot,t)}{\partial t}\bigg|_{t=t_0}\right],$$

where $\partial h/\partial t|_{t=t_0}\in\Gamma\big(\Omega(t_0), T(X)\big)$ and $[\partial h/\partial t|_{t=t_0}]$ is its image by the canonical mapping $\Gamma\big(\Omega(t_0), T(X)\big)\to T_{\Omega(t_0)}(\mathscr{P}_m)$. Plainly, the right-hand side of $(*)$ does not depend on the particular choice of homotopy h.

COROLLARY. *If v is a vector field defined in some neighbourhood $O\subset X$ of $\Omega\subset X$, then the mapping $\exp(t\cdot v)$ defines the curve in $\mathscr{P}_m$, the tangent vector of which at the point Ω is $[v|\Omega]$ (by $v|\Omega$ we denote the restriction of v to Ω).*

Remark. This corollary gives us the simple method of construction of the curve in $\mathscr{P}_m(X)$, the tangent vector $u\in T_\Omega(\mathscr{P}_m)$ of which is given: if $u=[v]$, then one can take any representative $v\in\Gamma\big(\Omega, T(X)\big)$, and its differentiable extension on the neighbourhood of $\Omega\subset X$. Then the one-parameter group of transformations, given by this field, defines the required curve.

(b) *Integral functionals as differentiable functions on $\mathscr{P}_m(X)$.* The most important examples of differentiable functions on $\mathscr{P}_m$ are the functions given by integrals. E.g. let Ω be an oriented compact m-submanifold. There exists, of course, an open neighbourhood $\mathscr{V}\subset\mathscr{P}_m$ of Ω which consists of orientable c.m-s. Moreover, $\mathscr{V}$ can be such, that one can choose "continuously" the orientation of each submanifold. Such sets $\mathscr{V}$ will be called *orientable sets.*

So let $\mathscr{V}\subset\mathscr{P}_m$ be orientable and let its orientation be fixed. Then every m-exterior differential form $\omega\in\Gamma\big(X, \overset{m}{\wedge} T^*(X)\big)$ defines in $\mathscr{V}$ the function

$$P_m(X)\ni\Omega\to f_\omega(\Omega) := \int_\Omega\omega.$$

Let $u \in T_\Omega(\mathscr{P}_m)$ and let Ω_t be a differentiable curve passing by $\Omega_0 = \Omega$, the tangent vector of which is

$$\frac{d}{dt}\,\Omega_t\Big|_{t=0} = u.$$

Then

(4) $$f'_\omega(\Omega)u = \frac{d}{dt}\Big(\int_{\Omega_t}\omega\Big)\Big|_{t=0}.$$

It is very easy to compute the right-hand side of (4), taking e.g. Ω_t as in last Remark (cf. [3]),

(5) $$f'_\omega(\Omega)u = \int_\Omega v \lrcorner d\omega - \int_{\partial\Omega} v \lrcorner \omega,$$

where $v \in [v] = u$ is any representative of u. The right-hand side of (5) does not depend on the choice of v because the first integral depend on v only by terms $\langle W, v \lrcorner d\omega\rangle = \langle W \wedge v, d\omega\rangle$, where $W \in \Gamma(\Omega, \overset{m}{\wedge} T(\Omega))$. So if $v_0 \in \Gamma(\Omega, T(\Omega))$, then $\langle W, v \lrcorner d\omega\rangle = \langle W, (v+v_0) \lrcorner d\omega\rangle$ because of the equality $W \wedge v_0 = 0$.

Analogically the second integral does not depend on the choice of v. So we can write symbolically:

(6) $$f'_\omega(\Omega)u = \int_\Omega u \lrcorner d\omega - \int_{\partial\Omega} u \lrcorner \omega.$$

(c) *Variational problems.* In the calculus of variations the most important are one-parameter families of submanifolds with the common boundary. Such curves correspond to homotopies "restreintes" (cf. [3]). It follows from Theorem 3 that the set of vectors tangent to such curves constitutes a closed subspace of the tangent space:

$$R_\Omega := \{u \in T_\Omega(\mathscr{P}_m) : u \,|\, \partial\Omega = 0\}$$

(it means that if $[v] = u$, then $v(p) \in T_p(\partial\Omega)$ for $p \in \partial\Omega$). So for $u \in R_\Omega$

(7) $$f'_\omega(\Omega)u = \int_\Omega u \lrcorner d\omega.$$

Definition. Let f be a C^1-function defined in the domain $\mathscr{U} \subset \mathscr{P}_m$. The compact m-submanifold $\Omega \in \mathscr{U}$ is a *critical point* of f if

$$f'(\Omega) \,|\, R_\Omega = 0.$$

Using (6) and treating every $v \in \Gamma(\Omega, T(X))$ as a limit of such v_k for which $v_k \,|\, \partial\Omega = 0$ one can check the following sufficient and necessary condition of being a critical point for functions f_ω:

$$\int_\Omega v \lrcorner d\omega = 0 \quad \text{for every } v \in \Gamma(\Omega, T(X)).$$

Exept of R ($R = \{R_\Omega\}_{\Omega \in \mathscr{P}_m(X)}$) the following distribution tends to be defined:

$$L_\Omega = \{u \in T_\Omega(\mathscr{P}_m) : u \,|\, \operatorname{int}\Omega = 0\}.$$

One must not imagine that the condition $u \,|\, \operatorname{int}\Omega = 0$ gives, by continuity, $u = 0$. This condition says only that if $v \in [v] = u$, then $v(x) \in T_x(\Omega)$ for all $x \in \operatorname{int}\Omega$; so by continuity $v(x) \in T_x(\Omega)$ for all $x \in \Omega$. For $u \in L_\Omega$ we have

(8) $$f'_\omega(\Omega)u = -\int_{\partial\Omega} u \lrcorner \omega.$$

Of course $R_\Omega \cap L_\Omega = 0$. Unfortunately the spaces R_Ω and L_Ω are not complementary: $R_\Omega \oplus L_\Omega \neq T_\Omega$. So the natural dream of everyone who only started to see the ins and outs of the calculus of variations — to decompose the variational derivative into two parts: the lagrangian one, dependent on the variation of $\operatorname{int}\Omega$, and the other one, dependent on the variation of the boundary — can not come true.

It is worth noticing, that the distribution R is absolutely integrable. It means that for every $\Omega_0 \in \mathscr{P}_m(X)$ there exists unique submanifold $M = M(\Omega_0) \subset \mathscr{P}_m(X)$, containing Ω_0 and such that $T_\Omega(M) = R_\Omega$ for every $\Omega \subset M$. It is very easy to find such M:

$$M(\Omega_0) := \{\Omega \in \mathscr{P}_m(X) : \partial\Omega = \partial\Omega_0\}.$$

The distribution L is not absolutely integrable, although through every $\Omega_0 \subset \mathscr{P}_m(X)$ there pass an integral manifold of L, e.g.:

$$\mathscr{P}_m(\tilde{\Omega}_0) = \{\Omega \in \mathscr{P}_m(X) : \Omega \subset \tilde{\Omega}_0\}$$

($\tilde{\Omega}_0$ is an arbitrary extension of Ω_0). But this integral manifold is not uniquely determined: for different extensions $\tilde{\Omega}_0$ we obtain different integral manifolds.

Remark. To every m-dimensional submanifold $Y \subset X$ corresponds a submanifold $\mathscr{P}_m(Y)$ of $\mathscr{P}_m(X)$, which is an integral submanifold of the distribution L.

(d) *Problems with free boundary and with constraints.* In the course of many years the mathematicians studying the calculus of variations were guided by the idea of geometrization of this branch of mathematics. This idea was one of the causes of foundation of functional analysis. Already in the thirties the classical works by Lusternik (cf. [10]) sought to geometrize the problems with free boundary.

In the present formulation those problems, as well as problems with constraints (e.g. isoperimetric problem) can be formulated in a natural way.

Definition. The element $\Omega \subset \mathscr{P}_m(X)$ is a *critical point of the functional f for the problem with free boundary* if $f'(\Omega) = 0$.

Definition. The element $\Omega \subset \mathscr{P}_m(X)$ is a *critical point of the functional f for the problem with free boundary and with constraints g*, if

$$g'(\Omega)u = 0 \Rightarrow f'(\Omega)u = 0$$

(e.g. for the isoperimetric problem $m = 1$ and g is the length of the curve).

7. Proofs of theorems. Before the proof of Lemma 2 we shall show the following

LEMMA 3. *Every c.m-s. $\Omega \subset \mathscr{P}_m(X)$ has an extension, and even the maximal extension.*

Proof. Let $\{O_i\}_{i=1}^k$ be a covering of the set $\partial\Omega \subset X$ with the domains of coordinate charts (O_i, μ_i), having property (2). Let us define in O_i the vector field $\overset{(i)}{v}$, which in the coordinate chart (O_i, μ_i) has the following coordinates: $(0, \ldots, 0, 1, 0, \ldots, 0)$ (unity on the m-th place). If $\{f_\lambda\}_{\lambda\in A}$ is a partition of unity subordinated to the covering $\{O_i\}_{i=1}^k$, we take

$$v = \sum_{\lambda\in A} f_\lambda \cdot \overset{(i)}{v},$$

where i is such that $\operatorname{supp} f_\lambda \subset O_i$.

For $p \in \partial\Omega$ we have $v(p) \in T_p(\Omega)$ and $v(p) \notin T_p(\partial\Omega)$ as a convex linear combination of vectors lying in one, convex half-space of the space $T_p(\Omega)$ (this half-space can be characterized by the condition $x^m > 0$ in every coordinate chart of type (2)).

There exists such $\varepsilon > 0$ that for every $t \in]-\varepsilon, \varepsilon[$ the Cauchy-problem

$$\frac{dx(p,t)}{dt} = v\big(x(p,t)\big)$$

with the initial condition $x(p, 0) = p$ has the solution for all $p \in \partial\Omega$. The set $\Omega_1 := \{x(p,t) \in X : t \in]-\varepsilon, \varepsilon[, p \in \partial\Omega\}$ is a C^∞-differentiable manifold: any coordinate chart $p = (p^1, \ldots, p^{m-1})$ in $\partial\Omega$ gives the coordinate chart in Ω_1 by the mapping $(p, t) \to x(p, t)$. Such coordinate charts are compatible with the differentiable structure of Ω in the set $\Omega_1 \cap \Omega = \{x(p,t) \in \Omega_1 : t \leqslant 0\}$. So $\tilde{\Omega} := \Omega \cup \Omega_1$ is the extension of Ω.

The set of all extensions of Ω satisfies the hypotheses of Zorn-Kuratowski lemma, so it contains maximal elements.

Proof of Lemma 2. Let us take any riemannian metric in the neighbourhood of $\tilde{\Omega} \subset X$. The bundle N can be given by the geodesic system connected with $\tilde{\Omega}$, which means that the fiber over $p \in \tilde{\Omega}$ is the $(n-m)$-dimensional manifold composed of all geodesic lines passing by p and orthogonal to $\tilde{\Omega}$.

Proof of Theorem 2. (a) The first axiom is satisfied in virtue of the definition of $\varkappa$.

(b) Let $\Omega_i \in \mathscr{P}_m(X)$, $i = 1, 2$. Let the fixed systems $(N_i, \Phi_i, H_i, \Psi_i, C_i, \xi_i)$ define the mappings $\varkappa_i: \mathscr{P}_m(X) \supset \mathscr{V}_i \to T_{\Omega_i}(\mathscr{P}_m)$.

The mapping $(\varkappa_2 \circ \varkappa_1^{-1})$ can be factorized as follows:

$$T_{\Omega_1}(\mathscr{P}_m) \supset V_1 \xrightarrow{F_1} \Gamma(\Omega_1, N_1) \times \Gamma(\partial\Omega_1, H_1) \xrightarrow{F_2}$$
$$\xrightarrow{F_2} \Gamma(\Omega_2, N_2) \times \Gamma(\partial\Omega_2, H_2) \xrightarrow{F_3} T_{\Omega_2}(\mathscr{P}_m).$$

The mappings F_1 and F_3 are of the class C^∞ as linear isomorphisms (cf. Proposition 1). So it remains to show the C^∞-differentiability of F_2. Let us trivialize the bundles H_i. Let $(v, \varphi) \in V \subset \Gamma(\Omega_1, N_1) \times C^\infty(\partial\Omega_1)$. Write $F_2(v, \varphi) =: \big(\Theta(v, \varphi), \chi(v, \varphi)\big) \in \Gamma(\Omega_2, N_2) \times C^\infty(\partial\Omega_2)$. It is sufficient to show that both Θ and χ are of the class C^∞. Let us take the mapping (for the sake of simplicity we identify objects isomorphic by Φ)

$$\partial\Omega_1 \ni p \to \beta(v, \varphi)(p) := \pi_{H_2} \circ \pi_{N_2} \circ \tau_\varphi\big(v(p)\big) \in \partial\Omega_2,$$

which is a diffeomorphism of $\partial\Omega_1$ on $\partial\Omega_2$. Of course the mapping

$$\Gamma(\Omega_1, N_1) \times C^\infty(\partial\Omega_1) \ni (v, \varphi) \to \beta(v, \varphi) \in C^\infty(\partial\Omega_1, \partial\Omega_2)$$

is continuous if each space is equipped with the topology of uniform convergence of all derivatives. Also the mapping

$$(v, \varphi) \to \beta(v, \varphi)^{-1} \in C^\infty(\partial\Omega_2, \partial\Omega_1)$$

is continuous in this sense.

Let us take the following mapping in the bundles H_i:

$$H_i \ni (p, t) = x \to \gamma_i(x) = t \in R^1.$$

Then

$$\chi(v, \varphi)\big(\beta(v, \varphi)(p)\big) = \gamma_2 \circ \pi_{N_2} \circ \tau_\varphi\big(v(p)\big) =: \delta\big(p, \varphi(p), v(p)\big). \tag{9}$$

If to trivialize locally the bundles N_1 in the neighbourhood of $\partial\Omega_1$, then the mapping

$$\partial\Omega_1 \times]-\varepsilon, \varepsilon[\times \pi_{N_1}^{-1}(p) \ni (p, t, a) \to \delta(p, t, a) \in R^1$$

is of the class C^∞.

Let us define the mapping

$$\Gamma(\Omega_1 N_1) \times C^\infty(\partial\Omega_1) \ni (v, \varphi) \to \tilde{\delta}(v, \varphi) \in C^\infty(\partial\Omega_1)$$

by the formula $\tilde{\delta}(v, \varphi)(p) := \delta\big(p, \varphi(p), v(p)\big)$.

Now (9) can be written in the form

$$\chi(v, \varphi) = K(v, \varphi)\,\delta(v, \varphi), \tag{10}$$

where $K(v,\varphi)(\psi) := \psi\circ\beta(v,\varphi)^{-1}\in C^\infty(\partial\Omega_2)$ for every $\psi\in C^\infty(\partial\Omega_1)$. But it was shown in [8] that mappings of type (10) are C^∞-differentiable, so χ is of the class C^∞.

Now we shall prove the C^∞-differentiability of Θ.

Let us take the mapping

$$\Gamma(\Omega_1, N_1)\times C^\infty(\partial\Omega_1)\ni(v,\varphi)\to\beta'(v,\varphi)\in C^\infty(\Omega_1,\Omega_2)$$

given by

$$\beta'(v,\varphi)(p) = \tau_2\big(\chi(v,\varphi)\big)^{-1}\circ\pi_{N_2}\circ\tau_\varphi\big(v(p)\big),$$

where $\tau_2(\psi)$ for $\psi\in C^\infty(\partial\Omega_2)$ denotes the translation in the bundle N_2 along the fibres of H_2 (analogous to τ_φ in N_1). The mappings $\beta'(v,\varphi)$ are diffeomorphisms of Ω_1 on Ω_2.

The mappings $(v,\varphi)\to\beta'(v,\varphi)\in C^\infty(\Omega_1,\Omega_2)$ and $(v,\varphi)\to\beta'(v,\varphi)^{-1}\in C^\infty(\Omega_2,\Omega_1)$ are continuous in the topology of uniform convergence of all derivatives. Let us notice that

$$\Theta(v,\varphi)\big(\beta'(v,\varphi)(p)\big) = \tau_2\big(\chi(v,\varphi)\big)^{-1}\circ I\circ\tau_\varphi\big(v(p)\big)\subset\pi_{N_2}^{-1}\big(\beta'(v,\varphi)(p)\big),$$

where I is a diffeomorphism $N_1\to N_2$, induced by the identity $O\to O$ (i.e. $I = \Phi_2\circ\Phi_1^{-1}$).

Suppose for an instant, that the bundle N_2 is trivial,

$$N_2 = \tilde\Omega_2\times R^{n-m}\ni(p,a) = x,$$

and that cartesian product structure of N_2 is compatible with the parallelism in the sets $\pi_{N_2}^{-1}(A_p)$. Let us introduce the following C^∞-differentiable mappings:

1° $N_2\ni x = (p,a)\to\gamma(x) := a\in R^{n-m}$;

2° $C^\infty(\partial\Omega_1)\ni\varphi\to\tilde\varphi\in C^\infty(\Omega_1)$, where

$$\tilde\varphi(p) = \begin{cases}\varphi\big(\pi_{H_1}(p)\big)\cdot\xi_1\big(\gamma_1(p)\big) & \text{for } p\in H_1\cap\Omega_1,\\ 0 & \text{elsewhere;}\end{cases}$$

3° $N_1\times R^1\ni(a',s)\to\delta'(a',s)\in R^{n-m}$, where $\delta'(a',s) = \gamma\circ I\circ\tau_s(a')$.

Here τ_s denotes, as usually, the parallel translation in N_1 along curves of H_1.

If $\tilde\delta'(v,\varphi)(p) := \delta'\big(v(p),\tilde\varphi(p)\big)$, then

$$\Theta(v,\varphi)(q) = \big(q, K'(v,\varphi)\tilde\delta'(v,\varphi)(q)\big)\in N_2, \tag{11}$$

where $K'(v,\varphi)\psi := \psi\circ\beta'(v,\varphi)^{-1}\in C^\infty(\Omega_2, R^{n-m})$ for every $\psi\in C^\infty(\Omega_1, R^{n-m})$.

But

$$\Theta'(v,\varphi) = K'(v,\varphi)\,\tilde\delta'(v,\varphi) \tag{12}$$

is C^∞-differentiable (see [8]), so that same is true about Θ.

If the bundle N_2 is not trivial, then taking the covering of N_2 by sets which locally trivialize N_2 and using the partition of unity connected with this covering we can show that Θ is a finite sum of mappings of type (11), so it is C^∞-differentiable mapping.

(c) Let $\Omega\in\mathscr{P}_m(X)$, and two system $(N_i, \Phi_i, H_i, \Psi_i, C_i, \xi_i)$, $i = 1, 2$, defining two mappings $\varkappa_1, \varkappa_2\in K_\Omega$ be given:

$$(\varkappa_2\circ\varkappa_1^{-1})'(0)u = \lim_{s\to0}\frac{1}{s}(\varkappa_2\circ\varkappa_1^{-1})(su)$$

$$= \lim_{s\to0}\frac{1}{s}F_3\circ F_2\circ F_1(su) = F_3\Big(\lim_{s\to0}\frac{1}{s}F_2(s\cdot F_1u)\Big).$$

Let $F_1u = (v,\varphi)\in\Gamma(\Omega, N_1)\times\Gamma(\partial\Omega, H_1)$.

The above written limit exists in the sense of the uniform convergence of all derivatives. So it suffices to calculate the limit at every point $p\in\Omega$. Let be first $p\in\operatorname{int}\Omega$.

Take the following point:

$$q := \beta'(sv, s\varphi)^{-1}(p)\xrightarrow[s\to0]{} p.$$

Now we can write

$$\lim_{s\to0}\frac{1}{s}F_2(sv, s\varphi)(p) = \lim_{s\to0}\frac{1}{s}\tau_2\big(\chi(sv,s\varphi)\big)^{-1}\circ I\circ\tau_{s\varphi}\big(sv(q)\big) \tag{13}$$

$$= \lim_{s\to0}\tau_2\big(\chi(sv,s\varphi)\big)^{-1}\Big[\frac{1}{s}I\Big(s\cdot\tau_{s\varphi}\big(v(q)\big)\Big)\Big].$$

But $\tau_{s\varphi}\big(v(q)\big)\xrightarrow[s\to0]{} v(p)$, so the vector tangent at p to the curve $]-\varepsilon,\varepsilon[\ni s\to s\cdot\tau_{s\varphi}\big(v(q)\big)$ is equal to $v(p)$ modulo $T_p(\Omega)$. But

$$\lim_{s\to0}\frac{1}{s}I\Big(s\cdot\tau_{s\varphi}\big(v(q)\big)\Big)$$

is the vertical (in N_2) component of the tangent vector at p to the curve $]-\varepsilon,\varepsilon[\ni s\to I\Big(s\cdot\tau_{s\varphi}\big(v(q)\big)\Big)$, so it is equal to $v(p)$ modulo $T_p(\Omega)$ (we recall that $I'(p) = 1$ — the identity in the space $T_p(X)$).

Because of the convergence $\tau_2\big(\chi(sv,s\varphi)\big)^{-1}\xrightarrow[s\to0]{} 1$ (uniformly on compact sets), we get that the right-hand side of (13) is equal to $v(p)$ modulo $T_p(\Omega)$. This means that for $p\in\operatorname{int}\Omega$

$$[(\varkappa_2\circ\varkappa_1^{-1})'(0)u](p) = u(p).$$

For $p\in\partial\Omega$ let us notice that the mapping

$$H\times(N\,|\,\partial\Omega)\ni v(p)+\varphi(p)\to\Delta\big(v(p)+\varphi(p)\big) = \Phi^{-1}\circ\tau_\varphi\big(v(p)\big)\in X$$

(here $N|\partial\Omega$ denotes the restriction of the bundle) satisfies $\Delta'(p) = 1$ — identity in $T_p(X)$ (of course $T_p(H\times N|\partial\Omega) = T_p(X)$). But $F_2(sv, s\varphi)(p) = \Delta_2^{-1}\cdot\Delta_1\big(sv(q)+s\varphi(q)\big)$, where $q := \beta(sv, s\varphi)^{-1}(p) \xrightarrow[s\to 0]{} p$. So the later proof goes as for $p \in \operatorname{int}\Omega$. We get

$$\lim_{s\to 0}\frac{1}{s}F_2(sv, s\varphi)(p) = v(p)+\varphi(p) \text{ modulo } T_p(\partial\Omega)$$

and so, for $p\in\partial\Omega$, $[(\varkappa_2\circ\varkappa_1^{-1})'(0)u](p) = u(p)$.

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Reçu par la Rédaction le 5. 6. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

On the class $L\log L$, martingales, and singular integrals*

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In a recent paper, Stein [9] has characterized the class of functions $f(x)$ such that

$$\int |f|\log^+|f|\,dx < +\infty,$$

the class $L\log L$, in terms of the Hardy-Littlewood maximal function as follows:

(S) *The Hardy-Littlewood maximal function Mf is integrable if and only if f belongs to $L\log L$.*

On the other hand, Burkholder [1] has characterized $L\log L$ as follows:

(B) *Let $f_1, f_2, \ldots$ be a sequence of stochastically independent, identically distributed random variables; let $A_n = \big(\sum_{k=1}^{n} f_k\big)/n$. Then $A^* = \sup_n |A_n|$ is integrable if and only if f_1 belongs to $L\log L$.*

Stein proves Theorem (S) by first obtaining the converse to a well-known inequality, due to Calderón and Zygmund, for the distribution function of Mf. The final result is then obtained by integrating both sides of this converse inequality. Burkholder's method is entirely different. He derives the final result without benefit of a converse inequality.

In the first section of this paper, we show that both theorems may be viewed as facts about special martingales. The converse inequality for Burkholder's problem is stated as Theorem 1. Theorem 2 extends Stein's converse inequality to a class of martingales, in which his result is a special case. While the martingale approach reveals that (S) and (B) are essentially the same theorem, there are differences. Theorem (S), in the martingale setting, holds for nonnegative functions only. (This fact is obscured in Stein's paper because the Hardy-Littlewood maximal function is always non-negative). Theorem (B), however, is stronger in the sense that the function f_1 is not assumed to be bounded below.

* This research was supported in part by NSF Grant GP 8056.