Remark. Examples of almost reflexive spaces that have properties V and Dunford-Pettis are c, c_0, and $C(S)$, where S is a compact Hausdorff dispersed space [5]. Hence we see that any uc operator $T\colon c_0 \to Y$ is compact. Note that T is a uc operator and hence compact if Y contains no subspace isomorphic to c_0.

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Principal ideals which are maximal ideals in Banach algebras

by

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1. Introduction. Let A be a commutative semisimple Banach algebra with identity. If for some $f \in A$, the principal ideal Af is a maximal ideal, then in a natural way there is associated with each element $g \in A$, a formal power series $\sum_{n=0}^{\infty} a_n f^n$ with complex coefficients. Indeed, as shown in Theorem 3 below, if Af is not in the Šilov boundary Γ, then for each $g \in A$, the Gelfand transform $\hat{g}$ is given by the power series

$$\hat{g}(y) = \sum_{n=0}^{\infty} a_n \hat{f}^n(y)$$

for all y in the maximal ideal space satisfying $|\hat{f}(y)| < \min\{|\hat{f}(t)|: t \in \Gamma\}$.

The phenomenon of a principal ideal being a maximal ideal occurs in the familiar "disk algebra" consisting of the continuous complex-valued functions on the plane disk $\{z: |z| \leqslant 1\}$ which are analytic in the interior. The ideal Az is maximal, and each $g \in A$ has a power series expansion $\sum_{n=0}^{\infty} a_n z^n$ holding in the interior of the disk.

We wish to acknowledge the work of Phillip E. Parker([1]) concerning the relation of the norm and Gelfand topologies on the maximal ideal space when Af is a maximal ideal, and present a result of his in section 3.

2. Let us suppose throughout this section that A is a sup norm function algebra on a compact Hausdorff space X. This means that A is a closed subalgebra of the algebra $C(X)$, that

(i) A separates points in X, and

(ii) $1 \in A$.

We wish also to impose the condition that

(iii) the maximal ideal space of A is X.

By saying that the maximal ideal space of A is X, we mean that

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each maximal ideal M in A is, for some $x \in X$, the set $M_x = \{g: g \in A$ and $g(x) = 0\}$. With or without condition (iii), the collection $\mathcal{M}$ of maximal ideals, endowed with the Gelfand topology, is a compact Hausdorff space, with X homeomorphically embedded. Moreover, each $g \in A$ has a norm preserving extension from X to $\mathcal{M}$. Thus condition (iii) amounts to saying that we shall consider the functions in A already to be extended to $\mathcal{M}$.

THEOREM 1. *Suppose the principal ideal Af is the maximal ideal M_x. Then x is not in the Šilov boundary Γ, unless x is an isolated point of X.*

Proof. Let $T\colon A \to M_x$ be defined by $T(g) = gf$. Then T is a bounded operator from one Banach space onto another, and T is easily seen to be one-to-one unless x is isolated in X. By the open map theorem T^{-1} is bounded, and we let $K = \|T^{-1}\| = \sup\{\|g\|;\ \|gf\| \leqslant 1\}$. Let $0 < \varepsilon < 1/K$. Since $f(x) = 0$ and f is continuous, there is a neighborhood U of x such that $|f| < \varepsilon$ on U. Since $x \in \Gamma$, there is a $g \in A$ such that somewhere in U, $|g| = \|g\| = 1$ and $|g| < 1$ outside U. It follows that for some positive integer n, $\|g^n f\| < \varepsilon$, $\|(g^n/\varepsilon)f\| \leqslant 1$, and hence that $\|g^n/\varepsilon\| \leqslant K$. Thus $\|g^n\| \leqslant K\varepsilon < 1$. But this cannot be true since $\|g^n\| = \|g\|^n = 1$.

THEOREM 2. *Suppose the principal ideal Af is the maximal ideal M_x, and that x is not isolated in X. Then for each $g \in A$, there is a power series expansion*

$$g(y) = \sum_{n=0}^{\infty} a_n f^n(y) \tag{1}$$

with complex coefficients, valid for all $y \in X$ such that $|f(y)| < m = \min\{|f(t)|: t \in \Gamma\}$, where Γ is the Šilov boundary of A.

Proof. Let $E = \{y: |f(y)| < m\}$, and let $A(\bar{E})$ denote the sup norm closure of the restrictions of functions in A to $\bar{E}$. Since E is an open subset of $X - \Gamma$, the local maximum principle (see [3]) says that the Šilov boundary Γ_m of $A(\bar{E})$ is a subset of the topological boundary ∂E. In particular, $\Gamma_m \subset \{y: |f(y)| = m\}$.

Now evaluation of elements in $A(\bar{E})$ at x gives a linear functional of norm 1, and by the usual application of the Hahn-Banach theorem and Riesz representation theorem, there is a complex Baire measure μ situated on Γ_m such that for all $h \in A(\bar{E})$

$$h(x) = \int_{\Gamma_m} h\,d\mu,$$

and $\|\mu\| = 1$. Since $1 \in A$, then $\mu(\Gamma_m) = 1$, and this with $\|\mu\| = 1$ implies that μ is real valued. Also, since $f(x) = 0$ and $|f| = m$ on Γ_m, then

$$0 = \int_{\Gamma_m} f^n\,d\mu = \overline{\int_{\Gamma_m} f^n\,d\mu} = \int_{\Gamma_m} \bar{f}^n\,d\mu = m^n \int_{\Gamma_m} f^{-n}\,d\mu,$$

and hence

$$\int_{\Gamma_m} f^{-n}\,d\mu = 0 \quad \text{for } n = 1, 2, 3, \ldots \tag{2}$$

Let $g \in A$, and $\|g\| = 1$. Then $g - g(x) \in M_x$ and so there is a function $g_1 \in A$ such that

$$g - g(x) = g_1 f.$$

In general, there is a function $g_{n+1} \in A$ such that

$$g_n - g_n(x) = g_{n+1} f. \tag{3}$$

It follows that

$$g = g(x) + g_1(x) f + \ldots + g_n(x) f^n + g_{n+1} f^{n+1}. \tag{4}$$

We think of equation (4) as an abstract Taylor formula. Upon dividing both sides of equation (4) by f^n, and integrating with respect to μ on Γ_m, we obtain with the aid of equation (2) above, that

$$g_n(x) = \int_{\Gamma_m} g/f^n\,d\mu.$$

Since $|f| = m$ on Γ_m,

$$|g_n(x)| \leqslant 1/m^n. \tag{5}$$

We now establish the estimate

$$|g_n(y)| \leqslant (n+1)/m^n \quad \text{for } y \in E. \tag{6}$$

For $n = 1$, (6) follows because on Γ_m

$$|g_1| = |g - g(x)|/|f| \leqslant 2\|g\|/m = 2/m.$$

Now suppose that (6) holds for $n = k$, and consider $n = k+1$. Fix $y \in E$, and let ϱ be chosen so that $0 < |f(y)| < \varrho < m$. Let $Q = \{s: |f(s)| < \varrho\}$, and let Γ_ϱ be the Šilov boundary for $A(Q)$. By the same arguments for the measure μ "representing" x, there is a non-negative Baire measure μ_y situated on Γ_ϱ, with $\|\mu_y\| = 1$, such that for $h \in A(Q)$,

$$h(y) = \int_{\Gamma_\varrho} h\,d\mu_y,$$

and $\Gamma_\varrho \subset \{s: |f(s)| = \varrho\}$. We obtain

$$g_{k+1}(y) = \int_{\Gamma_\varrho} (f)^{-1}\big(g_k - g_k(x)\big)\,d\mu_y = \int_{\Gamma_\varrho} (f)^{-1} g_k\,d\mu_y - g_k(x) \int_{\Gamma_\varrho} (f)^{-1}\,d\mu_y.$$

Since $|f| = \varrho$ on Γ_ϱ, we have

$$\int_{\Gamma_\varrho} (f)^{-1}\,d\mu_y = \varrho^{-2} \int_{\Gamma_\varrho} \bar{f}\,d\mu_y = \varrho^{-2}\overline{f(y)}.$$

It follows with the aid of (5) that

$$|g_{k+1}(y)| \leqslant \big((k+1)/m^k\big)(1/\varrho)+(1/m^k)(m/\varrho^2),$$

and as ϱ can be arbitrarily close to m, that

$$|g_{k+1}(y)| \leqslant (k+2)/m^{k+1},$$

so that (6) is established.

Now we return to the Taylor formula (4), and observe that by inequality (6), if $y \in E$, then

$$|g_{n+1}(y) f^{n+1}(y)| \leqslant (n+2)\big(|f(y)|/m\big)^{n+1}.$$

Since $|f(y)|/m < 1$, we infer that $(n+2)\big(|f(y)|/m\big)^{n+1}$ has limit zero as $n \to \infty$, and this establishes the convergence of the power series (1).

Remark. It is not difficult to show that if T is the operator introduced in the proof of Theorem 1 (namely $T(g) = gf$), and $x \notin \Gamma$, then $\|T^{-1}\| = 1/m$, where as before $m = \min\{|f(t)|: t \in \Gamma\}$. Using this fact, one can readily establish *via* formula (3), that $\|g_{n+1}\| \leqslant 2^{n+1}/m^{n+1}$, and hence *via* the Taylor formula (4), that the power series (1) converges for all y such that $|f(y)| < m/2$. But this, of course, gives us only half the radius of convergence.

Definition. A subset E of X is an "analytic disk" if there is a one-to-one continuous mapping τ from an open disk in the plane onto E such that for each $g \in A$, $g \circ \tau$ is analytic.

Corollary. *The set $E = \{y: |f(y)| < m\}$ is an analytic disk.*

Proof. Since for each $g \in A$ there is the power series expansion

$$g(y) = \sum_{k=0}^{\infty} a_k f^k(y) \quad \text{for } |f(y)| < m,$$

and A separates points in E, f must be one-to-one in E.

Let $E' = \{y: |f(y)| \leqslant m\}$. Then E' is a closed subset of A, and is A-convex in the sense of [3]. Hence, by [3], the maximal ideal space of $A(E')$ is E' itself, and the Šilov boundary of $A(E')$ is contained in $\{y: |f(y)| = m\}$. Now by Theorem 3.3.23 of [2] the set $f(E')$ contains the disk $\{z: |z| < m\}$, and since $|f| = m$ on $E'-E$, $f(E)$ contains, and hence equals, $\{z: |z| < m\}$. The desired function τ can thus be taken to be $(f|E)^{-1}$.

3. In this section we extend the results of the previous section to an arbitrary commutative semisimple Banach algebra A with identity. Let X be the maximal ideal space of A, and for $g \in A$, let $\hat{g}$ be the Gelfand transform of g. Then X is a compact Hausdorff space, and A is isomorphic to a subalgebra of $C(X)$ by the mapping $g \to \hat{g}$. This subalgebra may not be closed in sup norm, and we shall denote its sup norm closure by $\bar{A}$. Then $\bar{A}$, as a closed subalgebra of $C(X)$, satisfies conditions (i), (ii), and (iii) of the previous section.

Lemma. *Suppose the principal ideal Af is the maximal ideal $\{g: g \in A$ and $\hat{g}(x) = 0\}$ and x is not in the Šilov boundary Γ of A. Then the principal ideal $\bar{A}\hat{f}$ is the maximal ideal $\{g: g \in \bar{A}$ and $g(x) = 0\}$.*

Proof. Let $m = \min\{|\hat{f}(t)|: t \in \Gamma\}$. Then $m > 0$. If $g \in A$ and $\|\hat{g}\hat{f}\|_\infty \leqslant 1$, then for $t \in \Gamma$, $|\hat{g}(t)\hat{f}(t)| \leqslant 1$,

$$|\hat{g}(t)| \leqslant 1/|\hat{f}(t)| \leqslant 1/m,$$

and hence $\|\hat{g}\|_\infty \leqslant 1/m$.

Now let $g \in \bar{A}$ and $g(x) = 0$. There exists a sequence $\langle g_n\rangle_{n=1}^{\infty}$ in A such that $\hat{g}_n \to g$ uniformly on X. We may assume without loss of generality that $\hat{g}_n(x) = 0$, and thus that $g_n \in Af$. Let h_n be chosen so that $g_n = h_n f$. Then $\langle \hat{h}_n\hat{f}\rangle_{n=1}^{\infty}$ is a Cauchy sequence in sup norm, and it follows from the estimate in the previous paragraph that $\langle \hat{h}_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence. Let $h = \lim \hat{h}_n$. Then $h \in \bar{A}$, and $g = h \cdot \hat{f}$, proving that each element in the maximal ideal is in the principal ideal.

Theorem 3. *Suppose the principal ideal Af is the maximal ideal $\{g: g \in A$ and $\hat{g}(x) = 0\}$, and that x is not in the Šilov boundary Γ of A. Let $m = \min\{|\hat{f}(t)|: t \in \Gamma\}$. Then for each $g \in A$ there is a power series expansion*

$$\hat{g}(y) = \sum_{k=0}^{\infty} a_k \hat{f}^k(y)$$

valid for all $y \in X$ such that $|\hat{f}(y)| < m$.

Proof. This theorem follows directly from the previous lemma and Theorem 2.

We now consider a result concerning the norm topology of X, i.e. the norm topology induced on X *via* the canonical embedding of X into the adjoint space A^*. We denote the distance between x and y by $\|x-y\|$, and have the formula

$$\|x-y\| = \sup\{|\hat{g}(y)-\hat{g}(x)|: g \in A \text{ and } \|g\| = 1\}.$$

The following theorem was first obtained by Phillip E. Parker. The second hypothesis is satisfied, for instance, if there are no "point derivations" on A at x (cf. [1]).

Theorem 4. *If*

(1) *the principal ideal Af is the maximal ideal $\{g: g \in A$ and $\hat{g}(x) = 0\}$, and*

(2) *x is isolated in the norm topology of X, then x is isolated in the Gelfand topology of X.*

Proof. As previously observed, the inverse of the operator $T\colon A \to M_x$ defined by $T(g) = gf$ is bounded. Let $K = \|T^{-1}\|$. For $g \in A$ with $\|g\| = 1$ there is an element $g_1 \in A$ such that

$$g - \hat{g}(x)e = g_1 f,$$

and it follows that $\|g_1\| \leqslant 2K$, and hence

$$\|y - x\| \leqslant 2K\,|\hat{f}(y)|.$$

If x is isolated in the norm topology, there is an $r > 0$ such that $\|y - x\| < r$ implies $y = x$. Thus if $|\hat{f}(y)| < r/2K$, then $\|y - x\| < r$ and $y = x$. But this implies that x is isolated in the Gelfand topology of X.

Finally, we mention an open question. It was proved for the sup norm algebra case that Af being a maximal ideal M_x in the Šilov boundary implies that x is isolated in the Gelfand topology. Does this result hold true for an arbitrary commutative semisimple Banach algebra with identity? The technique for obtaining power series mentioned in the remark after Theorem 2 applies, and thus a negative answer would allow for power series representations on the Šilov boundary.

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A note on quasi-analytic vectors *

by

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In [3] we introduced the notion of a quasi-analytic vector. Let S be a symmetric operator in a Hilbert space $\mathscr{H}$ and x an element in $\bigcap_{n\geqslant 1}\mathscr{D}(S^n)$ ($\mathscr{D}(A)$ denotes the domain of an operator A acting in $\mathscr{H}$); then x is said to be a *quasi-analytic vector* for S if

$$\sum_{n=1}^{\infty}\|S^n x\|^{-1/n} = \infty$$

(this condition is equivalent to $\sum_{n=1}^{\infty}\|S^n x\|/\|S^{n+1}x\| = \infty$).

We proved in [3] that a closed symmetric operator S is self-adjoint if and only if it has a total set of quasi-analytic vectors. The purpose of this note is to prove a slightly stronger theorem for the case S is semi-bounded and then derive the original theorem as a corollary. The idea of the proof is essentially the same as in [3], but it seems not possible to derive Theorem 1 from our previous result. The reason is, as we shall see, that if a Stieltjes moment sequence is determined, the corresponding Hamburger moment sequence need not be determined.

THEOREM 1. *Let S be a semi-bounded, closed, symmetric operator in a Hilbert space $\mathscr{H}$. Then S is self-adjoint if and only if there exists a total set of vectors $x \in \bigcap_{n\geqslant 1}\mathscr{D}(S^n)$ such that*

$$\sum_{n=1}^{\infty}\|S^n x\|^{-1/2n} = \infty.$$

Proof. The necessity of the condition is a trivial consequence of the spectral theorem. In fact, if S is self-adjoint and $\{E(\sigma)\}$ its canonical spectral measure, let $E_c = E([-c, c])$ for $c > 0$. If $x \in \mathscr{R}(E_c)$ (range of E_c), then

$$\|S^n x\|^2 = \int_{-c}^{c}\lambda^{2n}\,d\|E(\lambda)x\|^2 \leqslant c^{2n}\|x\|^2,$$

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