Formal expansion of the product $(x+i0)^{-n}(x-i0)^{-r}$ gives

$$\left\{x^{-n}+\frac{i\pi(-1)^n}{(n-1)!}\delta^{(n-1)}\right\}\left\{x^{-r}-\frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}\right\}$$
$$=\left\{x^{-n}x^{-r}+\frac{\pi^2(-1)^r}{(n-1)!(r-1)!}\delta^{(n-1)}\delta^{(r-1)}\right\}+$$
$$+i\pi\left\{\frac{(-1)^n}{(n-1)!}\delta^{(n-1)}x^{-r}-\frac{(-1)^r}{(r-1)!}\delta^{(r-1)}x^{-n}\right\}$$

and so both real and imaginary parts are divergent except when $n=r$ and in this case the imaginary part is zero. We will, however, have

$$2x^{-n}x^{-r}+\frac{2i\pi(-1)^n}{(n-1)!}\delta^{(n-1)}x^{-r}-(x+i0)^{-n}(x-i0)^{-r}$$
$$=x^{-n-r}+\frac{i\pi(-1)^{n+r}}{(n+r-1)!}\delta^{(n+r-1)}$$

and in particular when $n=r$

$$2(x^{-n})^2-(x+i0)^{-n}(x-i0)^{-n}=x^{-2n}.$$

References

[1] I. M. Gelfand and G. E. Shilov, *Generalised functions*, Vol. I, 1964.

[2] J. Mikusiński, *On the square of the Dirac delta-distribution*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 14 (1966), p. 511-513.

Reçu par la Rédaction le 28. 8. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

Two renorming constructions related to a question of Anselone

by

V. KLEE (Seattle)*

To Professors S. Mazur and W. Orlicz on the fortieth anniversary of their scientific research

INTRODUCTION

Let X denote a normed linear space and X^* its conjugate space. For any point x of X let x^c denote the set of all points of X^* conjugate to x; that is, $y\in x^c$ if and only if $y\in X^*$, $\|y\|=\|x\|$, and $\langle x,y\rangle=\|x\|^2$. Let us say that X has the *A-property* provided that for each totally bounded subset T of X, the restriction of c to T admits a selection with totally bounded range; that is, there is a function s on T to X^* such that $s(t)\in t^c$ for all $t\in T$ and the set sT is totally bounded. This property was introduced by Anselone [1] in studing the total boundedness of sets of linear operators into X. Plainly, every finite-dimensional X has the A-property. Anselone [1] noted that X has the A-property if X^* is uniformly rotund and asked whether all normed spaces have the A-property. Here the question is resolved with the aid of an adaptation of a construction of Mazur and Sternbach [4] by showing that

Every infinite-dimensional Banach space can be renormed so as to lack the A-property.

On the other hand, the following problem is unsettled:

Can every Banach space (or at least every separable one) be renormed so as to have the A-property?

When X is complete the closure of any totally bounded subset of X is compact. For the A-property it then suffices to assume that the function c is single-valued and continuous or, equivalently, that the unit sphere $S=\{x: \|x\|=1\}$ is Fréchet-smooth at each point. This is weaker than uniform rotundity of X^*, which is equivalent to uniform Fréchet-smooth-

* Research supported in part by the Office of Naval Research, U. S. A. (NSF-GP-3579).

ness of S. (See Mazur [3] and Day [2] for a general discussion of smoothness of unit spheres and differentiability of norms.) It is natural to ask whether Gateaux-smoothness is sufficient. Here the question is resolved by showing that

There is a renormed version of l^2 which lacks the A-property even though its unit sphere is everywhere Gateaux-smooth and is Fréchet-smooth except at two points.

Along with Phelps's example [5] of a renormed version of l^1 whose unit sphere is everywhere Gateaux-smooth but nowhere Frechet-smooth, this is of interest in connection with Mazur's question [3] concerning the relationship between Fréchet-smoothness and Gateaux-smoothness.

The A-property fails in a very simple way for the spaces constructed here. In each case there is a convergent sequence $x_1, x_2, \ldots$ of points of the unit sphere such that $\|y_i - y_j\| > \frac{1}{2}$ whenever $y_i \in x_i^c$, $y_j \in x_j^c$, and $i \neq j$.

I am indebted for a helpful comment to Drs. E. Heil and P. Mani.

RENORMING SO AS TO LACK THE A-PROPERTY

Consider an arbitrary infinite-dimensional Banach space X and let V be a (closed) hyperplane through the origin in X. We want to produce a closed linear subspace W of V and infinite biorthogonal sequences $w_1, w_2, \ldots$, in W and $f_1'', f_2'', \ldots$ in W^* such that the following three conditions are satisfied:

(1) the linear hull of $\{w_1, w_2, \ldots\}$ is dense in W;

(2) $\|w_i\| = \|f_i''\| = \langle w_i, f_i''\rangle = 1$ for all i;

(3) $\langle w_i, f''\rangle = 0$ whenever $i \neq j$.

Let S denote the unit sphere of V for the original norm, and S^* the unit sphere of V^*. Choose $w_1 \in S$ and use the Hahn-Banach theorem to produce $f_1'' \in S^*$ with $\langle w, f''\rangle = 1$. Then proceed as follows. Having chosen $w_1, \ldots, w_n$ in S and $f_1'', \ldots, f_n''$ in S^* so that (2) and (3) hold for all $i, j \leqslant n$, let

$$L_n = \text{linear hull of } \{w_1, \ldots, w_n\},$$

$$M_n = \{x \in V: \langle x, f_1''\rangle = \ldots = \langle x, f_n''\rangle = 0\}.$$

As $\dim M_n > \dim L_n$, a theorem of Tikhomirov [6] guarantees the existence of $w_{n+1} \in S \cap M_n$ such that the flat $w_{n+1} + L_n$ includes no point of norm < 1. By the Hahn-Banach theorem there exists $f_{n+1}'' \in S^*$ such that $\langle w_{n+1} + x, f_{n+1}''\rangle = 1$ for all $x \in L_n$, and with this choice of w_{n+1} and f_{n+1}'' conditions (2) and (3) are satisfied for all $i, j \leqslant n+1$. Thus by induction there exist infinite sequences $w_1, w_2, \ldots$ in S and $f_1'', f_2'', \ldots$ in S^* satisfying (2) and (3). Let W be the closed linear hull of $\{w_1, w_2, \ldots\}$ and replace each f_i'' (without changing notation) by its restriction to W. Then (1), (2), and (3) are satisfied. This construction is an adaptation of one suggested by Mazur and Sternbach [4].

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of numbers with $0 < \varepsilon_i < 2^{-i}$ and let

$$C = \Big\{\sum_1^\infty \lambda_k w_k \colon |\lambda_i| \leqslant \varepsilon_i \text{ for all } i\Big\}.$$

Let U_V denote the (closed) unit ball of V for the original norm and let

$$U = \text{cl con}\big((w+C) \cup U_V \cup (-w-C)\big),$$

where w is a point of $X \sim V$. Then U is a bounded closed convex body in X with $U = -U$ and hence U is the unit ball for a new norm compatible with the original topology of X. Henceforth $\|\ \|$ will denote this new norm on X or subspaces of X, or conjugate norms induced by these. Let $\mu_1, \mu_2, \ldots$ be a sequence in $]0, 1[$ converging to 1 and let

$$q_i = \Big(\sum_{k \neq i} \mu_k \varepsilon_k w_k\Big) + \varepsilon_i w_i \in C.$$

Finally, let

$$x_i = \mu_i(w + q_i) + (1-\mu_i) w_i \in U.$$

The sequence $x_1, x_2, \ldots$ converges to the point $w + \sum_1^\infty \varepsilon_k w_k$. We show below that if $y_i \in x_i^c$ the restriction of y_i to W is equal to f_i''. It then follows from (2) and (3) that $\|y_i - y_j\| \geqslant 1$ whenever $i \neq j$ and thus the new norm has the properties claimed for it.

Note that $\|w\| = 1$, that each point x of X admits a unique expression in the form $x = v(x) + f(x) w$ with $v(x) \in V$ and $f(x)$ real, and that the functional f belongs to X^* with $\|f\| = 1$. Since $C \subset U_V$, it is easily verified that $U \cap V = U_V$ and hence $\|w_i\| = \|f_i''\| = 1$ in the new norm as well as the old. For each i the Hahn-Banach theorem guarantees the existence of $f_i' \in V^*$ with $f_i'' \subset f_i'$ and $\|f_i'\| = 1$. Let $f_i = f_i' \circ v$, so that $f_i \in X^*$ with $f_i'' \subset f_i$ and $\|f_i\| = 1$, and let $g_i = (1-\varepsilon_i) f + f_i$. By routine computation,

$$\langle w + q_i, g_i\rangle = \langle w_i, g_i\rangle = 1,$$

while

$$\langle x, g_i\rangle \leqslant 1 \quad \text{for all } x \in (w+C) \cup U_V \cup (-w-C).$$

Hence the set $\{x \in X: \langle x, g_i \rangle = 1\}$ is a supporting hyperplane of U and the segment $[w+q_i, w_i]$ lies in the unit sphere S for the new norm. In particular, $x_i \in S$. Now consider an arbitrary member y_i of x_i^c and note

(4) the U-maximum of y_i is 1, attained at x_i and hence also at $w+q_i$ and w_i.

From (4) it follows that the $(w+C)$-maximum of y_i is attained at $w+q_i$, whence the C-maximum of y_i is attained at q_i. This implies $\langle w_j, y_i \rangle = 0$ for all $j \neq i$, for q_i is the average of the points $q_i \pm (1-\mu_j)\varepsilon_j w_j$ of C. It then follows from (1)-(3) that the restriction of y_i to W is a multiple of f_i''. By (4), however, the $(U_V \cap W)$-maximum of y_i is 1, attained at w_i, whence the restriction of y_i to W is equal to f_i'' and the proof is complete.

GATEAUX-SMOOTHNESS AND FRÉCHET-SMOOTHNESS

Recall that a real-valued function γ on a normed space X is said to be *Gateaux-differentiable* (*weakly differentiable* in the sense of Mazur [3]) at a point z_0 provided that there exists a continuous linear functional $f \in X^*$ such that if

(5) $$\varepsilon(x) = \big(\gamma(z_0+x) - \gamma(z_0) - f(x)\big)/\|x\|,$$

then $\lim_{x \in R, x \to 0} \varepsilon(x) = 0$ for every ray R issuing from 0 in X. The function γ is said to be *Fréchet-differentiable* (*strongly differentiable* in the sense of Mazur [3]) at z_0 provided that there exists $f \in X^*$ with $\lim_{x \in X, x \to 0} \varepsilon(x) = 0$. For our present purposes it is convenient to work directly with smoothness properties of sets rather than differentiability properties of functions. Suppose that z_0 is a point of a subset Z of a normed linear space. A (closed) hyperplane H is said to be a *G-tangent* of Z at z_0 provided that if

(6) $$\sigma(h) = \delta(h, Z)/\|h - z_0\|,$$

then $\lim_{h \in R, h \to z_0} \sigma(h) = 0$ for every ray R issuing from 0 in X; and H is an *F-tangent* of Z at z_0 provided that $\lim_{h \in H, h \to z_0} \sigma(h) = 0$. (Here $\delta(h, Z) = \inf_{z \in Z} \|h - z\|$, the distance from the point h to the set Z). The set Z is said to be *G-smooth* (or *Gateaux-smooth*) at z_0 provided that Z admits a unique G-tangent at z_0 and to be *F-smooth* (or *Fréchet-smooth*) at z_0 provided that Z admits a unique F-tangent at z_0.

THEOREM. *Suppose that z_0 is a point of a convex subset Z of a normed linear space X. Then Z is G-smooth at z_0 if and only if there is a unique hyperplane H supporting Z at z_0. (H is then the G-tangent of Z at z_0.) If*

Z has an interior point p, then Z is G-smooth at z_0 if and only if z_0 is in the boundary of Z and the gauge functional of Z relative to p is Gateaux-differentiable at z_0. (The G-tangent of Z at z_0 is then $\{x: f(x) = 1\}$, where f is as in (5).)

Proof. Assume for notational simplicity that $z_0 = 0$. Note that

(7) If a set Z is supported at z_0 by a hyperplane H, then no other hyperplane is a G-tangent of Z at z_0.

To prove (7), let Q be an open halfspace which misses S and has boundary H. Any hyperplane through z_0 other than H includes a point q of Q and hence contains the ray $\{\lambda q: \lambda > 0\}$. But then

$$\frac{\delta(\lambda q, Z)}{\|\lambda q - z_0\|} \geqslant \frac{\delta(\lambda q, H)}{\|\lambda q\|} = \frac{\delta(q, H)}{\|q\|} > 0$$

and the desired conclusion follows.

Now suppose that Z is G-smooth at z_0, whence there exists $q \in X \sim \{0\}$ and $\varepsilon > 0$ such that

(8) $$\delta(\lambda q, Z)/\|\lambda q\| > \varepsilon$$

for positive values of λ arbitrarily close to 0. (Otherwise *every* hyperplane through z_0 would be an F-tangent of Z at z_0.) Suppose that Z is convex, consider an arbitrary $\mu > 0$, and choose $\lambda \in]0, \mu[$ such that (8) holds. Then $(\lambda/\mu)Z \subset Z$ by convexity (for $0 = z_0 \in Z$) and it follows that

$$\frac{\delta(\mu q, Z)}{\|\mu q\|} = \frac{\delta\big(\lambda q, (\lambda/\mu)Z\big)}{\|\lambda q\|} \geqslant \frac{\delta(\lambda q, Z)}{\|\lambda q\|} > \varepsilon.$$

Hence the convex set Z is disjoint from the open convex cone

$$\bigcup_{\mu > 0} \{x \in X: \|x - \mu q\| < \varepsilon \|\mu q\|\} =]0, \infty[\{x: \|x - q\| < \varepsilon \|q\|\}$$

and the two convex sets are separated by a hyperplane. As any such hyperplane supports Z at z_0, it follows from (7) and the G-smoothness of Z at z_0 that there is a unique hyperplane H supporting Z at z_0 and H is the G-tangent of Z at z_0.

Now suppose, conversely, that Z is convex and there is a unique hyperplane H supporting Z at z_0. It follows from (7) that Z admits at most one G-tangent at z_0. And H itself is such a tangent, for otherwise the reasoning of the preceding paragraph applies to a point q of $H \sim \{z_0\}$ and the resulting separating hyperplane contradicts the uniqueness of H. It follows that Z is G-smooth at z_0.

The remainder of the theorem follows from the well-known equivalence between Gateaux-differentiability of gauge functionals and uniqueness of supporting hyperplanes (Mazur [3], Day [2]).

THEOREM. *Suppose that z_0 is a point of a convex subset Z of a normed linear space X. Then Z is F-smooth at z_0 if and only if one of the following two conditions is satisfied:*

(9) *Z is contained in a hyperplane H and z_0 is interior to* $\mathrm{cl}\,Z$ *relative to H;*

(10) $\mathrm{cl}\,Z$ *has an interior point p, z_0 is in the boundary of* $\mathrm{cl}\,Z$, *and the gauge functional of* $\mathrm{cl}\,Z$ *relative to p is Fréchet-differentiable at z_0.*

Proof. Suppose that Z is convex and F-smooth at z_0, let H be the F-tangent of Z at z_0, and assume as before that $z_0 = 0$. We claim that

(11) if $q \in X \sim H$ and if $f_1, f_2, \ldots$ is a sequence in X^* with $f_n(q) \to 0$ and $\|f_n\| \to 1$ as $n \to \infty$, then $\liminf\limits_{n\to\infty}(\sup f_n Z) > 0$.

Indeed, from $f_n(q) \to 0$ and $\|f_n\| \to 1$ it follows that the norm of f_n's restriction to H converges to 1 as $n \to \infty$; hence there is a sequence $h_1, h_2, \ldots$ in H such that $\|h_n\| = 1$ and $f_n(h_n) \to 1$. Now for each $\lambda > 0$,

$$\delta(\lambda h_n, Z) \geqslant f(\lambda h_n) - \sup f_n Z$$

for all n, and if the limit inferior of $\sup f_n Z$ is 0 there exists $n(\lambda)$ such that $\delta(\lambda h_{n(\lambda)}, Z) > \lambda/2$. This contradicts the fact that H is an F-tangent of Z at z_0.

Now suppose that Z is not contained in H and choose $q \in Z \sim H$. Then the point $q/2$ is interior to $\mathrm{cl}\,Z$. For, if not, $q/2$ is the limit of a sequence $p_1, p_2, \ldots$ in $X \sim \mathrm{cl}\,Z$, and by a standard separation theorem there is a sequence $f_1, f_2, \ldots$ in X^* such that $\|f_n\| = 1$ and $\sup f_n Z < f_n(p_n)$. Since

$$f_n(q/2) = \tfrac{1}{2} f_n(q) < f_n(q) < f_n(p_n)$$

and $f_n(p_n) \to f_n(q/2)$, it follows that $f_n(q) \to 0, f_n(p_n) \to 0$, and (11) is contradicted. Hence $q/2 \in \mathrm{int}\,\mathrm{cl}\,Z$. A similar but simpler argument, also based on (11), shows that if $Z \subset H$, then z_0 is interior to $\mathrm{cl}\,Z$ relative to H.

The preceding two paragraphs show that if Z is F-smooth at z_0, then (9) holds or $\mathrm{cl}\,Z$ has non-empty interior. Plainly, (9) implies the F-smoothness of Z at z_0 and the latter implies z_0 is a boundary point of Z. To complete the proof it suffices to show that if z_0 is a boundary point and p an interior point of a closed convex body Z, then Z is F-smooth at z_0 if and only if the gauge-functional γ of Z relative to p is Fréchet-differentiable at z_0; in doing this we assume for notational convenience that $p = 0$.

Suppose first that γ is Fréchet-differentiable at z_0. Let f and ε be as in (5) and let $H = \{x\colon f(x) = 1\}$, the unique supporting hyperplane of Z at z_0. For $h \in H$ we have $\gamma(h) \geqslant 1$ and $h/\gamma(h) \in Z$, whence

$$\sigma(h) = \frac{\delta(h, Z)}{\|h - z_0\|} \leqslant \frac{\|h - h/\gamma(h)\|}{\|h - z_0\|} = \frac{\gamma(h) - \gamma(z_0)}{\|h - z\|}\,\frac{\|h\|}{\gamma(h)}$$
$$= \varepsilon(h - z_0) + \frac{f(h - z_0)}{\|h - z_0\|}\,\frac{\|h\|}{\gamma(h)}.$$

But $f(h) = f(z_0)$ for $h \in H$, and $\varepsilon(h - z_0) \to 0$, $\|h\| \to \|z_0\|$, and $\gamma(h) \to \gamma(z_0)$ as $h \to z_0$. Hence $\sigma(h) \to 0$ as $h \to z_0$ and Z is *F*-smooth at z_0.

Now suppose, conversely, that Z is F-smooth at z_0, let H be the unique hyperplane supporting Z at z_0, and let $f \in X^*$ with $H = \{x\colon f(x) = 1\}$. Defining ε by (5), we want to show

(12) $$\varepsilon(x) \to 0 \quad \text{as } x \to 0.$$

For each point x of X, let $v(x) = x - f(x) z_0$, whence $f(v(x)) = 0$. As X is both algebraically and topologically the direct sum of the hyperplane $\{x\colon f(x) = 0\}$ and the line $\{x\colon v(x) = 0\}$, there is a finite M such that

(13) $$(\|v(x)\| + \|f(x)\|)/\|x\| < M \quad \text{for all } x \in X \sim \{0\}.$$

Note also that

(14) $$f \leqslant \gamma,$$

for this inequality plainly holds on H, while f is homogeneous and γ is non-negative and positively homogeneous. For all x such that $f(x) > 0$, it is a consequence of (13), (14), the positive homogeneity and subadditivity of γ, and the fact that $\gamma(z_0) = 1 = f(z_0)$, that

$$0 \leqslant \varepsilon(x) = \frac{\gamma\big(v(x) + f(x) z_0 + z_0\big) - \gamma(z_0) - f\big(v(x) + f(x) z_0\big)}{\|x\|}$$
$$\leqslant \frac{\gamma\big(v(x) + z_0\big) - \gamma(z_0) - f\big(v(x)\big)}{\|v(x)\|}\,\frac{\|v(x)\|}{\|v(x)\| + \|f(x) v_0\|}\,\frac{\|v(x)\| + \|f(x) v_0\|}{\|x\|}$$
$$< M\varepsilon\big(v(x)\big).$$

Since $v(x) \to 0$ as $x \to 0$, it therefore suffices in proving (12) to consider those x for which $f(x) = 0$. For each such x, choose $z(x)$ in the boundary of Z such that

$$\|z_0 + x - z(x)\| \leqslant 2\delta(z_0 + x, Z);$$

note that $z_0 + x \in H$ and hence, by F-smoothness,

(15) $$\|z_0 + x - z(x)\|/\|x\| \to 0 \quad \text{as } x \to 0.$$

By the subadditivity of γ,

$$\gamma\big(z(x)\big)-\gamma\big(z(x)-z_0-x\big) \leqslant \gamma(z_0+x) \leqslant \gamma\big(z(x)\big)+\gamma\big(z_0+x-z(x)\big).$$

As $\gamma(z_0)=1=\gamma\big(z(x)\big)$ and $f(x)=0$, it follows from the definition (5) that

$$(16)\qquad -\gamma\big(z(x)-z_0-x\big) \leqslant \|x\|\,\varepsilon(x) \leqslant \gamma\big(z(x)-z_0-x\big)$$

Being convex and continuous, γ is majorized by a multiple of $\|\ \|$, whence it follows from (15) and (16) that $\varepsilon(x)\to 0$ as $x\to 0$. This completes the proof of the theorem.

The following is an immediate consequence of the preceding two theorems:

COROLLARY. *If a convex set Z is F-smooth at a point z_0, then it is also G-smooth at z_0.*

Note that the corollary does not apply to all sets Z. Indeed, let Z' be a convex set which has at z_0 a unique G-tangent H' but no F-tangent, and let Z'' be a convex set which has at z_0 a unique F-tangent H'' different from H'. Then the set $Z'\cup Z''$ is F-smooth at z_0 but it is not G-smooth, for both H' and H'' are G-tangents of $Z'\cup Z''$ at z_0.

A set Z is said to be *G-smooth* [resp. *F-smooth*] at a subset Z_0 of Z provided that it is G-smooth [resp. F-smooth] at each point of Z_0. And Z is said to be *uniformly F-smooth at* Z_0 provided that Z admits a unique F-tangent $H(z_0)$ at each point z_0 of Z_0 and there exists ξ such that

(17) ξ is a function on $]0,\infty[$ to $]0,\infty[$ with $\lim\limits_{\lambda\to 0^+}\xi(\lambda)=0$

and for all $z_0\in Z_0$ and $h\in H(z_0)$ it is true that

$$(18)\qquad \delta(h,Z)\leqslant \xi(\|h-z_0\|)\,\|h-z_0\|.$$

This situation is also described by saying that Z is *ξ-smooth* at Z_0.

THEOREM. *Suppose that Z is a weakly compact subset of a Banach space, $C=\mathrm{cl\,con}\,Z$, and H is a hyperplane supporting C. Then G-smoothness or uniform F-smoothness of Z at $Z\cap H$ implies that of C at $C\cap H$.*

Proof. By Phillips' version of a theorem of Krein (see [2], p. 55), the set C is weakly compact and hence of course $C\cap H$ is weakly compact. By the Krein-Milman theorem, $C\cap H$ is the closed convex hull of its extreme points. Each extreme point of $C\cap H$ is an extreme point of C and hence, by Milman's theorem, belongs to Z. It follows then that $C\cap H=\mathrm{cl\,con}(Z\cap H)$. If C is not G-smooth at $C\cap H$ there is a point of $C\cap H$ which lies on another supporting hyperplane H' of C. Relative to H, $H'\cap H$ is a supporting hyperplane of $C\cap H$ and the preceding reasoning shows

$$C\cap H'\cap H=\mathrm{cl\,con}\,(Z\cap H'\cap H).$$

In particular, $H'\cap H$ includes a point z_0 of Z and Z is supported at z_0 by both H' and H. This contradicts the assumption that Z is G-smooth at $Z\cap H$.

Now suppose that Z is uniformly F-smooth at $Z\cap H$ and let ξ be as above. Since Z is supported by H at each point of $Z\cap H$, and since F-smoothness implies G-smoothness, the $H(z_0)$ above (in the definition of uniform F-smoothness) is in fact equal to H for all $z_0\in Z\cap H$. To show that C is uniformly F-smooth at $C\cap H$ we show

$$(19)\qquad \delta(h,C)<\xi(\|h-c_0\|)\,\|h-c_0\|+2\varepsilon$$

for all $c_0\in C\cap H$, $h\in H$, and $\varepsilon>0$. As $C\cap H=\mathrm{cl\,con}\,(Z\cap H)$, there are points $z_1,\ldots,z_n$ of $Z\cap H$ and positive numbers $\mu_1,\ldots,\mu_n$ such that

$$(20)\qquad \sum_1^n \mu_k=1,$$

$$(21)\qquad \Big\|c_0-\sum_1^n \mu_k z_k\Big\|<\varepsilon.$$

Since $z_k+h-c_0\in H$, it follows from (18) (with the roles of h and z_0 in (18) played by z_k+h-c_0 and z_i respectively) that

$$\delta(z_k+h-c_0,Z)\leqslant \xi(\|h-c_0\|)\,\|h-c_0\|$$

and hence there exists $w_k\in Z$ such that

$$(22)\qquad \|z_k+h-c_0-w_k\|\leqslant \xi(\|h-c_0\|)\,\|h-c_0\|+\varepsilon.$$

Now use (20), (21), and (22) to show that

$$\Big\|h-\sum_1^n \mu_k w_k\Big\|\leqslant\Big\|h-\sum_1^n \mu_k w_k-c_0+\sum_1^n \mu_k z_k\Big\|+\varepsilon$$
$$\leqslant \sum_1^n \mu_k\|z_k+h-c_0-w_k\|+\varepsilon\leqslant \sigma(\|h-c_0\|)\,\|h-c_0\|+2\varepsilon,$$

whence (19) follows from the fact that $\sum\limits_1^n \mu_i w_i\in C$.

A SMOOTH RENORMING OF HILBERT SPACE WHICH LACKS THE A-PROPERTY

We now proceed with the promised renorming of l^2, whose points are sequences $x=(x_0,x_1,x_2,\ldots)$ of real numbers with $\sum\limits_0^\infty x_k^2<\infty$. The new norm is described in detail but the proof that it has the stated properties is given somewhat sketchily, for several of its steps are routine. Let the hyperplane $\{x\in l^2\colon x_0=0\}$ be denoted by V, its unit ball and unit sphere by U_V and S_V respectively. For each bounded sequence $a = (a_1, a_2, \ldots)$ of real numbers, let T_a denote the linear transformation of V into V given by

$$T_a(x) = (0, a_1x_1, a_2x_2, \ldots) \quad (x \in V);$$

it is a self-homeomorphism of V if $\inf_k |a_k| > 0$. For each $\lambda \in [-1, 1]$ and for each sequence $\eta = (\eta_1, \eta_2, \ldots)$ of even functions on $[-1, 1]$ to $[0, 1]$ with $\eta_i(0) = 1$ for all i, let $\eta(\lambda) = (\eta_1(\lambda), \eta_2(\lambda), \ldots)$. Then let

$$U_\eta = \bigcup_{|\lambda| \leqslant 1} \lambda\delta_0 + T_{\eta(\lambda)} U_V, \qquad S_\eta = \bigcup_{|\lambda| \leqslant 1} \lambda\delta_0 + T_{\eta(\lambda)} S_V,$$

$$U = \operatorname{cl} \operatorname{con} U_\eta, \qquad S = \text{boundary of } U.$$

(Here $\delta_0 = (1, 0, 0, \ldots)$.) As U is a bounded closed convex body in l^2 with $U = -U$, U and S are respectively the unit ball and the unit sphere of l^2 with respect to a new norm compatible with the original topology. Note that $\delta_0 \in S$. If $\eta_i(\lambda) = \sqrt{1-\lambda^2}$ for all i and λ, then $S = S_\eta$, S is the usual unit sphere of l^2, and S is uniformly F-smooth. We shall describe a sequence $\eta_1, \eta_2, \ldots$ for which the resulting S is G-smooth but not F-smooth at δ_0 and $-\delta_0$, is F-smooth at all other points (in fact, uniformly F-smooth at every closed subset of $S \sim \{\delta_0, -\delta_0\}$), and yet the renormed version of l^2 lacks the A-property.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence in $]0, \frac{1}{6}[$ converging to 0 and let $\eta_1, \eta_2, \ldots$ be even functions on $[0, 1]$ to $[0, 1]$ such that the following conditions are satisfied:

(23) η_i is continuous and concave, with $\eta_i(0) = 1$, $\eta_i(1-\varepsilon_i) = 2\varepsilon_i$, and $\eta_i(1) = 0$;

(24) η_i is differentiable on $[0, 1]$, with $\eta_i'(0) = 0$ and $\eta_i'(1-\varepsilon_i) = -1$;

(25) η_i has a vertical tangent at 1; that is, $\lim_{\lambda \to 1^-} \eta_i'(\lambda) = -\infty$.

As η_i is strictly positive on $[0, 1[$, it follows that

$$U_\eta = \{-\delta_0, \delta_0\} \cup \{x \in l^2\colon |x_0| < 1 \text{ and } \sum_1^\infty (x_k/\eta_k(x_0))^2 \leqslant 1\}$$

and from this that the set U_η is weakly closed. Hence U_η is weakly compact and it follows that

$$U \cap H = \operatorname{cl} \operatorname{con}(U_\eta \cap H) \tag{26}$$

for every supporting hyperplane H of U. In particular,

$$U \cap (\pm\delta_0 + V) = \{\pm\delta_0\}$$

and it follows from (25) that U_η and U are both G-smooth at δ_0 and $-\delta_0$.

The remainder of the proof requires an examination of the intersections of S_η with the various planes (2-flats) through the line $L = \{\lambda\delta_0\colon -\infty < \lambda < \infty\}$ and with the various hyperplanes parallel to V. Consider, for an arbitrary $s \in S_V$, the intersection of S_η with the halfplane $P_s = L + [0, \infty[s$. It is

$$\{\lambda\delta_0 + \tau_s(\lambda)s\colon |\lambda| \leqslant 1\},$$

where $\tau_s(\pm 1) = 0$ and for $|\lambda| < 1$ the number $\tau_s(\lambda)$ is the positive solution of $\sum_1^\infty (\tau_s(\lambda)s_k/\eta_k(\lambda))^2 = 1$; that is,

$$\tau_s = \Big(\sum_1^\infty s_k^2\eta_k^{-2}\Big)^{-1/2} \quad \text{on }]-1, 1[. \tag{27}$$

Fixing our attention on an arbitrary number $\bar\lambda \in]0, 1[$, we claim

(28) the derivatives τ_s', for $s \in S_V$, exist and are equicontinuous on $]-\bar\lambda, \bar\lambda[$.

To verify (28), let $\varrho_s = \tau_s^{-2} = \sum_1^\infty s_k^2\eta_k^{-2}$. It follows from (24) and (25) that on $]-\bar\lambda, \bar\lambda[$ the functions $\eta_1, \eta_2, \ldots$ are equicontinuous and uniformly bounded away from both 0 and ∞, and the derivatives $\eta_1', \eta_2', \ldots$ are equicontinuous and uniformly bounded. Hence ϱ_s is differentiable and

$$\varrho_s' = -2\sum_1^\infty s_k^2\eta_k^{-3}\eta_k',$$

whence the functions ϱ_s' are equicontinuous for $s \in S_V$. Then (28) follows from the fact that $\tau_s' = -\frac{1}{2}\varrho_s^{-3/2}\varrho_s'$. A consequence of (28) is

(29) the curves τ_s, for $s \in S_V$, are equi-F-smooth on $]-\bar\lambda, \bar\lambda[$;

more specifically, there is a function ξ satisfying (17) such that each curve τ_s is ξ-smooth (relative to the plane containing P_s) at each point $\lambda\delta_0 + \tau_s(\lambda)s$ with $|\lambda| < \bar\lambda$. Note also that

(30) the "spheres" $T_{\eta(\lambda)}S_V$, for $|\lambda| < \bar\lambda$, are equi-F-smooth;

this smoothness (relative to V) follows from the fact that S_V is uniformly F-smooth and the linear homeomorphisms $T_{\eta(\lambda)}$, for $|\lambda| < \bar\lambda$, are uniformly bounded with uniformly bounded inverses.

To establish the F-smoothness of S at each point p of $S \sim \{\delta_0, -\delta_0\}$, note that by (29) and (30) there is a unique hyperplane H supporting S at p. From U's weak compactness, the G-smoothness of U at $\pm\delta_0$, and the fact that $U \cap (\pm\delta_0 + V) = \{\pm\delta_0\}$, there follows the existence of $\bar\lambda \in]0, 1[$ such that

$$H \cap U \subset \{x\colon |x_0| < \bar\lambda\}.$$

The uniform F-smoothness of S at H then follows from (29), (30), and the last theorem of the preceding section.

It remains only to show that the renormed version of l^2 lacks the A-property. For $0 \leqslant i < \infty$, let δ_i denote the point of l^2 such that $\delta_{ij} = 1$ or 0 according as $j = i$ or $j \neq i$; let δ_i^* denote the same point considered as a member of the conjugate space $(l^2)^*$. Note that for $i = 1, 2, \ldots$ and for $|\lambda| < 1$, any hyperplane in V parallel to the hyperplane $V_i = \{x \in V\colon x_i = 0\}$ is carried onto such a parallel hyperplane by the transformation $T_{\eta(\lambda)}$. Note also that $\tau_{\delta_i} = \eta_i$. Since η_i is concave, and since S_V is supported at δ_i in V by a translate of V_i, it follows that U is supported at the point $\lambda\delta_0 + \eta_i(\lambda)\delta_i$ by a hyperplane which contains a translate of V_i and also contains the tangent to η_i at this point. In particular (using (23) and (24)), with $x_i = (1-\varepsilon_i)\delta_0 + 2\varepsilon_i\delta_i \in S$ and $\{y_i\} = x_i^c$ relative to the new norm $\|\ \|$, we have

$$y_0 = (1-3\varepsilon_i)^{-1}(\delta_i^* - \delta_i^*).$$

As $\varepsilon_i \in]0, \frac{1}{6}[$ and as $\delta_1, \delta_2, \ldots \in S$ it follows that $\|y_i - y_j\| > \frac{1}{2}$ for $i \neq j$. But of course $x_1, x_2, \ldots \to \delta_0$, so the proof is complete.

References

[1] P. M. Anselone, *A criterion for a set of linear operators to be totally bounded*, to appear.

[2] M. M. Day, *Normed linear spaces*, Berlin 1958.

[3] S. Mazur, *Über schwach Konvergenz in den Räumen* (L^p), Studia Math. 4 (1933), p. 128-133.

[4] — and L. Sternbach, *Über die Borelschen Typen von linearen Mengen*, ibidem 4 (1933), p. 48-53.

[5] R. R. Phelps, *A representation theorem for bounded convex sets*, Proc. Amer. Math. Soc. 11 (1960), p. 976-983.

[6] V. M. Tikhomirov, *Diameters of sets in function spaces and the theory of best approximations*, Russ. Math. Surveys 15 (1960), no. 3, p. 75-111. (Translated from Uspehi Mat. Nauk 15 (1960), no. 3 (93), p. 81-120.)

UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON, U. S. A.

Reçu par la Rédaction le 5. 10. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

A construction of basis in $C^{(1)}(I^2)$

by

Z. CIESIELSKI (Gdańsk)

The sequence $\{x_n, n = 1, 2, \ldots\}$ of elements of a given real Banach space $[X, \|\ \|]$ is called *basis* in X whenever each $x \in X$ has unique, convergent in the norm $\|\ \|$, expansion

$$x = \sum_{n=1}^{\infty} a_n x_n$$

with real coefficients $a_1, a_2, \ldots$ It is well known that the coefficients $a_n = a_n(x)$ are linear functionals over $[X, \|\ \|]$ and they are called *coefficient functionals* for the basis $\{x_n\}$.

There were two examples of seperable Banach spaces mentioned in the Banach monograph [1] (p. 238) for which it was not known how to construct bases. One of the examples is the space A of holomorphic functions in the interior and continuous on the boundary of the unit disc with uniform norm. The second example is the space $C^{(1)}(I^2)$, $I = \langle 0, 1\rangle$, of all functions with continuous partial derivatives of the first order on I^2 with the norm

$$\|x\|^{(1)} = \|x\| + \|D_1 x\| + \|D_2 x\|$$

where

$$\|x\| = \max\{|x(s,t)|\colon s, t \in I^2\},$$

$$D_1 x(s,t) = \frac{\partial x}{\partial s}(s,t) \quad \text{and} \quad D_2 x(s,t) = \frac{\partial x}{\partial t}(s,t).$$

The aim of this paper is to give an effective construction of a basis in the Banach space $[C^{(1)}(I^2), \|\ \|^{(1)}]$. It follows immediately from the construction that this result can be extended to the case of $C^{(1)}(I^n)$ with arbitrary $n \geqslant 1$.

The construction depends heavily on the properties of the Franklin orthonormal system $\{f_n, n = 0, 1, \ldots\}$.

To define the orthonormal Franklin system we need to recall the definition of the Schauder functions: $s_0 = 1$, $s_1(t) = t$ for $t \in I$, and for