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On Abel summability of multiple Laguerre series

by

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INTRODUCTION

The purpose of the present paper is to extend the results in [1] concerning Abel Summability of Multiple Hermite Series to the case of Multiple Laguerre Series. The 1-dimensional case has been studied in [3]-[7]. The novelty of our method in the 1-dimensional case is the statement of weighted maximal theorems.

1. NOTATION AND DEFINITIONS

1.1. $L^p_{e,m}(\alpha)$ denotes the family of Lebesgue measurable functions defined on $\boldsymbol{R}^m_+ = R_+\times\ldots\times R_+$ such that

$$(1.1.1)\qquad \int\limits_{\boldsymbol{R}^m_+} |f|^p e^{-\sum_1^m x_j}\prod_{j=1}^m x_j^{\alpha_j}dx_1\ldots dx_m = \int\limits_{\boldsymbol{R}^m_+}|f|^p e^{-X}X^\alpha dX<\infty,$$

where $1\leqslant p<\infty$ and the α_j $(j=1,\ldots,m)$ are such that $-\frac{1}{2}<\alpha_j<+\infty$. The $L^p_{e,m}(\alpha)$-norm is defined in the following way:

$$(1.1.2)\qquad \|f\|_p(e,\alpha)\overset{\text{def}}{=}\Big(\int\limits_{\boldsymbol{R}^m_+}|f|^p e^{-X}X^\alpha dX\Big)^{1/p},\quad 1\leqslant p<\infty.$$

1.2. $\tilde{L}^{(\alpha)}_{(n)}(X)$ denotes a family of m-dimensional polynomials defined as follows:

Let $n=(n_1,\ldots,n_m)$, where each n_j $(j=1,\ldots,m)$ is a non-negative integer, and let $\alpha=(\alpha_1,\ldots,\alpha_m)$, where each α_j $(j=1,\ldots,m)$ is a real parameter such that $-\frac{1}{2}<\alpha_j<\infty$ (see footnote (1)). Now

$$(1.2.1)\qquad \tilde{L}^{(\alpha)}_{(n)}(X)\overset{\text{def}}{=}\prod_{j=1}^m \Gamma(n_j+1)^{1/2}\Gamma^{-1/2}(n_j+\alpha_j+1)L^{(\alpha_j)}_{n_j}(x_j).$$

Here $L_{n_j}^{(\alpha_j)}(x_j)$ is the n_j-th Laguerre Polynomial of parameter α_j on the variable x_j (see [7], p. 99); $L_0^{(\alpha_j)}(x_j) = 1$, therefore

$$\tilde{L}_{(0)}^{(\alpha)}(X) = \prod_{j=1}^{m} \big(\Gamma(\alpha_j+1)\big)^{-1/2}.$$

If we fix α, $L_{(n)}^{(\alpha)}(X)$ is a closed orthonormal system in $L^2_{e,m}(\alpha)$ [1].

1.3. If $f \sim C_n \tilde{L}_{(n)}^{(\alpha)}(X)$, we shall denote by $f(r, X)$ its Abel Approximating, that is

$$f(r, X) \overset{\text{def}}{=} \sum_n r^n C_n \tilde{L}_{(n)}^{(\alpha)}(X) = \sum_{n_1,\dots n_m} r_1^{n_1} \dots r_m^{n_m} \times$$

$$\times \left(\frac{\Gamma(n_1+1)}{\Gamma(n_1+\alpha_1+1)}\right)^{1/2} L_{n_1}^{(\alpha_1)}(x_1) \dots \left(\frac{\Gamma(n_m+1)}{\Gamma(n_m+\alpha_m+1)}\right)^{1/2} L_{n_m}^{(\alpha_m)}(x_m) C_{n_1\dots n_m}.$$

1.4. By $f^{**}(X)$ we denote the maximal function associated with $f(r, X)$, that is

$$f^{**}(X) \overset{\text{def}}{=} \sup_{r_1,\dots,r_m} |f(r, X)|, \quad 0 < r_j < 1;\, j = 1, \dots, m.$$

1.5. We say that $\mu = \mu(J)$ is an *elementary real measure defined on R_+^m* with bounded variation there if the following two conditions are satisfied:

(A) $\mu(\bigcup J_k) = \sum \mu(J_k)$ and $J_k \cap J_j = \emptyset$ if $j \neq k$ and the J_k are a finite union of m-dimensional intervals.

(B) $\sup \sum_k |\mu(I_k)| < \infty$, where the sup is taken over all possible finite systems of non-degenerate pairwise disjoint m-dimensional intervals contained in R_+^m.

1.6. The variation $W(J)$ of μ is defined as

$$W(J) \overset{\text{def}}{=} \sup \sum_k |\mu(I_k)|.$$

The sup is taken over all possible finite systems of non-degenerate pairwise disjoint m-dimensional intervals contained in J.

[1] Actually, if we ask $-1 < \alpha_j < \infty$ $(j = 1, \dots, m)$, we will also have an orthonormal system; nevertheless, in this paper we shall only be concerned with the case $-1/2 < \alpha_j < \infty$.

1.7. The Fourier-Laguerre coefficients C_n of an elementary measure are defined as

$$C_n \overset{\text{def}}{=} \int_{R_+^m} \tilde{L}_{(n)}^{(\alpha)}(X)\, d\mu$$

provided that the integrals exist. Also

$$\mu(r, X) = \sum r^n C_n L_{(n)}^{(\alpha)}(X), \quad 0 < r_j < 1;\ j = 1, \dots, m.$$

1.8. We say that r *tends restrictedly to* $(1, \dots, 1)$ if there exists a real positive number θ such that $r \to (1, \dots, 1)$ submitted to the conditions

$$\theta^{-1} < \frac{1-r_i}{1-r_j} < \theta, \quad 0 < r_j < 1;\ i, j = 1, \dots, m.$$

2. STATEMENT OF THE MAIN RESULTS

2.1. THEOREM 1. (i) *If $f \in L^p_{e,m}(\alpha)$, $p \geq 2$, $f(r, X)$ is well defined and we have*

(A) $$\|f(r, X) - f(X)\|_p(e, \alpha) \to 0 \quad \text{as } r \to (1, \dots, 1),$$

(B) $$f(r, X) \to f(X) \quad \text{a.e. as } r \to (1, \dots, 1),$$

(C) $$\|f^{**}\|_p(e, \alpha) \leq C_p \|f\|_p(e, \alpha),$$

where C_p depends on p only.

(ii) *If $f(x_1, \dots, x_m) e^{\gamma \sum_1^m x_j} \in L^p_{e,m}(\alpha)$ $(1 < p < 2)$ for some $\gamma > 0$ such that $1/2 > \gamma > (2-p)/2p$, then the same conclusions* (A), (B) *and* (C) *of* (i) *are valid for f.*

(iii) *If $|f|\{\log^+|f|\}^m e^{(1/2)\sum_1^m x_j} \in L^1_{e,m}(\alpha)$, then*

(A) $$f(r, X) \to f(X) \quad \text{a.e. as } r \to (1, \dots, 1),$$

(B) $$\|f^{**}\|_1(e, \alpha) \leq O_\alpha + O'_\alpha \| |f| \{\log^+|f|\}^m \|_1(e, \alpha).$$

Here O_α and O'_α depend on $\alpha = (\alpha_1, \dots, \alpha_m)$ only.

(iv) *If $|f|\{\log^+|f|\}^{m-1} e^{(1/2)\sum_1^m x_j} \in L^1_{e,m}(\alpha)$, we have*

(A) $$f(r, X) \to f(X) \quad \text{a.e. as } r \to (1, \dots, 1),$$

(B) $$\|\{f^{**}\}^\beta\|_1(e, \alpha) \leq D_{\alpha,\beta} + D_{\alpha,\beta} \| |f| \{\log^+|f|\}^{m-1} \|_1(e, \alpha),$$

where $0 < \beta < 1$; $D_{\alpha,\beta}$ and $D'_{\alpha,\beta}$ depend on (α, β) only.

(v) *If* $|f|e^{(1/2)\sum_1^m x_j} \in L^1_{e,m}(\alpha)$, *then*

$$\|f(r,X)-f(X)\|_1(e,\alpha)\to 0 \quad \text{as } r\to(1,\ldots,1).$$

2.2. THEOREM 2. *If μ is an elementary real measure defined on $\boldsymbol{R}_+^m$ with bounded variation there, such that*

$$\int_{\boldsymbol{R}_+^m} e^{X/2}\,dW<\infty \quad (dW \text{ denotes the variation of } \mu),$$

then $\mu(r,X)$ converges a.e. when r tends restrictedly to $(1,\ldots,1)$. The limit is the density function associated with μ with respect to the measure $e^{-Y}Y^\alpha dY$.

3. HILLE-HARDY FORMULA

3.1. The following identity has been established (see [7], p. 101):

$$(3.1.1)\qquad \sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)r^n$$
$$=(1-r)^{-1}\exp-\{(x+y)r/(1-r)\}(-xyr)^{-\alpha/2}J_\alpha\{2(-xyr)^{1/2}(1-r)^{-1}\};$$

$J_\alpha\{z\}$ denotes the Bessel function of order α.

A formal product leads to

$$(3.1.2)\qquad \sum_{n_1=0,\ldots,n_m=0}^{\infty,\ldots,\infty}\frac{\Gamma(n_1+1)\ldots\Gamma(n_m+1)}{\Gamma(n_1+\alpha_1+1)\ldots\Gamma(n_m+\alpha_m+1)}r_1^{n_1}\ldots r_m^{n_m}\times$$
$$\times L_{n_1}^{(\alpha_1)}(x_1)L_{n_1}^{(\alpha_1)}(y_1)\ldots L_{n_m}^{(\alpha_m)}(x_m)L_{n_m}^{(\alpha_m)}(y_m)$$
$$=\sum_n r^n\tilde{L}_{(n)}^{(\alpha)}(X)\tilde{L}_{(n)}^{(\alpha)}(Y)$$
$$=\prod_{j=1}^m\{(1-r_j)^{-1}(-x_jy_jr_j)^{-\alpha_j/2}J_{\alpha_j}\{2(-x_jy_jr_j)^{1/2}(1-r_j)^{-1}\}\}\times$$
$$\times\exp-\Big\{\sum_1^m(x_j+y_j)r_j(1-r_j)^{-1}\Big\}=K_\alpha(r,X,Y).$$

We shall refer to $K_\alpha(r,X,Y)$ as the Multiple Hille-Hardy Singular Kernel.

3.2. LEMMA. (i) $|\tilde{L}_{(n)}^{(\alpha)}(X)|\leqslant A_\alpha\prod_{j=1}^m\{e^{x_j/2}n_j^{(\alpha_j/2+1/4-1/12)}\}$ *for* $\alpha_j>-1/2$ *and* $x_j\geqslant 0$ $(j=1,\ldots,m)$.

Here A_α depends only on $\alpha=(\alpha_1,\ldots,\alpha_m)$.

(ii) $0\leqslant K_\alpha(r,X,Y)\leqslant M_\alpha(r,Y)<\infty$ *for* $0<r_j<1$, $0\leqslant x_j<\infty$, $j=1,\ldots,m$.

Proof. Let us consider the 1-dimensional case and take into consideration the following formula (see [7], p. 106):

$$(3.2.1)\qquad (n!)^{1/2}\{\Gamma(n+\alpha+1)\}^{-1/2}L_n^{(\alpha)}(x)$$
$$=(-1)^n\pi^{-1/2}\Gamma(\alpha+\tfrac12)^{-1}\big((2n)!\big)^{-1}\big(\Gamma(n+\alpha+1)\cdot n!\big)^{1/2}\times$$
$$\times\int_{-1}^{1}(1-t^2)^{\alpha-1/2}H_{2n}(x^{1/2}t)\,dt.$$

$H_{2n}(s)$ denotes the $2n$-th Hermite polynomial.

Since

$$|H_{2n}(s)|\leqslant B_0e^{s^2/2}(2n!)^{1/2}2^n(2n)^{-1/12}$$

(see [7], p. 240), where the bound B_0 does not depend on (n,s), we infer that $\{n!/\Gamma(n+\alpha+1\}^{1/2}|L_n^{(\alpha)}(x)|$ is dominated by

$$(3.2.2)\qquad C_\alpha\big(\Gamma(n+\alpha+1)n!\big)^{1/2}(2n!)^{-1/2}2^nn^{-1/12}\Big(\int_{-1}^{1}(1-t^2)^{\alpha-1/2}dt\Big)e^{x/2}.$$

Now, an application of Stirling's Formula gives

$$(3.2.3)\qquad (n!)^{1/2}\big(\Gamma(n+\alpha+1)\big)^{-1/2}|L_n^{(\alpha)}(x)|\leqslant A_\alpha e^{x/2}n^{\alpha/2+1/4-1/12}.$$

Taking into account that

$$\tilde{L}_{(n)}^{(\alpha)}(X)=\prod_{j=1}^m\{(n_j!)^{1/2}\big(\Gamma(n_j+\alpha_j+1)\big)^{-1/2}\}L_{n_j}^{(\alpha_j)}(x_j),$$

we obtain (i).

Consider now, for fixed r and y,

$$(3.2.4)\qquad (1-r)^{-1}\exp-\{(x+y)r/(1-r)\}(-xyr)^{-\alpha/2}J_\alpha\{2(-xyr)^{1/2}(1-r)^{-1}\}.$$

Since $\alpha>-1/2$, we can use the following well known formula:

$$(3.2.5)\qquad J_\alpha(s)=\{\Gamma(1/2)\Gamma(\alpha+1/2)\}^{-1}(s/2)^\alpha\int_{-1}^{1}(1-t^2)^{\alpha-1/2}e^{ist}dt.$$

Therefore

$$(-xyr)^{-\alpha/2}J_\alpha\{2(-xyr)^{1/2}(1-r)^{-1}\}$$
$$=\big(\Gamma(1/2)\Gamma(\alpha+1/2)\big)^{-1}(1-r)^{-\alpha-1}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}e^{2(xyr)^{1/2}t(1-r)^{-1}}dt.$$

Also

(3.2.6) $$|(-xyr)^{-a/2}J_a\{2(-xyr)^{1/2}(1-r)^{-1}\}| \leqslant C(a)(1-r)^{-a-1}e^{2(xyr)^{1/2}(1-r)^{-1}}.$$

Consequently, (3.2.4) is dominated by

(3.2.7) $$C(a)(1-r)^{-1-a}\exp-\{(1-r)^{-1}[xr+yr-2(xyr)^{1/2}]\}$$
$$= e^{y}C(a)(1-r)^{-1-a}\exp-\{(1-r)^{-1}[xr+y-2(xyr)^{1/2}]\}$$
$$\leqslant C(a)e^{y}(1-r)^{-1-a}.$$

Now by multiplication we obtain (ii).

3.3. Lemma. (A) *If* $f(x_1,\ldots,x_m)e^{(1/2)\sum_1^m x_j}\in L^1_{e,m}(a)$, *then*

(i) f *has Fourier coefficients with respect to the system* $\{\tilde{L}^{(a)}_{(n)}(X)\}$;

(ii) *if* $f\sim\sum_n C_n\tilde{L}^{(a)}_{(n)}(X)$ *and* $0<r_j<1$ $(j=1,\ldots,m)$, *then*

$$\sum_n r^n C_n\tilde{L}^{(a)}_{(n)}(X) = \sum_{n_1\ldots n_m} r_1^{n_1}\ldots r_m^{n_m}C_{n_1,\ldots,n_m}\times$$
$$\times\left\{\frac{\Gamma(n_1+1)}{\Gamma(n_1+a_1+1)}\right\}^{1/2}\ldots\left\{\frac{\Gamma(n_m+1)}{\Gamma(n_m+a_m+1)}\right\}^{1/2}L^{(a_1)}_{n_1}(x_1)\ldots L^{(a_m)}_{n_m}(x_m)$$
$$= \int_{\mathbf{R}^m_+}K_a(r,X,Y)f(Y)e^{-Y}Y^a\,dY.$$

As before, $K_a(r,X,Y)$ *denotes the Hille-Hardy Multiple Singular Kernel.*

(B) *If* $f\in L^p_{e,m}(a)$, $p\geqslant 2$, *then the same conclusions as in* (A) *hold.*

(C) *If* $f(x_1,\ldots,x_m)\exp\{\gamma\sum_1^m x_j\}=fe^{\gamma X}\in L^p_{e,m}(a)$, $1<p<2$, $1/2>\gamma>(2-p)/2p$, *then, the same conclusions as in* (A) *hold.*

(D) $\int_{\mathbf{R}^m_+}K_a(r,X,Y)e^{-Y}Y^a\,dY=1$.

Proof. Let f be under the assumptions of (A). The following integral exists by lemma (3.2):

$$\int_{\mathbf{R}^m_+}K_a(r,X,Y)f(Y)e^{-Y}Y^a\,dY.$$

Now let us observe that

(3.3.1) $$\sum_n r^n|\tilde{L}^{(a)}_{(n)}(X)\tilde{L}^{(a)}_{(n)}(Y)|$$
$$=\sum_{0,\ldots,0}^{\infty\ldots\infty} r_1^{n_1}\ldots r_m^{n_m}\frac{n_1!}{\Gamma(n_1+a_1+1)}|L^{(a_1)}_{n_1}(x_1)L^{(a_1)}(y_1)|\ldots\frac{n_m!}{\Gamma(n_m+a_m+1)}\times$$
$$\times|L^{(a_m)}_{n_m}(x_m)L^{(a_m)}_{n_m}(y_m)|$$
$$\leqslant\prod_{j=1}^m\exp\{1/2(x_j+y_j)\}\prod_{j=1}^m\Big\{\sum_{n_j=0}^\infty r_j^{n_j}n_j^{2(a_j/2+1/4-1/12)}\Big\}B_a$$
$$\leqslant B(r_1,\ldots,r_m,a_1,\ldots,a_m)\prod_{j=1}^m\exp\{1/2(x_j+y_j)\}$$
$$=B(r,a)e^{X/2}e^{Y/2}.$$

The inequalities of (3.3.1) hold from part (i) of lemma 3.2. On the other hand,

(3.3.2) $$\sum_n C_n r^n\tilde{L}^{(a)}_{(n)}(X)$$
$$=\sum_n r^n\tilde{L}^{(a)}_{(n)}(X)\Big\{\int_{\mathbf{R}^m_+}f\tilde{L}^{(a)}_{(n)}(Y)e^{-Y}Y^a\,dY\Big\}$$
$$=\sum_n\int_{\mathbf{R}^m_+}r^n|\tilde{L}^{(a)}_{(n)}(X)\tilde{L}^{(a)}_{(n)}(Y)|e^{-Y}|f(Y)|Y^a\,dY.$$

Since

(3.3.3) $$\int_{\mathbf{R}^m_+}\sum_n r^n|\tilde{L}^{(a)}_{(n)}(X)\tilde{L}^{(a)}_{(n)}(Y)|\,|f(Y)|e^{-Y}Y^a\,dY$$
$$\leqslant B(r,a)e^{X/2}\int_{\mathbf{R}^m_+}|f(Y)|e^{-Y/2}Y^a\,dY<\infty,$$

we can interchange the summation with the integration and obtain (A) (ii). Part (i) of (A) follows from the estimate (i) of lemma 3.2. Now, let f belong to $L^p_{e,m}(a)$, $p>2$. From Hölder's inequality we have

(3.3.4) $$\int_{\mathbf{R}^m_+}|f|e^{Y/2}e^{-Y}Y^a\,dY\leqslant\|f\|_p(e,a)\,\|e^{Y/2}\|_{p/(p-1)}(e,a)$$
$$\leqslant C(p,a)\|f\|_p(e,a).$$

The boundedness of $\|e^{Y/2}\|_{p/(p-1)}(e, a)$ follows from the fact that $p/(p-1) < 2$ if $p > 2$. Therefore, conclusions (i) and (ii) of (A) follow for all functions belonging to $L^p_{e,m}(a)$, $p > 2$.

Now, let f belong to $L^2_{e,m}(a)$. For fixed r and X, $0 < r_j < 1$ $(j = 1, \ldots, m)$,

$$\sum_n r^n \tilde{L}^{(a)}_{(n)}(X) C_n$$

is a continuous linear functional on $L^2_{e,m}(a)$, since from the estimate (i) of lemma (3.2) we know that

$$\sum_n (r^n \tilde{L}^{(a)}_{(n)}(X))^2 < \infty, \quad 0 < r_j < 1\ (j = 1, \ldots, m). \tag{3.3.5}$$

On the other hand, for a dense subset, namely $L^p_{e,m}(a)$, $p > 2$, the functional has the representation

$$\int_{R^m_+} K_a(r, X, Y) f(Y) e^{-Y} Y^a dY. \tag{3.3.6}$$

Since, from part (ii) of lemma 3.2, $K_a(r, X, Y)$ is a bounded function of Y, $K_a(r, X, Y)$ belongs to $L^2_{e,m}(a)$ and the representation (3.3.6) will hold for all $L^2_{e,m}(a)$.

To obtain part (C), we will show that in this case $|f| e^{X/2} \in L^1_{e,m}(a)$:

$$\begin{aligned} &\int_{R^m_+} |f| e^{X/2} e^{-X} X^a dX \\ &= \int_{R^m_+} |f| e^{\gamma X} e^{(1-2\gamma)X/2} e^{-X} X^a dX \\ &\leqslant \Big(\int_{R^m_+} e^{(1-2\gamma)pX/(2p-2)} e^{-X} X^a dX \Big)^{(p-1)/p} \|f e^{\gamma X}\|_p (e, m). \end{aligned} \tag{3.3.7}$$

Let us observe that $1-2\gamma < 1-(2-p)/p = 2(p-1)/p$. Therefore $(1-2\gamma)2^{-1}(p-1)^{-1}p < 1$ and consequently

$$\int_{R^m_+} \exp\{(1-2\gamma)2^{-1}(p-1)^{-1}pX\} e^{-X} X^a dX < \infty. \tag{3.3.8}$$

This proves part (C).

Part (B) asserts that if $f \in L^2_{e,m}(a)$, then

$$\sum_n C_n r^n \tilde{L}^{(a)}_{(n)}(X) = \int_{R^m_+} K_a(r, X, Y) f(Y) e^{-Y} Y^a dY. \tag{3.3.9}$$

Taking $f = 1$ and observing that $C_{n_1, \ldots, n_m} = 0$ if and only if $n_j \neq 0$ for some j, we have

$$1 = \int_{R^m_+} K_a(r, X, Y) e^{-Y} Y^a dY \tag{3.3.10}$$

as we wished to prove.

3.4. Remark. If μ is an elementary measure defined on R^m_+ such that $\int_{R^m_+} e^{X/2} dW < \infty$, where dW denotes the variation of μ, its Fourier-Laguerre coefficients are well defined:

$$C_n = \int_{R^m_+} \tilde{L}^{(a)}_{(n)}(X) d\mu.$$

Furthermore, for this case we have the same conclusions as in part (A) of lemma 3.3.

4. ESTIMATES FOR $K_a(r, X, Y)$

4.1. We shall begin with the single kernel, namely

$$\begin{aligned} k_a(r, x, y) = (1-r)^{-1-a} e^{-(x+y)r/(1-r)} \{\Gamma(1/2)\Gamma(a+1/2)\}^{-1} \times \\ \times \int_{-1}^{1} (1-t^2)^{a-1/2} \exp\{2(xyr)^{1/2}(1-r)^{-1}t\} dt. \end{aligned} \tag{4.1.1}$$

4.2. Lemma. *If $a > -1/2$, there exists a function $k^*_a(s, r, x, y)$ defined on the set $\{(s, r, x, y)/ 0 \leqslant s \leqslant 1, 0 < r < 1, 0 < x < \infty, 0 < y \leqslant \infty\}$ having the following properties:*

(i) *If we fix the pair (s, r), $k^*_a(s, r, x, y)$, as a function of y, is non-increasing on $x < y < \infty$ and non-decreasing on $0 \leqslant y < x$.*

(ii) $k_a(r, x, y) \leqslant \int_0^1 (1-s^2)^{a-1/2} k^*_a(s, r, x, y) ds.$

(iii) $\int_0^\infty e^{-y} y^a dy \Big\{ \int_0^1 (1-s^2)^{a-1/2} k^*_a(s, r, x, y) ds \Big\} \leqslant A_a.$

Here the constant A_a depends on a only.

Proof. From (4.1.1) we have

$$k_a(r, x, y) \leqslant 2 \frac{e^{-(x+y)\frac{r}{1-r}}}{(1-r)^{1+a}\Gamma(\frac{1}{2})\Gamma(a+\frac{1}{2})} \int_0^1 (1-s^2)^{a-1/2} e^{\frac{2(xyr)^{1/2}}{1-r}s} ds. \tag{4.2.1}$$

Therefore

$$(4.2.2)\qquad k_a(r,x,y)\leqslant 2\big(\Gamma(1/2)\Gamma(a+1/2)\big)^{-1}\times$$
$$\times\int_0^1(1-s^2)^{a-1/2}(1-r)^{-1-a}\exp\{-(x+y)r/(1-r)+2(xyr)^{1/2}s/(1-r)\}\,ds$$
$$=2\big(\Gamma(1/2)\Gamma(a+1/2)\big)^{-1}\int_0^1(1-s^2)^{a-1/2}(1-r)^{-1-a}\exp\{(2s(xyr)^{1/2}-$$
$$-(x+y)r)/(1-r)\}\,ds.$$

Let us consider the kernel

$$h_a(s,r,x,y)=\big(\Gamma(1/2)\Gamma(a+1/2)(1-r)^{1+a}\big)^{-1}\times$$
$$\times\exp\{[2s(xyr)^{1/2}-(x+y)r]/(1-r)\}.$$

If we fix the parameters (s,r,x), a differentiation with respect to y shows that $h_a(s,r,x,y)$ is non-increasing if $y\geqslant s^2x/r$ and non-decreasing if $0\leqslant y<s^2x/r$.

Now we are going to define $k_a^*(s,r,x,y)$.

(4.2.3) If $s^2\geqslant r$

$$k_a^*(s,r,x,y)=h_a(s,r,x,y)\quad\text{for } 0\leqslant y<x \text{ or } s^2x/r<y<\infty,$$
$$k_a^*(s,r,x,y)=h_a(s,r,x,s^2x/r)$$
$$=2\frac{\exp\{-(r-s^2)x/(1-r)\}}{\Gamma(1/2)\Gamma(a+\frac12)(1-r)^{1+a}}\quad\text{for } x\leqslant y\leqslant s^2x/r.$$

(4.2.4) If $s^2<r$

$$k_a^*(s,r,x,y)=h_a(s,r,x,y)\quad\text{for } 0\leqslant y<s^2x/r \text{ or } x<y<\infty,$$
$$k_a^*(s,r,x,y)=h_a(s,r,x,s^2x/r)$$
$$=2\frac{\exp\{-(r-s^2)x/(1-r)\}}{\Gamma(1/2)\Gamma(a+\frac12)(1-r)^{1+a}}\quad\text{for } s^2x/r\leqslant y\leqslant x.$$

An easy verification shows that $k_a^*(s,r,x,y)$ is under the conditions of (i) and (ii). It only remains to prove (iii) (the non-trivial part of the lemma).

(A) Let us suppose that $1>r\geqslant 1/2$ and consider the integral

$$(4.2.5)\qquad \int_0^\infty e^{-y}y^a\,dy\int_0^1 k_a^*(s,r,x,y)(1-s^2)^{a-1/2}\,ds$$
$$=\int_0^\infty e^{-y}y^a\,dy\int_0^{r^{1/2}}k_a^*(s,r,x,y)(1-s^2)^{a-1/2}\,ds+$$
$$+\int_0^\infty e^{-y}y^a\,dy\int_{r^{1/2}}^1 k_a^*(s,r,x,y)(1-s^2)^{a-1/2}\,ds.$$

(A_1) Bound for $\int_0^\infty e^{-y}y^a\,dy\int_{r^{1/2}}^1 k_a^*(s,r,x,y)(1-s^2)^{a-1/2}\,ds$.

Taking into account the definition of $k_a^*(s,r,x,y)$, the integral under consideration is readily seen to be equal or less than

$$(4.2.6)\qquad \int_0^\infty e^{-y}y^a\,dy\int_{r^{1/2}}^1 h_a(s,r,x,y)(1-s^2)^{a-1/2}\,ds+$$
$$+\int_{r^{1/2}}^1(1-s^2)^{a-1/2}\big(\Gamma(1/2)\Gamma(a+1/2)(1-r)^{1+a}\big)^{-1}\times$$
$$\times\exp\{-(r-s^2)x/(1-r)\}\Big\{\int_x^{s^2x/r}e^{-y}y^a\,dy\Big\}\,ds$$
$$\leqslant 2+\int_{r^{1/2}}^1(1-s^2)^{a-1/2}\big(\Gamma(1/2)\Gamma(a+1/2)(1-r)^{1+a}\big)^{-1}\times$$
$$\times\exp\{-(r-s^2)x/(1-r)\}\Big\{\int_x^{s^2x/r}e^{-y}y^a\,dy\Big\}\,ds$$

since $\int_0^1 h_a(s,r,x,y)(1-s^2)^{a-1/2}\,ds\leqslant 2k_a(r,x,y)$.

Setting $s^2=u$, we have

$$(4.2.7)\qquad \int_{r^{1/2}}^1(1-s^2)^{a-1/2}\big(\Gamma(1/2)\Gamma(a+1/2)(1-r)^{1+a}\big)^{-1}\times$$
$$\times\exp\{-(r-s^2)x/(1-r)\}\Big\{\int_x^{s^2x/r}e^{-y}y^a\,dy\Big\}\,ds$$
$$=C(a)\int_r^1(1-u)^{a-1/2}u^{-1/2}(1-r)^{-1-a}\exp\{-(r-u)x/(1-r)\}\times$$
$$\times\Big\{\int_x^{ux/r}e^{-y}y^a\,dy\Big\}\,du.$$

Observing that $x\leqslant(u/r)x\leqslant 2x$ (since $r\geqslant 1/2$ and $1\geqslant u\geqslant r$), if $x\leqslant y\leqslant(u/r)x$, there exists a constant D_a, depending on a only, such that

$$(4.2.8)\qquad y^a\leqslant D_ax^a,\quad x\geqslant 0,\quad x\leqslant y\leqslant 2x.$$

The precedinginequality yields

$$(4.2.9)\qquad \int_x^{ux/r}e^{-y}y^a\,dy\leqslant D_ax^a\int_x^{ux/r}e^{-y}\,dy=D_ax^ae^{-x}\{1-e^{-x(u-r)/r}\}.$$

Therefore, (4.2.7) is dominated by

$$(4.2.10)\quad D_\alpha C_\alpha \int_r^1 (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} x^\alpha e^{-x} \times$$
$$\times e^{-x(r-u)/(1-r)} \{1-e^{-x(u-r)/r}\} du$$
$$= D_\alpha C_\alpha \int_r^1 x^{1/2} (1-r)^{-1-1/2} u^{-1/2} e^{-x(1-u)/(1-r)} \times$$
$$\times \big(x(1-u)/(1-r)\big)^{\alpha-1/2} \{1-e^{-x(u-r)/r}\} du.$$

Observing that

$$(4.2.11)\quad \sup_{x\geqslant 0, u\geqslant r>0} (1-e^{-x(u-r)/r}) x^{-1/2} (u/r-1)^{-1/2} \leqslant M_0$$

and that

$$(4.2.12)\quad u/r-1 = (u-r)/r \leqslant (1-r)/r \quad \text{for } 1\geqslant u\geqslant r,$$

that is $u/r-1 \leqslant 2(1-r)$ since $1>r\geqslant \frac{1}{2}$.

Now taking into account (4.2.11) and (4.2.12), the right-hand member of (4.2.10) is readily seen to be equal or less than

$$(4.2.13)\quad C_\alpha D_\alpha M_0 2^{1/2} \sup_{\lambda>0} \lambda \int_{1/2}^1 e^{-|\lambda(1-u)|} \{\lambda(1-u)\}^{\alpha-1/2} u^{-1/2} du$$
$$\leqslant 2C_\alpha D_\alpha M_0 \int_{-\infty}^{\infty} e^{-|x|} |x|^{\alpha-1/2} dx.$$

This completes part (A_1).

(A_2) Bound for $\int_0^\infty e^{-y} y^\alpha dy \int_0^{r^{1/2}} k_\alpha^*(s, r, x, y)(1-s^2)^{\alpha-1/2} ds$.

As in case (A_1), after a change of variables the following inequality is valid:

$$(4.2.14)\quad \int_0^\infty e^{-y} y^\alpha dy \int_0^{r^{1/2}} k_\alpha^*(s, r, x, y)(1-s^2)^{\alpha-1/2} ds$$
$$\leqslant 2 + C_\alpha \int_0^r (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} \Big\{\int_{xu/r}^x y^\alpha e^{-y} dy\Big\} du.$$

Suppose now that $\alpha<0$; therefore we have

$$(4.2.15)\quad C_\alpha \int_0^r (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} \Big\{\int_{xu/r}^x y^\alpha e^{-y} dy\Big\} du$$
$$\leqslant C_\alpha \int_0^r (1-u)^{\alpha-1/2} u^{-1/2} u^\alpha r^{-\alpha} x^\alpha (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} x(1-u/r) du.$$

But $1-u/r = (r-u)/r \leqslant (1-u)/r \leqslant 2(1-u)$ since $0\leqslant u\leqslant r<1$ and $r\geqslant 2^{-1}$. Thus the right-hand member of inequality (4.2.15) is dominated by

$$(4.2.16)\quad C_\alpha \int_0^r (1-u)^{1/2} u^{\alpha-1/2} \big(x/(1-r)\big) e^{-|x(r-u)/(1-r)|} |x(r-u)/(1-r)|^\alpha du$$
$$\leqslant C_\alpha \sup_{1/2\leqslant r\leqslant 1} \sup_{\lambda>0} \lambda \int_0^1 u^{\alpha-1/2} e^{-|\lambda(r-u)|} |\lambda(r-u)|^\alpha du.$$

Calling $\Phi(u)$ to be the maximal function associated to the function equal to $u^{\alpha-1/2}$ if $0\leqslant u\leqslant 1$ and zero otherwise, the right-hand member of inequality (4.2.16) is readily seen to be dominated by

$$(4.2.17)\quad C_\alpha \Big(\int_{-\infty}^{\infty} e^{-|s|} |s|^\alpha ds\Big) \sup_{1/2\leqslant r\leqslant 1} \Phi(r).$$

This gives (A_2) when $\alpha<0$.

Suppose now that $\alpha\geqslant 0$. In this case we have

$$(4.2.18)\quad C_\alpha \int_0^r (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} \Big(\int_{xu/r}^x y^\alpha e^{-y} dy\Big) du$$
$$\leqslant C_\alpha \int_0^{r-\varepsilon} (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} x^{\alpha+1} (1-u/r) du +$$
$$+ C_\alpha \int_{r-\varepsilon}^r (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} \Big(\int_{xu/r}^x y^\alpha e^{-y} dy\Big) du,$$

where $0<\varepsilon<1/2\leqslant r<1$.

On the other hand, $(1-u/r) = (r-u)/r \leqslant 2(1-u)$ (since $1>r\geqslant u\geqslant 0$). Thus we have

$$(4.2.19)\quad C_\alpha \int_0^{r-\varepsilon} (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} x^{\alpha+1} (1-u/r) du$$
$$\leqslant 2C_\alpha \int_0^{r-\varepsilon} u^{-1/2} (1-u)^{-1/2} e^{-x(r-u)/(1-r)} x^{\alpha+1} (1-u)^{\alpha+1} (1-r)^{-\alpha-1} du.$$

Observing now that $r-u\geqslant\varepsilon$ or equivalently $1\leqslant(1/\varepsilon)(r-u)$, therefore $1-u\leqslant\varepsilon^{-1}(r-u)$. Thus we have

$$(4.2.20)\quad \int_0^{r-\varepsilon} (1-u)^{\alpha-1/2} u^{-1/2} (1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} x^{\alpha+1} (1-u/r) du$$
$$\leqslant 2\, C_\alpha \varepsilon^{-1-\alpha} \{\sup_s e^{-|s|} |s|^{\alpha+1}\} \int_0^1 \big(u(1-u)\big)^{-1/2} du.$$

Putting $a = 1/2-\varepsilon$, we have

(4.2.21) $$C_\alpha \int_{r-\varepsilon}^{r} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} \Big(\int_{xu/r}^{x} y^\alpha e^{-y}\, dy\Big) du$$

$$\leqslant C_\alpha \int_{a}^{r} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} x^\alpha e^{-ax} x(1-u/r)\, du$$

$$\leqslant 2C_\alpha \int_{a}^{r} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)} x^{\alpha+1} e^{-ax}(1-u)\, du.$$

Since $0<a<1$, we see that $e^{-x(r-u)/(1-r)} \leqslant e^{-ax(r-u)/(1-r)}$ for $0 \leqslant u \leqslant r < 1$, $x>0$. Therefore the last term of (4.2.21) is dominated by

(4.2.22) $$2C_\alpha x(1-r)^{-1} \int_{a}^{1} u^{1/2}(1-u)^{1/2} e^{-ax(1-u)/(1-r)} \{x(1-u)/(1-r)\}^\alpha du$$

$$\leqslant 2C_\alpha a^{-1/2} \int_{-\infty}^{\infty} e^{-a|s|}|s|^\alpha ds.$$

This, together with (4.2.20) gives (A_2) for $\alpha \geqslant 0$.

Now we shall be concerned with the case $0<r<1/2$, that is, the boundedness of the following integrals for $0<r<1/2$:

(4.2.23) $$C_\alpha \int_{r}^{1} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)}\, du \Big\{\int_{x}^{xu/r} y^\alpha e^{-y} dy\Big\}$$

and

$$C_\alpha \int_{0}^{r} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(r-u)/(1-r)}\, du \Big\{\int_{xu/r}^{x} y^\alpha e^{-y} dy\Big\}.$$

The second integral is readily seen to be equal or less than

$$C_\alpha 2^{1+\alpha}\Gamma(\alpha+1)\int_0^1 (1-u)^{\alpha-1/2}u^{-1/2}du.$$

The first integral is

(4.2.24) $$C_\alpha \int_{r}^{1} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(1-u)/(1-r)} e^x \Big(\int_{x}^{xu/r} y^\alpha e^{-y} dy\Big) du.$$

For $0 \leqslant r \leqslant 1/2$, $0 \leqslant x \leqslant 1$ (4.2.24) is uniformely bounded; therefore we shall consider $x \geqslant 1$ only.

For $\alpha \leqslant 0$ we have

(4.2.25) $$C_\alpha \int_{r}^{1} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(1-u)/(1-r)} e^x \Big(\int_{x}^{xu/r} y^\alpha e^{-y} dy\Big) du$$

$$\leqslant C_\alpha \int_{r}^{1} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(1-u)/(1-r)} e^x \Big(\int_{x}^{\infty} e^{-y} dy\Big) du$$

$$\leqslant 2^{1+\alpha} C_\alpha \int_0^1 (1-u)^{\alpha-1/2}u^{-1/2}du.$$

If $\alpha>0$, there exists a bound M_α (depending on α only) such that

(4.2.26) $$\sup_{x \geqslant 1} x^{-\alpha} e^x \int_x^\infty e^{-y} y^\alpha dy \leqslant M_\alpha.$$

To see this, let us consider the closest integer m to α, such that $m \geqslant \alpha$. An integration by parts m times yields

(4.2.27) $$\int_x^\infty e^{-y}y^\alpha dy = \sum_{k=0}^{m-1} C_k(\alpha) e^{-x}x^{\alpha-k} + C_m(\alpha)\int_x^\infty e^{-y}y^{\alpha-m}dy.$$

Since $m \geqslant \alpha$ and $x \geqslant 1$,

$$\int_x^\infty e^{-y}y^{\alpha-m}dy \leqslant e^{-x}.$$

On the other hand, $(x^{-\alpha}e^x)e^{-x}x^{\alpha-k} \leqslant 1$ for $x \geqslant 1$. Therefore (4.2.26) holds.

Taking into account (4.2.26), (4.2.24) is dominated by

(4.2.28) $$C_\alpha \int_{0}^{1} (1-u)^{\alpha-1/2} u^{-1/2}(1-r)^{-1-\alpha} e^{-x(1-u)/(1-r)} x^\alpha \Big(x^{-\alpha}e^x \int_{x}^{\infty} e^{-y} y^\alpha dy\Big) du$$

$$\leqslant 2C_\alpha M_\alpha \int_0^1 \big(u(1-u)\big)^{-1/2} e^{-x(1-u)/(1-r)} \{x(1-u)/(1-r)\}^\alpha du$$

$$\leqslant 2C_\alpha M_\alpha \{\sup_s e^{-|s|}|s|^\alpha\} \int_0^1 \big(u(1-u)\big)^{-1/2} du.$$

This finishes part (iii) of the lemma.

4.3. Remark. If $x=0$, the Hille-Hardy Singular Kernel takes the form

(4.3.1) $$P_\alpha(1-r)^{-1-\alpha}e^{-yr/(1-r)}.$$

Therefore, it already has the desired form.

4.4. Definition. Let f belong to $L^1_{e,m}(\alpha)$, and consider a point X belonging to $\boldsymbol{R}^m_+$. We say that the integral $\int_J fe^{-Y}Y^\alpha dY$ is *strongly differentiable at the point X with respect to the measure*

$$d\nu = e^{-Y}Y^\alpha dY = e^{-(y_1+\ldots+y_m)}y_1^{\alpha_1}\ldots y_m^{\alpha_m}dy_1\ldots dy_m$$

if the limit

$$\lim_{d(I_X)\to 0}\big(1/\nu(I_X)\big)\int_{I_X} fd\nu \tag{4.4.1}$$

exists.

The I_X are m-dimensional, non-degenerate intervals with edges parallel to the coordinate axes, containing the point X. $d(I_X)$ denotes the diameter of I_X. All the I_X must be taken contained in $\boldsymbol{R}^m_+$.

We shall also define f^*, the *strong maximal function* associated to f as

$$f^*(X) \overset{\text{def}}{=} \sup_{\boldsymbol{R}^m_+\supset I_X\supset\{X\}}\big(1/\nu(I_X)\big)\Big|\int_{I_X} fd\nu\Big|, \tag{4.4.2}$$

where the I_X, as in the preceding definition, are non-degenerate intervals with edges parallel to the coordinate axes.

4.5. Lemma. *f^* has the following properties:*

(i) *If $f\in L^p_{e,m}(\alpha)$, $p>1$, then*

$$\|f^*\|_p(e,\alpha)\leqslant C(p)\|f\|_p,$$

where $C(p)$ depends on p only.

(ii) *If $|f|\{\log^+|f|\}^m$ belongs to $L^1_{e,m}(\alpha)$, then*

$$\|f^*\|_1(e,\alpha)\leqslant A(\alpha)+B(\alpha)\,\||f|(\log^+|f|)^m\|_1(e,\alpha),$$

where the constants depend on α only.

(iii) *If $|f|(\log^+|f|)^{m-1}$ belongs to $L^1_{e,m}(\alpha)$, then for $0<\beta<1$*

$$\|(f^*)^\beta\|_1(e,\alpha)\leqslant C(\alpha,\beta)+D(\alpha,\beta)\,\||f|(\log^+|f|)^{m-1}\|_1(e,\alpha),$$

where the constants depend on α and on β.

For the proof of this lemma see [1], Part I, Theorem (1.8) and take as measures μ_j those generated by the density functions $e^{-x_j}x_j^{\alpha_j}$ if $x_j\geqslant 0$ and zero otherwise.

4.6. Lemma. *Let $H(r,x,y)=\prod_{j=1}^m h_j(r_j,x_j,y_j)$ be a family of non-negative real functions defined on $\boldsymbol{R}^m_+\times\boldsymbol{R}^m_+$ depending on the parameter $r=(r_1,\ldots,r_m)\in\Delta$, such that the following two conditions are verified:*

(A) *For each pair (r_j,x_j), $h_j(r_j,x_j,y_j)$ as a function of y_j is defined on $\boldsymbol{R}_+$ and is non-decreasing if $y_j\leqslant x_j$ and non-increasing if $y_j>x_j$.*

(B) *$\int_0^\infty h_j(r_j,x_j,y_j)e^{-y_j}y_j^\alpha dy_j<M$, where the bound M does not depend neither on j nor on (r_j,x_j). Then if $f\in L^1_{e,m}(\alpha)$, we have*

(i) $\bar f(X)=\sup_{r\in\Delta}\Big|\int_{\boldsymbol{R}^m_+}H(r,X,Y)f(Y)e^{-Y}Y^\alpha dY\Big|\leqslant M^m(|f|)^*(X)$;

(ii) *$\bar f(X)$ verifies the same type of inequalities as those of $f^*(X)$ (with different constants).*

For the proof see [1], Part I, lemma (1.15), and take as measures μ_j those generated by the density functions $e^{-x_j}x_j^{\alpha_j}$ if $x_j\geqslant 0$ and zero otherwise.

5. PROOF OF THEOREM 1

5.1. It follows from lemma 3.3 that in all cases

$$f(r,X)=\int_{\boldsymbol{R}^m_+}K_\alpha(r,X,Y)f(Y)e^{-Y}Y^\alpha dY, \tag{5.1.1}$$

$$0<r_j<1\ (j=1,\ldots,m).$$

$K_\alpha(r,X,Y)$ denotes the Hille-Hardy Multiple Kernel.

On the other hand, setting

$$k^{**}_{\alpha_j}(r_j,x_j,y_j)=\int_0^1(1-s^2)^{\alpha-1/2}k^*_{\alpha_j}(s,r_j,x_j,y_j)\,ds\quad\text{if}\quad x_j>0, \tag{5.1.2}$$

$$k^{**}_{\alpha_j}(r_j,0,y_j)=k_{\alpha_j}(r_j,0,y_j),$$

where $k^*_{\alpha_j}(s,r_j,x_j,y_j)$ denotes the auxiliary kernel introduced in lemma 4.2 and $k_{\alpha_j}(r_j,x_j,y_j)$ denotes the single Hille-Hardy kernel. An easy verification shows that

(5.1.3)

(A) $\quad K^{**}_\alpha(r,X,Y)=\prod_{j=1}^m k^{**}_{\alpha_j}(r_j,x_j,y_j)\geqslant K_\alpha(r,X,Y)$,

(B) $\quad K^{**}_\alpha(r,X,Y)$ is under the conditions of lemma 4.6.

Therefore, the maximal inequalities for $f(r,X)$ are valid as a consequence of lemma 4.6.

The pointwise convergence in $L^2_{e,m}(\alpha)$ follows from the fact that $f(r,X)\to f(X)$ everywhere if f has only a finite number of non-vanishing Fourier-Laguerre coefficients. Such family of functions is dense in $L^2_{e,m}(\alpha)$. This fact together with the maximal inequality for $L^2_{e,m}(\alpha)$, implies the pointwise a.e. convergence of $f(r,X)$ in all $L^2_{e,m}(\alpha)$. Since $L^p_{e,m}(\alpha)\subset L^2_{e,m}(\alpha)$ for $p\geqslant 2$, we also have pointwise convergence a.e. in this case. On the other hand, we have dominated convergence (from the maximal inequality), therefore $f(r,X)$ converges in $L^p_{e,m}(\alpha)$-norm, for $p\geqslant 2$, to $f(X)$.

The set F of bounded functions vanishing outside a compact set is dense in $L^p_{e,m}(\alpha)$, $p \geqslant 1$, and also in $L^1_{e,m}(\alpha)\{\log^+ L^1_{e,m}(\alpha)\}^k$ for $k > 0$. The functions of such family have the property that $f(r, X) \to f(X)$ a.e. since this family is contained in $L^2_{e,m}(\alpha)$. This, together with the maximal inequalities, proves the a.e. convergence for the functions of (ii), (iii) and (iv). The norm convergence in case (ii) follows from the same argument used in the case $p \geqslant 2$.

For functions $f(X)$ under the conditions of case (v) we have

$$|f(r, X)| \leqslant \int_{R^m_+} K_\alpha(r, X, Y)\, |f(Y)|\, e^{-Y} Y^\alpha dY. \tag{5.1.4}$$

Taking into account that $\int_{R^m_+} K_\alpha(r, X, Y) e^{-Y} Y^\alpha dY = 1$ and that $K_\alpha(r, X, Y) = K_\alpha(r, Y, X)$, from Fubini's Theorem it follows

$$\|f(r, X)\|_1(e, \alpha) \leqslant \|f\|_1(e, \alpha). \tag{5.1.5}$$

If f is also in F, we have

$$\|f(r, X) - f(X)\|_1(e, \alpha) \leqslant \Big(\int_{R^m_+} e^{-Y} Y^\alpha dY\Big)^{1/2} \|f(r, X) - f(X)\|_2(e, \alpha). \tag{5.1.6}$$

Therefore, for functions of F we have $L^1_{e,m}(\alpha)$-convergence. This, together with inequality (5.1.5) implies that $f(r, X)$ converges to $f(X)$ in $L^1_{e,m}(\alpha)$-norm for every function under the conditions of case (v). Thus, the proof of Theorem 1 is completed.

6. RESTRICTED CONVERGENCE

6.1. Lemma. *Let μ be an elementary measure defined on R^m_+, with bounded variation there. Then if*

$$K(X) = \prod_{j=1}^m A_j (1 + |x_j|^{\beta_j})^{-1}, \qquad \beta_j > 1,$$

and defining

$$\tilde{\tilde{\mu}}(X) \overset{\text{def}}{=} \sup_{(\lambda_1, \ldots, \lambda_m)} \Big| \Big\{\prod_{j=1}^m \lambda_j\Big\} \int_{R^m_+} K\big(\lambda(X - Y)\big)\, d\mu(Y) \Big|,$$

where $\theta^{-1} \leqslant (\lambda_i/\lambda_k) \leqslant \theta$ $(i, k = 1, \ldots, m)$, we have the following properties:

(i) $|E(\tilde{\tilde{\mu}}(X), \varepsilon)| < C(\theta, m)\, \varepsilon^{-1} \int_{R^m_+} dW$.

dW denotes the variation of μ, and $C(\theta, m)$ depends on θ and m only. $|E(\tilde{\tilde{\mu}}(X), \varepsilon)|$ denotes the Lebesgue measure of the set $\{X$ such that $\tilde{\tilde{\mu}}(X) > \varepsilon\}$.

(ii) *If μ is a singular elementary measure defined on R^m_+ and with bounded variation there, we have*

$$\lim_{\lambda \to (\infty, \ldots, \infty)} \Big\{\prod_{j=1}^m \lambda_j\Big\} \int_{R^m_+} K\big(\lambda(X - Y)\big)\, d\mu(Y) = 0 \quad \text{a.e.}$$

as $\lambda \to (\infty, \ldots, \infty)$ restrictedly, that is submitted to the condition $\theta^{-1} \leqslant (\lambda_i/\lambda_k) \leqslant \theta$, $\theta > 0$ $(i, k = 1, \ldots, m)$.

For the proof see [1], Part I, lemma (1.5).

6.2. Lemma. *Let $f(Y)$ belong to $L^1_{e,m}(\alpha)$ and write*

$$f_K(r, X) = \int_{R^m_+} K_\alpha(r, X, Y) f(Y) e^{-Y} Y^\alpha dY;$$

then

(i) $|f_K(r, X^2)| \leqslant D_\alpha(M) \int_{R^m_+} \Big\{\prod_{j=1}^m r_j^{1/2}(1 - r_j)^{-1/2}\big(1 + (x_j - y_j)^2 r_j/(1 - r_j)\big)^{-1}\Big\} \times$ $\times |f(Y^2)|\, e^{-Y^2} Y^{2\alpha+1} dY$, *whenever* $1/M < x_j^2 < M$; $1/2 \leqslant r_j < 1$; $X^2 = (x_1^2, \ldots, x_m^2)$. *The bound $D_\alpha(M)$ depends on M and α only and is always finite for $M > 0$.*

Proof. Introducing the change of variables $Y = S^2$, $X = \dot{S}^2$ in the expression $\int_{R^m_+} K_\alpha(r, X, Y)\, |f(Y)|\, e^{-Y} Y^\alpha dY$, we obtain

$$\begin{aligned} &2^m \int_{R^m_+} C(\alpha) \prod_{j=1}^m \Big\{(1 - r_j)^{-1-\alpha_j} e^{-(\dot{s}_j^2 + s_j^2) r_j/(1 - r_j)} e^{-s_j^2} s_j^{2\alpha_j + 1} \times \\ &\qquad \times \Big(\int_{-1}^{1} (1 - t_j^2)^{\alpha_j - 1/2} e^{2\dot{s}_j s_j r_j^{1/2} t_j/(1 - r_j)} dt_j\Big)\Big\}\, |f(S^2)|\, dS \\ &\leqslant 2^{2m} C(\alpha) \int_{R^m_+} \prod_{j=1}^m \Big\{(1 - r_j)^{-1-\alpha_j} e^{-(\dot{s}_j^2 + s_j^2) r_j/(1 - r_j)} e^{-s_j^2} s_j^{2\alpha_j + 1} \times \\ &\qquad \times \Big(\int_0^1 (1 - t_j)^{\alpha_j - 1/2} e^{2\dot{s}_j s_j r_j^{1/2} t_j/(1 - r_j)} dt_j\Big)\Big\}\, |f(S^2)|\, dS. \end{aligned} \tag{6.2.1}$$

Let us observe now that

$$\begin{aligned} &(1 - r_j)^{-1-\alpha_j} e^{-(\dot{s}_j^2 + s_j^2) r_j/(1 - r_j)} e^{-s_j^2} s_j^{2\alpha_j + 1} \times \\ &\qquad \times \Big(\int_0^1 (1 - t_j^2)^{\alpha_j - 1/2} e^{2\dot{s}_j s_j r_j^{1/2} t_j/(1 - r_j)} dt_j\Big) \\ &= e^{\dot{s}_j^2} (1 - r_j)^{-1-\alpha_j} e^{-(\dot{s}_j - s_j r_j^{1/2})^2/(1 - r_j)} e^{-s_j^2} s_j^{2\alpha_j + 1} \times \\ &\qquad \times \Big(\int_0^1 (1 - t_j^2)^{\alpha_j - 1/2} e^{-2 r_j^{1/2} \dot{s}_j s_j (1 - t_j)/(1 - r_j)} dt_j\Big). \end{aligned} \tag{6.2.2}$$

Let us suppose that $(s_j/\dot{s}_j) \geqslant 1/2$, and consider the following inequality valid for $1/2 \leqslant r_j < 1$, $s_j > 0$, $\dot{s}_j > 0$:

$$(6.2.3) \quad \int_0^1 (1-t_j^2)^{\alpha_j-1/2} e^{-2\dot{s}_j s_j r_j^{1/2}(1-t_j)/(1-r_j)} dt_j$$
$$\leqslant \int_0^1 |1-t_j|^{\alpha_j-1/2} e^{-2^{1/2}\dot{s}_j s_j|1-t_j|/(1-r_j)} |1+t_j|^{\alpha_j-1/2} dt_j$$
$$\leqslant [\max(1, 2^{\alpha_j-1/2})] \big(\dot{s}_j s_j/(1-r_j)\big)^{-\alpha_j-1/2} \int_{-\infty}^{\infty} |t|^{\alpha_j-1/2} e^{-\sqrt{2}|t|} dt.$$

On the other hand, since $(s_j/\dot{s}_j) \geqslant 1/2$, we have

$$(6.2.4) \quad (s_j/\dot{s}_j)^{\alpha_j+1/2} \leqslant 2^{\alpha_j+1/2}(s_j/\dot{s}_j)^{2\alpha_j+1}.$$

Taking into account (6.2.3) and (6.2.4), the right-hand member of (6.2.2) is dominated by

$$(6.2.5) \quad F(\alpha_j) e^{\dot{s}_j^2} \dot{s}_j^{-(2\alpha_j+1)} (1-r_j)^{-1/2} e^{-(\dot{s}_j - r_j^{1/2}s_j)^2/(1-r_j)} e^{-s_j^2} s_j^{2\alpha_j+1},$$

whenever $(s_j/\dot{s}_j) \geqslant 1/2$. The bound $F(\alpha_j)$ depends on α_j only. Let us suppose that $0 < (s_j/\dot{s}_j) < 1/2$.

The right-hand member of (6.2.2) is dominated by

$$(6.2.6) \quad e^{\dot{s}_j^2}(1-r_j)^{-1-\alpha_j} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/(1-r_j)} \Big(\int_0^1 (1-t_j^2)^{\alpha_j-1/2} dt_j\Big) e^{-s_j^2} s_j^{2\alpha_j+1}.$$

For $1/2 \leqslant r_j < 1$ and $0 < (s_j/\dot{s}_j) < 1/2$ the following inequalities are valid:

$$(6.2.7) \quad e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)} = e^{-\dot{s}_j^2(1-r_j^{1/2}s_j/\dot{s}_j)^2/2(1-r_j)} \leqslant e^{-\dot{s}_j^2/4(1-r_j)}$$
$$\leqslant \big(\dot{s}_j^2/(1-r_j)\big)^{-\alpha_j-1/2} \sup_t \{e^{-|t|/4}|t|^{\alpha_j+1/2}\} = \big(\dot{s}_j^2/(1-r_j)\big)^{-\alpha_j-1/2} G(\alpha_j).$$

On the other hand, (6.2.6) can be written in the following way:

$$(6.2.8) \quad \Big(\int_0^1 (1-t_j^2)^{\alpha_j-1/2} dt\Big)(1-r_j)^{-\alpha_j-1/2} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)} \times$$
$$\times e^{\dot{s}_j^2}(1-r_j)^{-1/2} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)} e^{-s_j^2} s_j^{2\alpha_j+1}.$$

Taking into account (6.2.7), (6.2.8) is readily seen to be equal or less than

$$(6.2.9) \quad e^{\dot{s}_j^2} \dot{s}_j^{-(2\alpha_j+1)} H(\alpha_j)(1-r_j)^{-1/2} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)} e^{-s_j^2} s_j^{2\alpha_j+1},$$

whenever $0 < (s_j/\dot{s}_j) < 1/2$, $1/2 \leqslant r_j < 1 \cdot H(\alpha_j)$ depends on α_j only. From (6.2.5) and (6.2.9), the right-hand member of (6.2.2) is dominated by

$$(6.2.10) \quad e^{\dot{s}_j^2} \dot{s}_j^{-2\alpha_j-1} B(\alpha_j)(1-r_j)^{-1/2} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)} e^{-s_j^2} s_j^{2\alpha_j+1}$$

for $0 < s_j < \infty$, $0 < \dot{s}_j < \infty$ and $B(\alpha_j) = H(\alpha_j) + F(\alpha_j)$. For $1/2 \leqslant r < 1$ we have

$$(6.2.11) \quad (1-r_j)^{-1/2} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)}$$
$$\leqslant (2r_j)^{1/2}(1-r_j)^{-1/2} e^{-(\dot{s}_j-r_j^{1/2}s_j)^2/2(1-r_j)}$$
$$= (2r_j)^{1/2}(1-r_j)^{-1/2} \exp\{-(1/2)[\dot{s}_j(1-r_j^{1/2})/(1-r_j)^{1/2} +$$
$$+ \big(r_j^{1/2}/(1-r_j)^{-1/2}\big)(\dot{s}_j - s_j)]^2\}.$$

Setting $A(\dot{s}_j, r_j) = \dot{s}_j(1-r_j^{1/2})/(1-r_j)^{1/2} = (1-r_j^{1/2})^{1/2}(1+r_j^{1/2})^{-1/2}\dot{s}_j$, we have

$$(6.2.12) \quad 0 \leqslant A(\dot{s}_j, r_j) \leqslant 2^{-1/2}\dot{s}_j.$$

Therefore

$$(6.2.13) \quad \sup_{\substack{0<\dot{s}_j<N \\ 0<u<\infty \\ 1/2<r_j<1}} (1+u^2) e^{-(A(\dot{s}_j,r_j)+u)^2/2} \leqslant M(N),$$

where $M(N)$ depends on N only. Thus, the right-hand member of (6.2.11) is dominated by

$$(6.2.14) \quad 2^{1/2} r_j^{1/2}(1-r_j)^{-1/2} M(N)\big(1+r_j(\dot{s}_j-s_j)^2/(1-r_j)\big)^{-1}$$

for $1/2 \leqslant r_j < 1$, $0 < s_j < \infty$, $0 < \dot{s}_j < N$. This, together with (6.2.10) concludes the proof of the lemma.

6.3. Proof of Theorem 2. From lemma 3.3, see Remark 3.4, it follows that under the assumption $\int_{R_+^m} e^{X/2} dW < \infty$, $\mu(r, X)$ may be represented by $\int_{R_+^m} K_\alpha(r, X, Y) d\mu$.

Let μ be absolutely continuous with respect to the Lebesgue measure, that is $\mu(e^{-Y}Y^\alpha dY) = f(Y)e^{-Y}Y^\alpha dY$. The pointwise a.e. convergence of $\mu(r, X^2) = f(r, X^2)$ as $r \to (1, \ldots, 1)$ restrictedly on each set $Q_M = \{X/\ 1/M < x_j^2 < M\}$, $M > 0$, follows from lemmae 6.1 and 6.2 and from the fact that for a dense subset in $L^1_{e,m}(\alpha)$, $f(r, X) \to f(X)$ as $r \to (1, \ldots, 1)$ restrictedly.

For μ singular and non-negative we have

$$(6.3.1) \quad \mu(r, X^2)$$
$$\leqslant D_\alpha(M) \int_{R_+^m} \Big(\prod_{j=1}^m r_j^{1/2}(1-r_j)^{-1/2}\big(1+r_j(X-Y)^2/(1-r_j)\big)^{-1}\mu(e^{-Y^2}Y^{2\alpha+1}dY),$$

whenever $1/2 \leqslant r_j < 1$, $1/M < x_j^2 < M$ $(j = 1, \ldots, m)$.

Inequality (6.3.1) can be readily verified taking into account that it is valid for μ absolutely continuous and considering that there exists a sequence μ_n of such measures converging weakly to μ.

From lemma 6.1 it follows that $\mu(r, X^2) \to 0$ a.e. on each Q_M, $M > 0$. This ends the proof of Theorem 2.

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The comparison of an unconditionally converging operator *

by

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1. Preliminaries. In [3] A. Pełczyński shows that every weakly compact operator is an unconditionally converging operator. In the following we show that if an operator is strictly singular, almost weakly compact, or completely continuous (not the same as compact), then the operator is unconditionally converging; but not conversely.

Our notation and terminology will follow rather closely that used in [1]. Two common abbreviations used are uc for unconditionally converging or unconditionally convergent and wuc for weakly unconditionally convergent. All spaces are to be Banach spaces and all operators are to be linear and continuous. A linear operator $T: X \to Y$ is said to be *weakly compact* if it maps bounded sets in X into weakly sequentially compact sets.

Definition 1.1. (a) A series $\sum_n x_n$ of elements from a Banach space X is *uc* if for every bounded real sequence $\{t_n\}$ the series $\sum_n t_n x_n$ is convergent.

(b) A series $\sum_n x_n$ is *wuc* if for every real sequence $\{t_n\}$ with $\lim_n t_n = 0$ the series $\sum_n t_n x_n$ is convergent.

Definition 1.2. Let X and Y be Banach spaces. A linear operator $T: X \to Y$ is said to be *unconditionally converging* (*uc operator*) if it sends every wuc series in X into uc series in Y.

LEMMA 1.3. *Let $T: X \to Y$. Then T is a uc operator if and only if T has no bounded inverse on a subspace E of X isomorphic to c_0.*

Proof. Assume T is not a uc operator. Then T has a bounded inverse on a subspace isomorphic to c_0 by Lemma 1 of [4].

The converse implication is an obvious consequence of the fact that in the space c_0 the series consisting of unit vectors $e_n = (0, 0, \ldots, 1, 0, \ldots)$ is wuc but not uc.

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