as seen from Theorem 3.4 E_{10} of [4], where $A(G)$ is Eymard's algebra of Fourier transforms.

We also remark that presumably the results of Gaudry in [7] can be interpreted as the identification

$$\mathrm{Hom}_{C_c(G)}\big(C_c(G), (C_c(G))^*\big) \cong \big(C_c(G)\otimes_{C_c(G)} C_c(G)\big)^*$$

(where tensor products must now be defined for modules which are locally convex topological vector spaces) together with a concrete representation of $C_c(G)\otimes_{C_c(G)} C_c(G)$ as a function space analogous to the representations given in Theorems 3.3 and 5.5.

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Reçu par la Rédaction le 30. 4. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

On invariant measures for expanding differentiable mappings

by

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This note concerns expanding differentiable mappings first studied by M. Shub, see [5] and [6]. These mappings are closely connected with Anosov diffeomorphisms. But while it is not known whether there always exists a finite Lebesgue measure invariant with respect to an Anosov diffeomorphism (see [1] and [6]), it turns out that such a measure always exists for any expanding differentiable mapping. The purpose of this note is to prove this fact. It seems that this may be of some interest and that is why we publish the proof although the arguments used in it have some points of similarity with the proof of Theorem 1 in [3], p. 483.

The authors are very much indebted to Professor J. G. Sinai for his valuable remarks concerning this paper.

In the sequel M will always denote a compact, connected differentiable manifold of class C^∞ unless stated otherwise. If φ is a map of class C^1 of M into itself, then $d\varphi$ will denote the derivative of φ which is the map of the tangent bundle $T(M)$ into itself. We shall say that φ *is expanding if* there exist a Riemannian metric $\|\cdot\|$ on M, a positive real number a and a real number c greater than 1 and such that

$$\|(d\varphi^n)(\mathfrak{a})\| \geqslant ac^n\|\mathfrak{a}\| \tag{1}$$

for each $\mathfrak{a}\in T(M)$ and $n = 1, 2, \ldots$

EXAMPLE. Let φ be a differentiable mapping of the 2-dimensional torus into itself given by the formula

$$\varphi(x, y) = \big(mx+ny+\varepsilon\cdot f(x, y),\ px+qy+\varepsilon g(x, y)\big)(\mathrm{mod}\,1), \qquad 0\leqslant x<1,\ 0\leqslant y<1,$$

where

(i) m, n, p, q are integers;

(ii) the eigenvalues of the matrix $\begin{pmatrix} m, & n\\ p, & q\end{pmatrix}$ are real and their moduli are greater than 1;

(iii) f and g are the real functions of class C^1 on R^2, periodic with period 1 with respect to each variable;

(iv) ε is a real positive number.

It is easy to see that if ε is sufficiently small, then φ is expanding. For more general examples of expanding mappings see [5] and [6].

Let μ be a Borel measure on M. We shall say that μ is the *Lebesgue measure* if μ is equivalent to the Riemannian measure on M induced by a certain Riemannian metric on M. Now we may state the following

THEOREM. *If φ is an expanding map of class C^2 of M into itself, then*

(a) *there exists a normalised Lebesgue measure μ on M invariant with respect to φ;*

(b) *the dynamical system (μ, φ) is exact* [1] *and therefore ergodic;*

(c) *if $\bar{\mu}$ is any normalised Borel measure on M absolutely continuous with respect to a certain Riemannian measure and invariant with respect to φ, then $\bar{\mu} = \mu$.*

For the proof of the theorem the following lemmas will be needed.

LEMMA 1. *If φ is an expanding map of M into itself, there exist a Riemannian metric $\|\cdot\|_1$ of class C^∞ on M and a real number c_1 greater than 1 such that*

$$\|(d\varphi)(\mathfrak{a})\|_1 \geqslant c_1 \|\mathfrak{a}\|_1 \tag{2}$$

for each $\mathfrak{a} \in T(M)$.

Proof. First we shall prove that there exsits a Riemannian metric $\|\cdot\|_1$, not necessarily of class C^∞, such that (2) is satisfied. For this purpose let k be an integer such that $ac^k > 1$ and $k > 1$, where a and c are from (1). Then we may define the Riemannian metric $\|\cdot\|_1$ on M as follows:

$$\|\mathfrak{a}\|_1^2 = \|\mathfrak{a}\|^2 + \ldots + \|(d\varphi^{k-1})(\mathfrak{a})\|^2, \tag{3}$$

where $\mathfrak{a} \in T(M)$ and where $\|\cdot\|$ is from (1). Since M is compact, there exists a finite real number A such that

$$\|(d\varphi)(\mathfrak{a})\|^2 + \ldots + \|(d\varphi^{k-1})(\mathfrak{a})\|^2 \leqslant A \|\mathfrak{a}\|^2 \tag{4}$$

for $\mathfrak{a} \in T(M)$. Now let c_1 be any real number such that $1 < c_1 < ac^k$ and $a^2c^{2k} - c_1^2 \geqslant A(c_1^2 - 1)$. Then it is easy to see that (2) is satisfied. Now we shall show that $\|\cdot\|_1$ may be chosen to be of class C^∞. For this purpose let us assume that (2) is satisfied for the Riemannian metric $\|\cdot\|_1$. If we prove that for any positive number ε there exists a Riemannian metric $\|\cdot\|_2$ of class C^∞ on M such that

$$\big|\|\mathfrak{a}\|_2^2 - \|\mathfrak{a}\|_1^2\big| \leqslant \varepsilon \|\mathfrak{a}\|_1^2 \tag{5}$$

for each $\mathfrak{a} \in T(M)$, the proof of the lemma will be completed. In fact, if we choose ε such that $1+\varepsilon < c_1^2(1-\varepsilon)$, then $\|\cdot\|_2$ will satisfy (2) with the constant $c_1\sqrt{\dfrac{1-\varepsilon}{1+\varepsilon}}$.

[1] For the definition of exact dynamical systems see [4].

If we apply the standard method based on the C^∞-partition of unity, the proof of (5) may be reduced to the proving of the following:

(6) for any $\varepsilon > 0$ and any point $p \in M$ there exists a Riemannian metric $\|\cdot\|_{2,U}$ of class C^∞ on some neighborhood U of p such that

$$\big|\|\mathfrak{a}\|_1^2 - \|\mathfrak{a}\|_{2,U}^2\big| \leqslant \varepsilon \|\mathfrak{a}\|_1^2 \quad \text{for } \mathfrak{a} \in T(M)/U.$$

Condition (6) may easily be proved by a uniform approximation of the coordinates of $\|\cdot\|_1$ in some coordinate system on M containing p by real functions of class C^∞. Therefore the proof of (6) will be omitted. Thus the proof of the lemma is completed.

From now on, φ will denote, unless stated otherwise, an expanding map of class C^2, $\|\cdot\|_1$ will denote the Riemannian metric on M given by Lemma 1, and $d_1(\cdot,\cdot)$, μ_1 will denote the natural metric and the Riemannian measure induced by $\|\cdot\|_1$ on M respectively.

The following two lemmas may be proved in the standard way and therefore we shall omit the proofs:

LEMMA 2. *Each expanding map is an N-fold covering, where $1 < N < \infty$.*

LEMMA 3. *If $f(\cdot)$ is a real function of class C^1 on M, then there exists a finite real number L such that*

$$|f(x) - f(y)| \leqslant L d_1(x, y) \quad \textit{for } x, y \in M.$$

Let φ be a map of class C^1 of a Riemannian manifold $(X, \|\cdot\|)$ of class C^∞ into a Riemannian manifold $(X_*, \|\cdot\|_*)$ of class C^∞, where $\dim X = \dim X_*$. Then we may define on X the function $D\varphi$ as

$$(D\varphi)(x) = \mu_*^{\varphi(x)}\big((d\varphi)(A_x)\big)$$

for $x \in X$, where A_x is any Borel set in $T_x(X)$ of measure μ^x equal to 1 and μ^x, μ_*^y are the natural measures induced by the Riemannian metrics $\|\cdot\|$ and $\|\cdot\|_*$ on $T_x(X)$ and $T_y(X_*)$, respectively. This function, as is easy to see, is well-defined and will be termed the *scalar derivative* of φ. If φ is of class C^2, then $D\varphi$ is of class C^1. Further, if φ is a diffeomorphism X onto X_*, then

$$\mu_*\big(\varphi(A)\big) = \int_A (D\varphi)(x)\, d\mu(x)$$

for each Borel set $A \subset X$, where μ, μ_* are the Riemannian measures induced by $\|\cdot\|$, $\|\cdot\|_*$ on X, X_*, respectively.

Now we may prove the following

LEMMA 4. *There exists a real finite number a such that*

$$\mu_1\big(\varphi^{-n}(A)\big) \leqslant a\mu_1(A). \tag{7}$$

for each Borel set $A \subset M$ and $n = 1, 2, \ldots$

Proof. In view of Lemma 2 and the well-known property of the Riemannian metric of class C^∞, there exists an open cover $\{U_i\}_{1\leqslant i\leqslant p}$ of M such that

(8) φ is a diffeomorphism of each component of $\varphi^{-1}(U_i)$ onto U_i for $i=1,2,\ldots,p$;

(9) for each pair of points x, y belonging to U_i, there exists a regular curve joining x and y, contained in U_i and such that $d_1(x,y)$ is equal to its length, $i=1,\ldots,p$.

Let δ be the Lebesgue number of the cover $\{U_i\}_{1\leqslant i\leqslant p}$; then there exist a positive integer k and open sets $A_{i_0,\ldots,i_n}(1\leqslant i_0\leqslant k,\ 1\leqslant i_1\leqslant N, \ldots, 1\leqslant i_n\leqslant N)$ for $n=0,1,\ldots$, where N is from Lemma 2, such that

(10) $\mu_1\big(M-\bigcup_{i_0=1}^{k}A_{i_0}\big)=0$ and $A_{i_0}\neq\emptyset$ for $i_0=1,\ldots,k$

(11) $A_{i_0,\ldots,i_n}\cap A_{i'_0,\ldots,i'_n}=\emptyset$ for each pair $(i_0,\ldots,i_n)$, $(i'_0,\ldots,i'_n)$ of different admissible $(n+1)$-tuples of indices for $n=0,1,\ldots$;

(12) $\varphi^{-1}(A_{i_0,\ldots,i_n})=\bigcup_{i_{n+1}=1}^{N}A_{i_0,\ldots,i_n,i_{n+1}}$;

(13) φ is a diffeomorphism of $A_{i_0,\ldots,i_{n+1}}$ onto $A_{i_0,\ldots,i_n}$;

(14) $\operatorname{diam}(A_{i_0,\ldots,i_n})<\delta/c_1^n$;

(15) $d_1(\varphi(x),\varphi(y))\geqslant c_1 d_1(x,y)$ for each pair of points x, y belonging to $A_{i_0,\ldots,i_{n+1}}$ where $n=0,1,\ldots$

The above sets will be defined by induction. First we shall define A_{i_0} for $i_0=1,2,\ldots,k$. For this purpose let us remark that there exists a cover $\{B_i\}_{1\leqslant i\leqslant k}$ of M such that B_i are open balls of radii not greater than $\delta/2$ and such that $\mu_1(\operatorname{Fr}(B_i))=0$ for $i=1,2,\ldots,k$. Then the open sets defined as

$$A_1=B_1 \text{ and } A_s=B_s-\bigcup_{j=1}^{s-1}\bar{B}_j \quad \text{for } s=2,3,\ldots,k$$

have the required properties if one rejects empty sets. Let us now assume that the sets $A_{i_0,\ldots,i_j}$ have been defined for $1\leqslant i_0\leqslant k$, $1\leqslant i_1\leqslant N,\ldots,1\leqslant i_j\leqslant N$, where $j=0,1,\ldots,n$. If $(i_0,\ldots,i_n)$ is any admissible $(n+1)$-tuple of indices, then in view of (14) there exists an $r_{i_0,\ldots,i_n}$ $(1\leqslant r_{i_0,\ldots,i_n}\leqslant p)$ such that $A_{i_0,\ldots,i_n}\subset U_{r_{i_0,\ldots,i_n}}$. On account of (8) $\varphi^{-1}(A_{i_0,\ldots,i_n})$ is equal to $\bigcup_{i_{n+1}=1}^{N}A_{i_0,\ldots,i_n,i_{n+1}}$ where $A_{i_0,\ldots,i_n,i_{n+1}}$ $(i_{n+1}=1,\ldots,N)$ are the intersections of $\varphi^{-1}(A_{i_0,\ldots,i_n})$ with the components of $\varphi^{-1}(U_{r_{i_0,\ldots,i_n}})$. It is easy to see that (11) and (13) are satisfied if one replaces n by $n+1$, and in view of (9) one obtains (15) and therefore (14)

for $A_{i_0,\ldots,i_n,i_{n+1}}(i_{n+1}=1,\ldots,N)$. Now we shall prove that there exists a finite real number β such that

$$\frac{(D\varphi^n)(y)}{(D\varphi^n)(x)}\leqslant\beta \tag{16}$$

for $x,y\in A_{i_0,\ldots,i_n}$, $n=1,2,\ldots$ For this purpose let us assume that $x,y\in A_{i_0,\ldots,i_n}$, where n is any positive integer. Then, in view of the chain rule for scalar derivatives, one obtains

$$\begin{aligned}\frac{(D\varphi^n)(y)}{(D\varphi^n)(x)} &= \prod_{i=0}^{n-1}\frac{(D\varphi)(\varphi^i(y))}{(D\varphi)(\varphi^i(x))}\\ &\leqslant \prod_{i=0}^{n-1}\Big(1+\frac{|(D\varphi)(\varphi^i(x))-(D\varphi)(\varphi^i(y))|}{(D\varphi)(\varphi^i(x))}\Big)\\ &\leqslant \exp\Big\{\sum_{i=0}^{n-1}\frac{|(D\varphi)(\varphi^i(x))-(D\varphi)(\varphi^i(y))|}{(D\varphi)(\varphi^i(x))}\Big\}.\end{aligned} \tag{17}$$

On account of Lemma 3 inequality (17) implies that

$$\frac{(D\varphi^n)(y)}{(D\varphi^n)(x)}\leqslant\exp\Big\{\frac{L}{\gamma}\sum_{i=0}^{n-1}d_1(\varphi^i(x),\varphi^i(y))\Big\},$$

where $\gamma=\inf_{x\in M}(D\varphi)(x)$, and L is the constant given by Lemma 3. From (15) it follows that

$$d_1(\varphi^{i+1}(x),\varphi^{i+1}(y))\geqslant c_1 d_1(\varphi^i(x),\varphi^i(y))$$

for $i=0,1,\ldots,n-1$. This implies that

$$d_1(\varphi^{n-1}(x),\varphi^{n-1}(y))\geqslant c_1^{n-1-i}d_1(\varphi^i(x),\varphi^i(y)) \tag{18}$$

for $i=0,1,\ldots,n-1$. From (18) it follows that (16) is satisfied with the constant β equal to

$$\exp\Big\{\frac{L}{\gamma}\operatorname{diam}(M)\frac{c_1}{c_1-1}\Big\}.$$

Now we may prove (7). For this purpose let us remark that in view of (11), (12) and (13) it follows that

$$\mu_1(\varphi^{-n}(A))=\sum_{\substack{1\leqslant i_1\leqslant N\\ \cdots\cdots\\ 1\leqslant i_n\leqslant N}}\int_A(D\varphi^{-n}_{i_0,\ldots,i_n})(y)\,d\mu_1(y), \tag{19}$$

where A is any Borel set contained in A_{i_0} $(1\leqslant i_0\leqslant k)$, and $D\varphi^{-n}_{i_0,\ldots,i_n}$ denotes the scalar derivative of the inverse map $\varphi^{-n}_{i_0,\ldots,i_n}$ to $\varphi^n/A_{i_0,\ldots,i_n}$.

But (16) implies

$$(20)\qquad \frac{\sup\limits_{x\in A_{i_0}} (D\varphi^{-n}_{i_0,\ldots,i_n})(x)}{\inf\limits_{x\in A_{i_0}} (D\varphi^{-n}_{i_0,\ldots,i_n})(y)} \leqslant \beta .$$

From (19) and (20) one obtains

$$\mu_1(\varphi^{-n}(A)) \leqslant \beta\cdot\mu_1(A) \sum_{\substack{1\leqslant i_1\leqslant N\\ \cdots\cdots\\ 1\leqslant i_n\leqslant N}} \inf_{x\in A_{i_0}} (D\varphi^{-n}_{i_0,\ldots,i_n})(x).$$

Since, in view of (24),

$$\mu_1(\varphi^{-n}(A_{i_0})) \geqslant \mu_1(A_{i_0}) \sum_{\substack{1\leqslant i_1\leqslant N\\ \cdots\cdots\\ 1\leqslant i_n\leqslant N}} \inf_{x\in A_{i_0}} (D\varphi^{-n}_{i_0,\ldots,i_n})(x),$$

taking into account (10) we find that (8) is satisfied with a constant a equal to $\beta\mu_1(M)/\min\limits_{1\leqslant i_0\leqslant K}\mu_1(A_{i_0})$. Thus the lemma is completely proved.

LEMMA 5. *For each open non-empty set $U\subset M$ there exists a positive integer n_0 such that $\varphi^{n_0}(U) = M$.*

Proof. Let (M_*, π) denote the universal covering of M. Since π is a regular map of class C^∞, the Riemannian metric $\|\cdot\|_*$, defined on M_* as $\|\mathfrak{a}\|_* = \|(d\pi)(\mathfrak{a})\|_1$ for $\mathfrak{a}\in T(M_*)$ is of class C^∞. Let Γ denote the group of cover transformations of the covering (M_*, π). It is well known that Γ is the group of isometries of the Riemannian manifold $(M_*, \|\cdot\|_*)$. Now, since M is compact, there exsists an open set $Z\subset M_*$ such that

$$(21)\qquad \pi(Z) = M \quad\text{and}\quad \operatorname{diam}(Z) = \delta_* < \infty$$

(the diameter of Z is in the metric $d_*(\cdot,\cdot)$ induced on M_* by $\|\cdot\|_*$). Now let us remark that

$$(22)\qquad \pi(K_*(x_0, \delta_*)) = M,$$

where $K_*(x_0, \delta_*)$ is any closed ball of radius equal to δ_*. Indeed, in view of (21) there exists an $\bar{x}_0\in Z$ such that $\pi(x_0) = \pi(\bar{x}_0)$. Further, since Γ is transitive on each fibre of the covering (M_*, π), there exists a $g\in\Gamma$ such that $x_0 = g(\bar{x}_0)$. This implies that $g(K_*(\bar{x}_0, \delta_*)) = K_*(x_0, \delta_*)$ and in view of (21) we infer that (22) is satisfied.

Since (M_*, π) is universal, there exists a continuous map $\varphi_*\colon M_*\to M_*$ such that

$$(23)\qquad \varphi\pi = \pi\varphi_* .$$

It is easy to see that (M_*, φ_*) is the covering of M_* and φ_* is a regular map of class C^2. Since M_* is simply connected, it follows that φ_* is a homeomorphism, and therefore a diffeomorphism of class C^2 of M_* onto itself. In view of the definition of $\|\cdot\|_*$, we obtain

$$(24)\qquad \|(d\varphi_*)(\mathfrak{a})\|_* \geqslant c_1\|\mathfrak{a}\|_*$$

for $\mathfrak{a}\in T(M_*)$. Since φ_* is a diffeomorphism, (24) implies

$$(25)\qquad d_*(\varphi_*(x), \varphi_*(y)) \geqslant c_1 d_*(x, y) \quad \text{for } x, y\in M_*.$$

Now let us assume that U is any open non-empty set, $U\subset M$. There exists a closed ball $K_*(x_0, r)$ in M_* such that $\pi(K_*(x_0, r))\subset U$, $K_*(x_0, r)\subset Z$. Now it suffices to show that

$$(26)\qquad \varphi^{n_0}\big(\pi(K_*(x_0, r))\big) = M$$

for a certain positive integer n_0. In view of (23), (26) is equivalent to

$$(27)\qquad \pi\big(\varphi_*^{n_0}(K_*(x_0, r))\big) = M.$$

To prove (27), let n_0 be such a positive integer that

$$(28)\qquad c_1^{n_0} r > \delta_* .$$

Then, on account of (22) and (27) it suffices to show that

$$(29)\qquad K_*(\varphi_*^{n_0}(x_0), \delta_*) \subset \varphi_*^{n_0}(K_*(x_0, r)).$$

To prove (29) let us assume that, on the contrary, there exists a $y\in K_*(\varphi_*^{n_0}(x_0), \delta_*) - \varphi_*^{n_0}(K_*(x_0, r))$. Then there exists a regular curve $k\colon\langle 0, 1\rangle\to M_*$ such that $k(0) = \varphi_*^{n_0}(x_0)$, $k(1) = y$ and $L_*(k|_0^1) < \delta_*+\varepsilon$, where $L_*(k|_0^1)$ denote the length of k and ε is such that $c_1^{n_0}r\geqslant\delta_*+\varepsilon$, $\varepsilon > 0$. Further, it is easy to see that there exists a t_0, $0 < t_0 < 1$, such that $k(t_0)\in\operatorname{Fr}\big(\varphi_*^{n_0}(K_*(x_0, r))\big)$. This implies that $L_*(k|_0^{t_0}) < \delta_*+\varepsilon$ and therefore $d_*(\varphi_*^{n_0}(x_0), k(t_0)) < \delta_*+\varepsilon$, but in view of formula (25) we have $d_*(\varphi_*^{n_0}(x_0), k(t_0)) \geqslant \delta_*+\varepsilon$. Thus the proof of the lemma is completed.

The following lemma is, in fact, the theorem on p. 525 in [4], suitably modified for our purposes. Therefore its proof will be omitted.

LEMMA 6. *Let φ be an endomorphism of a Lebesque space (M, μ) with the measure vanishing on points. If*

(i) *there exists a family $\mathscr{A}$ of measurable sets of positive measure such that finite sums of disjoint sets belonging to $\mathscr{A}$ are dense in the space of all measurable sets;*

(ii) *there exist real finite numbers L_1, L_2 and for each $A\in\mathscr{A}$ there exist positive integers $n_1, n_2, n_1\leqslant n_2$, such that*

(ii$_a$) $\mu(\varphi^{n_1}(Z)) \leqslant L_1\dfrac{\mu(Z)}{\mu(A)}$ *for each measurable set $Z\subset A$,*

(ii$_b$) $\mu(\varphi^{n_2}(A)) = 1,$

(ii$_c$) $\mu(\varphi^{n_2-n_1}(Z)) \leqslant L_2\mu(Z)$ *for each measurable set* $Z \subset \varphi^{n_1}(A)$; *then* φ *is exact.*

Proof of the theorem. We may assume that the measure μ_1 is normalised. From Lemma 4 it follows that

$$\frac{\mu_1(A)+\mu_1(\varphi^{-1}(A))+\ldots+\mu_1(\varphi^{-n+1}(A))}{n} \leqslant a\cdot\mu_1(A) \tag{30}$$

for each Borel set A and $n = 1, 2, \ldots$ From (30), in view of Theorem 9 in [2] on p. 667, it easily follows that the sequence

$$\frac{1}{n}\sum_{i=1}^{n}\mu_1(\varphi^{-i+1}(A))$$

is convergent (see the proof of Theorem 1 on p. 483 in [3]). Let us put

$$\mu(A) = \lim_{n\to\infty}\frac{\mu_1(A)+\ldots+\mu_1(\varphi^{-n+1}(A))}{n}.$$

From Corollary 4 in [2] on p. 160 it follows that μ is a normalised Borel measure on M. It is evident that μ is absolutely continuous with respect to μ_1. It remains to show that μ_1 is absolutely continuous with respect to μ. For this purpose we shall prove that

(31) if $\mu_1(\varphi^{-n}(A_{i_0})) \nrightarrow 0$ for some A_{i_0}, then the condition $\mu(A) = 0$ implies $\mu_1(A) = 0$, where A is a Borel set contained in A_{i_0}.

For this purpose let us assume that $\mu_1(\varphi^{-n}(A_{i_0})) \nrightarrow 0$. Then, in view of Lemma 4, it follows that

$$\inf_n \mu_1(\varphi^{-n}(A_{i_0})) = a_{i_0} > 0. \tag{32}$$

Further, keeping in mind the notation from the proof of Lemma 4, one obtains

$$\mu_1(\varphi^{-n}(A)) \geqslant \mu_1(A)\sum_{\substack{1\leqslant i_1\leqslant N\\ \cdots\cdots\\ 1\leqslant i_n\leqslant N}}\inf_{x\in A_{i_0}}(D\varphi^{-n}_{i_0,\ldots,i_n})(x) \tag{33}$$
$$\geqslant \frac{1}{\beta}\mu_1(A)\sum_{\substack{1\leqslant i_1\leqslant N\\ \cdots\cdots\\ 1\leqslant i_n\leqslant N}}\sup_{x\in A_{i_0}}(D\varphi^{-n}_{i_0,\ldots,i_n})(x).$$

From (32) it follows that

$$\sum_{\substack{1\leqslant i_1\leqslant N\\ \cdots\cdots\\ 1\leqslant i_n\leqslant N}}\sup_{x\in A_{i_0}}(D\varphi^{-n}_{i_0,\ldots,i_n})(x) \geqslant \frac{a_{i_0}}{\mu_1(A_{i_0})}\mu_1(A). \tag{34}$$

From (33) and (34) we find that

$$\mu_1(\varphi^{-n}(A)) \geqslant \frac{1}{\beta}\frac{a_{i_0}}{\mu_1(A_{i_0})}\mu_1(A)$$

for $n = 1, 2, \ldots$ It is easy to see that this completes the proof of (31).

Part (a) of the theorem will be completely proved if we show that (31) is satisfied for each i_0, $1 \leqslant i_0 \leqslant k$. Suppose, on the contrary, that there exists an i_0', $1 \leqslant i_0' \leqslant k$, such that (31) is not satisfied for $A_{i_0'}$. Then, since

$$\mu_1(M) = \sum_{i_0=1}^{k}\mu_1(\varphi^{-n}(A_{i_0})),$$

there exists an i_0'' such that (31) is satisfied for $A_{i_0''}$. Then in view of Lemma 5, there exists a positive integer n_0 such that $\varphi^{n_0}(A_{i_0''}) \supset A_{i_0''}$. Let us put $B = \varphi^{-n_0}(A_{i_0'}) \cap A_{i_0''}$. The set B is open and non-empty; therefore in view of (31)

$$\mu_1(\varphi^{-n}(B)) \nrightarrow 0.$$

But $\varphi^{-n}(B) \subset \varphi^{-n-n_0}(A_{i_0'})$ for $n = 1, 2, \ldots$ This implies that $\mu_1(\varphi^{-n}(B)) \to 0$; thus we obtain a contradiction. This completes the proof of part (a) of the theorem.

Now we shall proceed to part (b). For this purpose it suffices to show that our dynamical system satisfies the hypothesis of Lemma 6 [2]. To do so let $\mathscr{A}$ be the family of all sets $A_{i_0,\ldots,i_n}$ $(n \geqslant 1)$, from the proof of Lemma 4. Then, to prove that (i) is satisfied, it suffices to show that for each $\varepsilon > 0$ and each open set G there exists a set G_ε, $G_\varepsilon \subset G$, equal to the finite sum of disjoint set belonging to $\mathscr{A}$ such that

$$\mu(G - G_\varepsilon) < \varepsilon. \tag{35}$$

For this purpose let us put $G_k = \{x\colon x \in G,\ d_1(\mathrm{Fr}\,G, x) > 1/k\}$, $k = 1, 2, \ldots$ Then $G_k \subset G_{k+1}$ for $k = 1, 2, \ldots$ and $\bigcup_{k=1}^{\infty} G_k = G$. Therefore there exists a k_0 such that $\mu(G - G_{k_0}) < \varepsilon$. In view of (14) there exists an n_0 such that

$$d_1(A_{i_0,\ldots,i_n}) < \frac{1}{k_0} \tag{36}$$

for each admissible (n_0+1)-tuple of indices. Now let us put $G_\varepsilon = \bigcup A_{i_0,\ldots,i_n}$, where the sum is over such admissible (n_0+1)-tuples of indices $(i_0, \ldots, i_n)$ that $G_{k_0} \cap A_{i_0,\ldots,i_{n_0}} \neq \emptyset$. Since

$$\mu(M - \bigcup_{(i_0,\ldots,i_{n_0})} A_{i_0,\ldots,i_{n_0}}) = 1$$

[2] In fact, to apply Lemma 6 one has to complete the measure.

and in view of (36), it is easy to see that (35) is satisfied. Now we shall prove that (ii) is also satisfied. To do so, let us remark that from the proof of part (a) it follows that there exist two real finite numbers a, $\bar{a}$ such that $\mu(A) \leqslant a\mu_1(A)$ and $\mu_1(A) \leqslant \bar{a}\mu(A)$ for each Borel set A. Therefore it suffices to show that (ii) is satisfied when we replace the measure μ by μ_1. For this purpose let us put $L_1 = \beta$, where β is from (16). Further, in view of Lemma 5, it follows that there exists a positive integer s_0 such that $\varphi^{s_0}(A_{i_0}) = M$ for $i_0 = 1, 2, \ldots, k$, where A_{i_0} is from the proof of Lemma 4. Since φ^{s_0} is a local diffeomorphism and in view of the remarks preceding Lemma 4, it easily follows that there exists a finite real number L_2 such that $\mu_1(\varphi^{s_0}(Z)) \leqslant L_2\mu_1(Z)$ for each Borel set Z. Now let us put for each set $A_{i_0,\ldots,i_n} (n \geqslant 1)$ $n_1 = n$ and $n_2 = n+s_0$. Then (ii_b) and (ii_c) are satisfied; to prove (ii_a) it suffices to apply (21) and act as in the proofs of Lemma 4 and (31) (see the proof of the theorem on p. 525 in [4]). Thus the proof of part (b) is completed.

It remains to prove (c). In fact, in view of (a) and (b), the part (c) follows from the well-known theorem which states that any invariant normalized measure absolutely continuous with respect to a normalized invariant ergodic measure is equal to this measure.

Added in proof. After the paper was submitted for publication, the paper [7] came to our attention. As we understand, there is announced the following result: for each expanding mapping of the compact manifold M into itself there exists an invariant regular Borel measure μ positive on each open set in M.

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Reçu par la Rédaction le 28. 5. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

Existence of differentiable structure in the set of submanifolds

An attempt of geometrization of calculus of variations

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1. Introduction. Vigorous development of calculus of variations we witness recently shows only too clearly the effectiveness of the functional analysis approach towards this branch of mathematics. Thanks to the systematic studies started by Eells and his school (see e.g. [4] or [5]) we know that many important families of mappings, encountered in the calculus of variations, can be naturally equipped with the structure of an infinitely dimensional differentiable Banach manifold. E.g. the set $C^r(K, X)$ of r times continuously differentiable mappings of a compact manifold K into a finite-dimensional manifold X, is a manifold modelled on some vector space $C^r(K, Y)$.

Such approach enabled Palais and Smale (see [11], [12] or [13]) to found the unified general Morse theory, the method of steepest descent (cf. also [2]) and to prove many beautiful global theorems about functionals of the calculus of variations. The excellent report by Eells [6], containing also wide bibliography on the subject shows how considerable is the bulk of work done in this field.

As it appears, however, for many purposes like

a) classic field theory

b) differential geometry

the Banach manifolds fail to suffice. E.g. let us consider the classic electrodynamics. In the relativistic Lagrange formulation of the theory the essential role is played by states of the field in bounded domains of the space-time, limited within the dimensions of the laboratory and the duration of an experiment. Such a state is therefore a section over compact set of the bundle $\overset{2}{\wedge} T^*(R^4)$, where R^4 is the space-time. (As we know, the electromagnetic field is an exterior differential 2-form). The range of such a section is a compact 4-dimensional submanifold with boundary of $\overset{2}{\wedge} T^*(R^4)$. Among all such submanifolds with common boundary only the one which is the critical point of action is of interest for physicists.