Proof. As previously observed, the inverse of the operator $T\colon A \to M_x$ defined by $T(g) = gf$ is bounded. Let $K = \|T^{-1}\|$. For $g \in A$ with $\|g\| = 1$ there is an element $g_1 \in A$ such that

$$g - \hat{g}(x)e = g_1 f,$$

and it follows that $\|g_1\| \leqslant 2K$, and hence

$$\|y - x\| \leqslant 2K|\hat{f}(y)|.$$

If x is isolated in the norm topology, there is an $r > 0$ such that $\|y-x\| < r$ implies $y = x$. Thus if $|\hat{f}(y)| < r/2K$, then $\|y-x\| < r$ and $y = x$. But this implies that x is isolated in the Gelfand topology of X.

Finally, we mention an open question. It was proved for the sup norm algebra case that Af being a maximal ideal M_x in the Šilov boundary implies that x is isolated in the Gelfand topology. Does this result hold true for an arbitrary commutative semisimple Banach algebra with identity? The technique for obtaining power series mentioned in the remark after Theorem 2 applies, and thus a negative answer would allow for power series representations on the Šilov boundary.

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A note on quasi-analytic vectors *

by

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In [3] we introduced the notion of a quasi-analytic vector. Let S be a symmetric operator in a Hilbert space $\mathscr{H}$ and x an element in $\bigcap_{n\geqslant 1} \mathscr{D}(S^n)$ ($\mathscr{D}(A)$ denotes the domain of an operator A acting in $\mathscr{H}$); then x is said to be a *quasi-analytic vector* for S if

$$\sum_{n=1}^{\infty} \|S^n x\|^{-1/n} = \infty$$

(this condition is equivalent to $\sum_{n=1}^{\infty} \|S^n x\|/\|S^{n+1}x\| = \infty$).

We proved in [3] that a closed symmetric operator S is self-adjoint if and only if it has a total set of quasi-analytic vectors. The purpose of this note is to prove a slightly stronger theorem for the case S is semi-bounded and then derive the original theorem as a corollary. The idea of the proof is essentially the same as in [3], but it seems not possible to derive Theorem 1 from our previous result. The reason is, as we shall see, that if a Stieltjes moment sequence is determined, the corresponding Hamburger moment sequence need not be determined.

THEOREM 1. *Let S be a semi-bounded, closed, symmetric operator in a Hilbert space $\mathscr{H}$. Then S is self-adjoint if and only if there exists a total set of vectors $x \in \bigcap_{n\geqslant 1} \mathscr{D}(S^n)$ such that*

$$\sum_{n=1}^{\infty} \|S^n x\|^{-1/2n} = \infty.$$

Proof. The necessity of the condition is a trivial consequence of the spectral theorem. In fact, if S is self-adjoint and $\{E(\sigma)\}$ its canonical spectral measure, let $E_c = E([-c, c])$ for $c > 0$. If $x \in \mathscr{R}(E_c)$ (range of E_c), then

$$\|S^n x\|^2 = \int_{-c}^{c} \lambda^{2n} d\|E(\lambda)x\|^2 \leqslant c^{2n}\|x\|^2,$$

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and therefore $\sum_{n=1}^{\infty}\|S^n x\|^{-1/2n}=\infty$. But the set of such vectors is dense in $\mathscr{H}$, since $E_c\to I$ strongly as $c\to\infty$.

We must prove that the condition is sufficient. Since $\|S^n x\| = \|(-S)^n x\|$ for all n and S is self-adjoint if and only if $-S$ is self-adjoint, we may assume that $(Sx|x)\geqslant c\|x\|^2$ for all $x\in\mathscr{D}(S)$.

Furthermore, we may assume that $c>0$, for otherwise replace S by $S+aI$, where $a>-c$, and observe that S is self-adjoint if and only if $S+aI$ (a real) is self-adjoint and

$$\sum_{n=1}^{\infty}\|(S+aI)^n x\|^{-1/2n}=\infty \Leftrightarrow \sum_{n=1}\|S^n x\|^{-1/2n}=\infty \quad \text{for any } a.$$

To see that this equivalence holds, it is sufficient to show that for any a (real or complex) and $x\in\bigcap_{n\geqslant 1}\mathscr{D}(S^n)$, $\|x\|=1$,

$$\sum_{n=1}^{\infty}\|S^n x\|^{-1/2n}=\infty \Rightarrow \sum_{n=1}^{\infty}\|(S+aI)^n x\|^{-1/2n}=\infty.$$

This follows from the fact that

$$\|(S+aI)^n x\| = \left\|\sum_{k=0}^{n}\binom{n}{k}a^{n-k}S^k x\right\| \leqslant \sum_{k=0}^{n}\binom{n}{k}|a|^{n-k}\|S^k x\|$$

$$\leqslant \sum_{k=0}^{n}\binom{n}{k}|a|^{n-k}\|S^n x\|^{k/n} = (\|S^n x\|^{1/n}+|a|)^n$$

$$\leqslant \|S^n x\|(1+|a|\,\|S^n x\|^{-1/n})^n \leqslant \|S^n x\|\left(1+\frac{|a|}{\|Sx\|}\right)^n \quad \text{for } n\geqslant 1,$$

since $\|S^k x\|^{1/k}$ is monotonically increasing with $k\geqslant 1$ (if $\|x\|=1$). This is a consequence of the inequality $\|S^k x\|^2 = (S^{k-1}x/S^{k+1}x)\leqslant \|S^{k-1}x\|\times \times\|S^{k+1}x\|$.

Suppose then that S is a closed, symmetric operator such that $(Sx|x)\geqslant c\|x\|^2$ for all $x\in\mathscr{D}(S)$, where $c>0$, and the set $\mathscr{D}$ of elements $x\in\bigcap_{n\geqslant 1}\mathscr{D}(S^n)$ such that

$$\sum_{n=1}^{\infty}\|S^n x\|^{-1/2n}=\infty$$

is total in $\mathscr{H}$.

If $x\in\bigcap_{n\geqslant 1}\mathscr{D}(S^n)$, then $\mu_n(x)=(S^n x|x)$, $n=0,1,\ldots$, is a Stieltjes moment sequence; i.e. there exists a bounded positive Radon measure μ on $[0,\infty)$ such that

$$\mu_n(x)=(S^n x|x)=\int_0^{\infty}t^n\,d\mu(t) \quad \text{for } n=0,1,\ldots,$$

because

$$\sum_{i=0}^{n}\sum_{j=0}^{n}a_i\bar{a}_j\mu_{i+j}(x)\geqslant 0 \quad \text{and} \quad \sum_{i=0}^{n}\sum_{j=0}^{n}a_i\bar{a}_j\mu_{i+j+1}(x)\geqslant 0$$

for any finite sequence of complex numbers $(a_0,a_1,\ldots,a_n)$ (cf. [5] and [6]). Now, T. Carleman has shown in [1] that a Stieltjes moment sequence (μ_n) is determined (that means that the bounded positive Radon measure μ such that

$$\mu_n=\int_0^{\infty}t^n\,d\mu(t) \quad \text{for } n=0,1,\ldots,$$

is unique) if $\sum_{n=1}^{\infty}\mu_n^{-1/2n}=\infty$. Now, $\mu_n(x)=(S^n x|x)\leqslant\|S^n x\|\,\|x\|$, and therefore

$$\sum_{n=1}^{\infty}\|S^n x\|^{-1/2n}=\infty \Rightarrow \sum_{n=1}^{\infty}\mu_n(x)^{-1/2n}=\infty.$$

(It is easily seen that the implication also goes the other way). Thus, if $x\in\mathscr{D}$, then there exists a unique bounded positive Radon measure μ_x on $[0,\infty)$ such that

$$(S^n x|x)=\int_0^{\infty}t^n\,d\mu_x(t) \quad \text{for } n=0,1,\ldots$$

From this point on, the proof is similar to the proof of Theorem 1 in [3], but since a determined Stieltjes moment sequence need not be a determined Hamburger moment sequence, there may exist for a given x in $\mathscr{D}$ more than one bounded positive Radon measure ν on the whole real line such that

$$(S^n x|x)=\int_{-\infty}^{\infty}t^n\,d\nu(t) \quad \text{for } n=0,1,\ldots$$

That means, that a vector x in $\mathscr{D}$ is not necessarely a vector of uniqueness for S in the sense of [3] and therefore Theorem 1 is not an immediate consequence of the results in [3].

Since S is positive definite, it has a positive definite self-adjoint extension, for example the Friedrichs extension. Let T be any positive definite self-adjoint extension of S and $\{E(\sigma)\}$ its canonical spectral measure. Then

$$(S^n x|x)=(T^n x|x)=\int_0^{\infty}\lambda^n\,d\|E(\lambda)x\|^2$$

for $n = 0, 1, \ldots$ and for any $x \in \bigcap_{n\geq 1} \mathcal{D}(S^n)$, and therefore $\|E(\sigma)x\|^2 = \mu_x(\sigma)$ for $x \in \mathcal{D}$ and σ any Borel set in $[0, \infty)$.

Suppose $S \neq T$, then there exists another positive definite self-adjoint extension T_1 of S which is different from T. This is true because $(Sx|x) \geqslant c\|x\|^2$ and $c > 0$, as was shown by Krein [2]. Let $\{E_1(\sigma)\}$ be the canonical spectral measure of T_1. Then as above $\|E_1(\sigma)x\|^2 = \mu_x(\sigma)$ for $x \in \mathcal{D}$ and any Borel set σ in $[0, \infty)$. Therefore $\|E_1(\sigma)x\|^2 = \|E(\sigma)x\|^2$ for all $x \in \mathcal{D}$ and Borel sets σ in $[0, \infty)$. Since $\mathcal{D}$ is total in $\mathcal{H}$, it follows that $\|E_1(\sigma)x\|^2 = \|E(\sigma)x\|^2$ for all x in $\mathcal{H}$ and all Borel sets σ in $[0, \infty)$. From this follows by the polarization identity that $E_1(\sigma) = E(\sigma)$ for all Borel sets σ in $[0, \infty)$. But since T and T_1 are positive definite, $E_1(\sigma) = E(\sigma) = 0$ for all Borel sets σ in $(-\infty, 0)$ and therefore $E_1 = E$. Hence $T_1 = T$. This contradiction shows that $S = T$.

As an immediate corollary we obtain

THEOREM 2. (cf. [3], p. 181). *A closed symmetric operator S in a Hilbert space $\mathcal{H}$ is self-adjoint if and only if S has a total set of quasi-analytic vectors.*

Proof. We first show that S^2 is a closed operator. Indeed, if $x_n \in \mathcal{D}(S^2)$, $x_n \to x$ and $S^2 x_n \to y$ as $n \to \infty$, then

$$\|S(x_n - x_m)\|^2 = \big(S^2(x_n - x_m)\,|\,x_n - x_m\big) \to 0 \quad \text{as } n, m \to \infty.$$

Hence, since S is closed, $x \in \mathcal{D}(S)$ and $Sx_n \to Sx$. Thus, $Sx_n \to Sx$ and $S^2 x_n \to y$ as $n \to \infty$. Therefore, since S is closed, $Sx \in \mathcal{D}(S)$ and $S^2 x = y$.

Let $x \in \bigcap_{n\geq 1} \mathcal{D}(S^n)$. Then $\|S^n x\|^{1/n}$ is monotonically increasing with n if $\|x\| = 1$. Hence

$$\sum_{n=1}^{\infty} \|S^n x\|^{-1/n} = \infty \Leftrightarrow \sum_{n=1}^{\infty} \|S^{2n} x\|^{-1/2n} = \infty$$

for any $x \in \bigcap_{n=1} \mathcal{D}(S^n)$. Therefore x is a quasi-analytic vector for S if and only if

$$\sum_{n=1}^{\infty} \|S^{2n} x\|^{-1/2n} = \infty.$$

Hence, if S is self-adjoint, S^2 is a positive definite self-adjoint operator and S has a total set of quasi-analytic vectors by Theorem 1. (This is the trivial part of the theorem and is also seen directly as in the proof of Theorem 1.)

Conversely, if S has a total set of quasi-analytic vectors, then S^2 is a closed densely defined operator and therefore $(S^2)^*$ exists and $(S^2)^* \supset S^{*2} \supset S^2$. That is, S^2 is a positive definite closed symmetric operator with a total set of vectors x such that

$$\sum_{n=1}^{\infty} \|S^{2n} x\|^{-1/2n} = \infty.$$

Hence, by Theorem 1, S^2 is self-adjoint. But this implies that S is self-adjoint. Indeed, $S^*S \supset S^2$ and hence, since S^2 and S^*S are both self-adjoint, $S^2 = S^*S$. Similarly $S^2 = SS^*$. Hence $SS^* = S^*S$, i.e. S is normal and symmetric and therefore self-adjoint.

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