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On the space of entire functions over certain non-archimedean fields and its dual

by

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1. Introduction. In [5], Theorem 3.1, we gave a characterization of the dual $\overline{K\langle x\rangle}$ of the class $K\langle x\rangle$ of all entire functions over a non-archimedean field K, with certain restrictions on the valuation of K, topologized by a suitable metric. The main object of the present paper is to give another characterization (Theorem 3.2) of $\overline{K\langle x\rangle}$. As an application of this characterization we prove in Section 4 the Hahn-Banach Theorem for $K\langle x\rangle$. We also give an alternative approach (in Theorem 3.1) to the topology of $K\langle x\rangle$.

2. Definitions and notations. A *valuation of rank* 1 of a field K is a mapping $|\ |$ from K into the reals such that for all $a, b \in K$

$$|a| \geqslant 0 \text{ and } = 0 \text{ if and only if } a = 0,$$

$$|ab| = |a|\,|b|, \qquad |a+b| \leqslant |a|+|b|.$$

If the valuation satisfies, in addition, the condition

$$|a+b| \leqslant \max(|a|, |b|),$$

then it will be called a *non-archimedean field* (Bruhat [1], p. 4, calls such a valuation a *real valuation.*) In the sequel, K denotes a complete non-archimedean non-trivial valued field. We recall that

$$\alpha = \alpha(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n \in K, \tag{2.1}$$

is an entire function (see [1], p. 114) over K if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in K$, or equivalently, if $|a_n|^{1/n} \to 0$ as $n \to \infty$. $K\langle x\rangle$ denotes the class of all entire functions topologized by the metric $|\alpha-\beta|$ ($\alpha, \beta \in K\langle x\rangle$), where

$$|\alpha| = \max_n [|a_0|, |a_n|^{1/n}, n \geqslant 1]. \tag{2.2}$$

In what follows the symbol $|\ |$ is used to denote the functional defined in (2.2) and also the valuation of K. With this topology, $K\langle x\rangle$ is a non-normable linear metric space over K (see [5]). We use $K\langle x\rangle$ for denoting the algebraic space of entire functions or this space with the metric topology defined earlier or the topology of the metric itself, as needed in the context. $\overline{K\langle x\rangle}$ denotes the dual of $K\langle x\rangle$. Every element $f\in\overline{K\langle x\rangle}$ is of the form (see [5], Theorem 3.1)

$$f(a) = \sum_{n=0}^{\infty} c_n a_n, \quad a = \sum_{n=0}^{\infty} a_n x^n, \tag{2.3}$$

where

$$\{|c_n|^{1/n}\} \text{ is bounded as } n\to\infty. \tag{2.4}$$

3. The main theorem. Given an entire function

$$a = a(x) = \sum_{n=0}^{\infty} a_n x^n$$

over K, we define, for each $r\neq 0$ and $r\in N_K = \{|a|;\, a\in K\}$, $|a:r|$ as follows:

$$|a:r| = \sup_n [|a_n| r^n, n\geqslant 0]. \tag{3.1}$$

It is easily seen that, for $r\neq 0$ in N_K, (3.1) defines a non-archimedean norm on the class $K\langle x\rangle$. We denote $K\langle x\rangle$ (or the topology on $K\langle x\rangle$) with this non-archimedean norm by $K_r\langle x\rangle$ and the corresponding dual of $K\langle x\rangle$ by $\overline{K_r\langle x\rangle}$. If $r_1 > r_2$, then $|a:r_1|\geqslant |a:r_2|$ and therefore $K_{r_1}\langle x\rangle$ is weaker (in the sense of Vaidyanathaswamy [6], p. 71) than $K_{r_2}\langle x\rangle$ (we actually mean by $K_{r_1}\langle x\rangle$ and $K_{r_2}\langle x\rangle$ the topologies induced on $K\langle x\rangle$ by the respective norms; for convenience we shall prefer to make such statements in the sequel also), and hence $\overline{K_{r_1}\langle x\rangle}\subseteq\overline{K_{r_2}\langle x\rangle}$. Further, if (a_q) is a sequence in $K\langle x\rangle$ such that $|a_q|\to 0$ as $q\to\infty$, then for each $r\neq 0$ in N_K,

$$|a_q:r|\to 0 \quad \text{as } q\to\infty.$$

Hence $K_r\langle x\rangle$ is stronger than $K\langle x\rangle$. Thus $\{K_r\langle x\rangle\}$ forms a decreasing family, as r increases, of normed topologies on $K\langle x\rangle$, each of which is stronger than $K\langle x\rangle$.

If S is any subset of $K\langle x\rangle$ and T any topology on $K\langle x\rangle$, we denote by $(\mathrm{Cl}\,S)_T$ the closure of S in the topology T. Theorem 3.1 characterizes $K\langle x\rangle$ in terms of the family $\{K_r\langle x\rangle\}$:

THEOREM 3.1. *Let S be a subset of $K\langle x\rangle$. Then*

$$(\mathrm{Cl}\,S)_{K\langle x\rangle} = \bigcap_{0\neq r\in N_K} (\mathrm{Cl}\,S)_{K_r\langle x\rangle}. \tag{3.2}$$

Proof. Since each $K_r\langle x\rangle$ is stronger than $K\langle x\rangle$, we have

$$(\mathrm{Cl}\,S)_{K\langle x\rangle}\subseteq(\mathrm{Cl}\,S)_{K_r\langle x\rangle}.$$

Therefore

$$(\mathrm{Cl}\,S)_{K\langle x\rangle}\subseteq B = \bigcap_{0\neq r\in N_K}(\mathrm{Cl}\,S)_{K_r\langle x\rangle}.$$

Thus to prove (3.2) we have only to prove that if a is at a positive distance from S in $K\langle x\rangle$, then it is so in $K_r\langle x\rangle$ for some $r\neq 0$ (and therefore also for all sufficiently large r). But this follows immediately from

LEMMA 3.1. *If $|a|\geqslant d>0$, then $|a:r|\geqslant d$ for all $r\in N_K$ such that $r>A(1/d)$, where, for $t>0$, $A(t) = \max(1, t)$.*

Proof. Let $a = \sum_{n=0}^{\infty} a_n x^n$ and $|a|\geqslant d>0$. If, for some $r>0$, $|a:r|<d$, then it follows that

$$|a_0|<d, \quad |a_n|^{1/n} r < d^{1/n}\leqslant A(d) \quad (n\geqslant 1).$$

This means that if $A(d)/r<d$, then $|a|<d$ which contradicts the hypothesis. Hence $A(d)/r\geqslant d$, i.e. $r\leqslant A(d)/d = A(1/d)$. Thus, if $r>A(1/d)$, then $|a:r|\geqslant d$.

The following is a parallel for $K_r\langle x\rangle$ of the result for $K\langle x\rangle$ quoted at the end of Section 2:

LEMMA 3.2. *Every functional in $\overline{K_r\langle x\rangle}$ is of the form*

$$f(a) = \sum_{n=0}^{\infty} c_n a_n, \quad a = \sum_{n=0}^{\infty} a_n x^n, \tag{3.3}$$

where

$$\left\{\frac{|c_n|}{r^n}\right\} \text{ is bounded as } n\to\infty, \tag{3.4}$$

and conversely.

Proof. Suppose $f(a)$ is a continuous linear functional on $K_r\langle x\rangle$. Then, by known results ([3], p. 1134-1135, Theorems 9 and 10), there exists an M such that

$$|f(a)|\leqslant M|a:r|.$$

Let

$$\delta_n = x^n \quad\text{and}\quad f(\delta_n) = c_n \quad (n\geqslant 0).$$

Then

$$f(a) = \lim_{n\to\infty}(c_0a_0+\ldots+c_na_n) = \sum_{n=0}^{\infty} c_n a_n,$$

since, in $K_r\langle x\rangle$, $a_0\delta_0+\ldots+a_n\delta_n\to a$ as $n\to\infty$. Also

$$|c_n|\leqslant M|\delta_n:r| = Mr^n.$$

Therefore (3.4) is true. Conversely, if (3.4) is true, the functional defined by (3.3) exists for all a and $|f(a)|\leqslant M|a:r|$ for some $M>0$. Hence, by a result of Monna ([3], Theorem 10) $f(a)$, which is obviously linear, is continuous on $K_r\langle x\rangle$.

We now prove the main result:

THEOREM 3.2. $\overline{K\langle x\rangle} = \bigcup_{0\neq r\in N_K}\overline{K_r\langle x\rangle}$.

Proof. Since a functional continuous in any topology will also be continuous in a weaker topology, we infer that $\overline{K_r\langle x\rangle}$ is contained in $\overline{K\langle x\rangle}$ for every $r\neq 0$ in N_K. Thus

$$(3.5)\qquad \bigcup_{r\neq 0}\overline{K_r\langle x\rangle}\subseteq\overline{K\langle x\rangle}.$$

We have to prove the reverse inclusion. For this, let $f\in\overline{K\langle x\rangle}$ be given by $f(a)=\sum_{n=0}^{\infty}c_na_n$, where $a=\sum_{n=0}^{\infty}a_nx^n$. Then, by (2.4), there exists a real number $M>0$ such that $|c_n|^{1/n}\leqslant M$, $n\geqslant 1$. Since K is non-trivial valued, we can find $r\neq 0$ in N_K such that $0<M\leqslant r$. Thus $|c_n|^{1/n}\leqslant r$, $n\geqslant 1$, $r\in N_K$. Hence $\{|c_n|/r^n\}$ is bounded and, by Lemma 3.2, $f\in\overline{K_r\langle x\rangle}$. Therefore

$$(3.6)\qquad \overline{K\langle x\rangle}\subseteq\bigcup_{r\neq 0}\overline{K_r\langle x\rangle}.$$

(3.5) and (3.6) together yield the result.

In view of Lemma 3.2, Theorem 3.2 is, in fact, equivalent to Theorem 3.1 of [5] quoted at the end of Section 2. For, if $f\in\overline{K\langle x\rangle}$, then $f\in\overline{K_r\langle x\rangle}$ for some $(0\neq)r\in N_K$. Then, by Lemma 3.2, for $a\in K\langle x\rangle$, $a=\sum_{n=0}^{\infty}a_nx^n$, $f(a)=\sum_{n=0}^{\infty}c_na_n$ and $|c_n|/r^n<k$ for all n. Hence $|c_n|^{1/n}<k^{1/n}r$ and is bounded.

4. Application. We need the following concepts defined by Ingleton [2].

Definition 4.1. A *nest of (closed) spheres* in K is a set of (closed) spheres totally ordered by inclusion.

Definition 4.2. A non-archimedean field is said to be *spherically complete* if every nest of spheres in the field has a common point.

Definition 4.3. K is said to have *Hahn-Banach property* if, for any normed space F over K, every linear functional defined on a subspace of F has an extension of the same norm defined on the whole of F.

We also need the following theorem of Ingleton [2]:

THEOREM 4.1. *A non-archimedean valued field has the Hahn-Banach property if and only if it is spherically complete.*

In the rest of this section we assume, that K is spherically complete. Then the Hahn-Banach theorem for $K\langle x\rangle$ can be stated as

THEOREM 4.2. *Let S be a linear subspace of $K\langle x\rangle$. Let $f(a)$ be a linear functional defined and continuous (in the topology of $K\langle x\rangle$) on S. Then there is a functional $F\in\overline{K\langle x\rangle}$ such that $F(a)=f(a)$ for $a\in S$.*

Theorem 4.2 is a special case of a more general result of Monna ([4], p. 400)(1). The proof given below differs in details considerably from that of the general result and is of independent interest as an application of Theorem 3.2.

Proof. By Theorems 4.1 and 3.2 it is enough to prove that if $f(a)$ satisfies the conditions of the theorem, then it is continuous on S in the topology of $K_r\langle x\rangle$ for some $r\neq 0$ in N_K. This we prove as

LEMMA 4.1. *Let S be a linear subspace of $K\langle x\rangle$. Let $f(a)$ be a linear functional defined and continuous on S in the topology of $K\langle x\rangle$. Then $f(a)$ is continuous on S in the topology of $K_r\langle x\rangle$ for some $r\neq 0$ in N_K.*

Proof. Suppose $f(a)$ is not continuous on S regarded as subspace of $K_r\langle x\rangle$ for any $r\neq 0$ in N_K. Since K has a non-trivial valuation, there exists a sequence $\{\lambda_i\}$ in K such that $|\lambda_i|\to\infty$ as $i\to\infty$. Then by Monna's results ([3], p. 1134-1135, Theorems 9 and 10), we can for each i choose an a_i' such that, given any M and λ_i,

$$(4.1)\qquad |f(a_i')|>M\big|a_i':|\lambda_i|\big|.$$

Now let $t_i\in K$ be such that

$$(4.2)\qquad 0<|t_i|\leqslant\frac{1}{|\lambda_i|}\cdot\frac{1}{\big|a_i':|\lambda_i|\big|},$$

this being possible since the valuation of K is non-trivial. We put $a_i=t_ia_i'$. Then $|f(a_i)|=|f(t_ia_i')|=|t_i||f(a_i')|$. Therefore, by (4.1) and (4.2), we have

$$(4.3)\qquad |f(a_i)|>M|t_i|\big|a_i':|\lambda_i|\big|.$$

(1) The author is grateful to Prof. Dr. A. F. Monna for having pointed out this fact and for having kindly scrutinized an earlier version of this paper.

Now, we choose M such that $M|t_i|\,\big|a_i'\colon|\lambda_i|\big| > 1$. Hence, by (4.3), $|f(a_i)| > 1$, and, by (4.2),

$$\big|a_i\colon|\lambda_i|\big| = |t_i|\,\big|a_i'\colon|\lambda_i|\big| \leqslant \frac{1}{|\lambda_i|}.$$

This implies, by definition of $\big|a_i\colon|\lambda_i|\big|$ in (3.1) that

$$|a_{i0}| \leqslant \frac{1}{|\lambda_i|}, \qquad |a_{in}|^{1/n}\,|\lambda_i| \leqslant \frac{1}{|\lambda_i|^{1/n}}, \qquad n \geqslant 1,$$

i.e.

$$|a_{in}|^{1/n}\,|\lambda_i| \leqslant A\left(\frac{1}{|\lambda_i|}\right). \tag{4.4}$$

Since $|\lambda_i| \to \infty$, we can assume that $|\lambda_i| \geqslant 1$ for all i. Then (4.4) gives that $|a_{in}|^{1/n}|\lambda_i| \leqslant 1$, i.e. $|a_{in}|^{1/n} \leqslant 1/|\lambda_i|$ for each $n \geqslant 1$. Thus we have

$$|a_{i0}| \leqslant \frac{1}{|\lambda_i|}, \qquad |a_{in}|^{1/n} \leqslant \frac{1}{|\lambda_i|}, \qquad n \geqslant 1.$$

Therefore, by definition of $|a_i|$ in (2.2), $|a_i| \leqslant 1/|\lambda_i|$. But $1/|\lambda_i| \to 0$ as $i \to \infty$. Therefore $|a_i| \to 0$ as $i \to \infty$. On the other hand, $|f(a_i)| > 1$, i.e. $f(a)$ is not continuous on S in the topology of $K\langle x\rangle$. This proves the result.

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An L^1-algebra for algebraically irreducible semigroups *

by

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1.1. Introduction. This paper is another chapter in the theory of L^1-algebras of linearly quasi-ordered semigroups. Algebraically irreducible commutative semigroups are known to be a special case of linearly quasi-ordered semigroups, but the structure of the semigroup over an idempotent arc in the decomposition space $S/\mathscr{L}$ is more amenable for the algebraically irreducible semigroups (see Theorem 1.3). Adapting the work of Lardy [4] on $L^1(a, b)$, where (a, b) is an idempotent commutative semigroup and using Lebesgue measure on (a, b) we introduce a measure M on the algebraically irreducible semigroups S for which $S/\mathscr{L}$ is an idempotent semigroup. We show that $L^1(S, M)$ is semisimple and that the multiplicative linear functionals (maximal ideal space) of this algebra is in one-to-one correspondence with the measurable semicharacters on S. We conclude the paper with some remarks as to the extension of the results here to a wider class of linearly quasi-ordered semigroups. Our work here was motivated by Lardy [4] and the remarks in Rothman [7] about assigning measure zero to idempotent arcs in $S/\mathscr{L}$. The methods of [6] and [7] are used. While the notation here is different, it is clear that it is in agreement with that of [6] and [7] when passing from functions in $L^1(S, M)$ to the corresponding measures in $M(S)$.

1.2. Definitions and basic theorems. In what follows, a *semigroup* S is a Hausdorff topological space together with a continuous associative multiplication. We shall use 1 to denote the identity element, K to denote the minimal ideal (which exists in S is compact [9]), and H to denote the maximal subgroup of S with identity 1.

A compact connected semigroup S is *algebraically irreducible about* $B \subset S$ if S contains no proper closed connected subsemigroup containing B. In particular, a compact connected abelian semigroup with an identity element, 1, algebraically irreducible about $K \cup H$ will be called an *A-I semigroup* [5]. We use the left equivalence of Green [2] defined for S

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