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Products of generalized functions

by

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In this paper we will consider products of the generalised functions $(x\pm i0)^n$ which are defined for integer values of n by

$$(x\pm i0)^n = x^n$$

for $n = 0, 1, 2, \ldots$ and

$$(x\pm i0)^{-n} = x^{-n} \pm \frac{i\pi(-1)^n}{(n-1)!}\,\delta^{(n-1)}$$

for $n = 1, 2, \ldots$, see [1]. It follows immediately that

$$\frac{d}{dx}(x\pm i0)^n = n(x\pm i0)^{n-1}$$

for $n = 0, \pm 1, \pm 2, \ldots$

First of all if $r, n \geqslant 0$ we obviously have

$$(x\pm i0)^n (x\pm i0)^r = (x\pm i0)^{n+r}$$

and we have

$$\begin{aligned}(x\pm i0)^n (x\pm i0)^{-r} &= x^n\left\{x^{-r} \pm \frac{i\pi(-1)^r}{(r-1)!}\,\delta^{(r-1)}\right\}\\ &= x^{n-r} \pm \frac{i\pi(-1)^r}{(r-1)!}\,x^n\,\delta^{(r-1)}\\ &= \begin{cases} x^{n-r} & \text{for } n \geqslant r\\ x^{n-r} \pm \dfrac{i\pi(-1)^r}{(r-n-1)!}\,\delta^{(r-n-1)} & \text{for } r > n\end{cases}\\ &= (x\pm i0)^{n-r}.\end{aligned}$$

Now let us consider the square of $(x\pm i0)^{-1}$. Formally, we have

$$(x\pm i0)^{-1}(x\pm i0)^{-1} = (x^{-1} \mp i\pi\delta)^2 = \{(x^{-1})^2 - \pi^2\delta^2\} \mp 2i\pi\delta x^{-1}.$$

Although $(x^{-1})^2$ and δ^2 do not exist on their own, Mikusiński [2] has shown that $\{(x^{-1})^2 - \pi^2\delta^2\}$ exists if considered as a single entity and gives the result

$$(x^{-1})^2 - \pi^2\delta^2 = x^{-2}.$$

At the same time he gives the result

$$\delta x^{-1} = -\tfrac{1}{2}\delta'.$$

Thus, using these results, we have

$$(x \pm i0)^{-1}(x \pm i0)^{-1} = (x \pm i0)^{-2}.$$

We define further products by formal differentiation to obtain the result that

$$(x \pm i0)^{-n}(x \pm i0)^{-r} = (x \pm i0)^{-n-r},$$

for $n, r = 1, 2, \ldots$ We prove this result by induction since assuming this result for $n + r \leqslant m$ we have

$$\{(x \pm i0)^{-1}\}^m = (x \pm i0)^{-m}.$$

Formal differentiation gives

$$m\{(x \pm i0)^{-1}\}^{m-1}\frac{d}{dx}(x \pm i0)^{-1} = -m(x \pm i0)^{-m-1}$$

and hence we define

$$(x \pm i0)^{-m+1}(x \pm i0)^{-2} = (x \pm i0).^{-m-1}$$

Formal differentiation of the equation

$$(x \pm i0)^{-m+s}(x \pm i0)^{-s} = (x \pm i0)^{-m}$$

gives

$$(m-s)(x \pm i0)^{-m-1+s}(x \pm i0)^{-s} + s(x \pm i0)^{-m+s}(x \pm i0)^{-s-1} = m(x \pm i0)^{-m-1}.$$

Assuming the result that

$$(x \pm i0)^{-m-1+s}(x \pm i0)^{-s} = (x \pm i0)^{-m-1}$$

for some s it follows that

$$(x \pm i0)^{-m+s}(x \pm i0)^{-s-1} = (x \pm i0)^{-m-1}.$$

Since the assumption is true for $s = 2$ it follows by induction that the result is true for $s = 2, \ldots, m$. But the case $s = m$ is equivalent to the case $s = 1$ and so we have

$$(x \pm i0)^{-m-1+s}(x \pm i0)^{-s} = (x \pm i0)^{-m-1}$$

for $s = 1, 2, \ldots, m$ and by our first assumption we have

$$(x \pm i0)^{-n}(x \pm i0)^{-r} = (x \pm i0)^{-n-r}$$

for $n + r \leqslant m+1$. Our result now follows by induction. Thus

$$(x \pm i0)^{n}(x \pm i0)^{r} = (x \pm i0)^{n+r}$$

for $n, r = 0, \pm 1, \pm 2, \ldots$

Thus considering negative integers only we have

$$\begin{aligned}(x+i0)^{-n}(x+i0)^{-r} &= \left\{x^{-n} + \frac{i\pi(-1)^n}{(n-1)!}\delta^{(n-1)}\right\}\left\{x^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}\right\} \\ &= x^{-n}x^{-r} - \frac{\pi^2(-1)^{n+r}}{(n-1)!(r-1)!}\delta^{(n-1)}\delta^{(r-1)} + \\ &\quad + i\pi\left\{\frac{(-1)^n}{(n-1)!}\delta^{(n-1)}x^{-r} + \frac{(-1)^r}{(r-1)!}\delta^{(r-1)}x^{-n}\right\} \\ &= (x+i0)^{-n-r} \\ &= x^{-n-r} + \frac{i\pi(-1)^{n+r}}{(n+r-1)!}\delta^{(n+r-1)}.\end{aligned}$$

Equating real and imaginary parts we get

$$x^{-n}x^{-r} - \frac{\pi^2(-1)^{n+r}}{(n-1)!(r-1)!}\delta^{(n-1)}\delta^{(r-1)} = x^{-n-r},$$

$$\frac{(-1)^n}{(n-1)!}\delta^{(n-1)}x^{-r} + \frac{(-1)^r}{(r-1)!}\delta^{(r-1)}x^{-n} = \frac{(-1)^{n+r}}{(n+r-1)!}\delta^{(n+r-1)}$$

for $n, r = 1, 2, \ldots$

In particular, when $r = n$ we have

$$(x^{-n})^2 - \pi^2\{(n-1)!\}^{-2}\{\delta^{(n-1)}\}^2 = x^{-2n},$$

$$\delta^{(n-1)}x^{-n} = \frac{(-1)^n(n-1)!}{2(2n-1)!}\delta^{(2n-1)}.$$

It follows that

$$x^{-n}(x \pm i0)^{-n} - (x^{-n})^2 = \pm\frac{i\pi}{2(2n-1)!}\delta^{(2n-1)},$$

$$\delta^{(n-1)}(x \pm i0)^{-n} \mp \frac{i\pi(-1)^n}{(n-1)!}\{\delta^{(n-1)}\}^2 = \frac{(-1)^n(n-1)!}{2(2n-1)!}\delta^{(2n-1)}.$$

Formal expansion of the product $(x+i0)^{-n}(x-i0)^{-r}$ gives

$$\left\{x^{-n}+\frac{i\pi(-1)^n}{(n-1)!}\delta^{(n-1)}\right\}\left\{x^{-r}-\frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}\right\} = \left\{x^{-n}x^{-r}+\frac{\pi^2(-1)^r}{(n-1)!(r-1)!}\delta^{(n-1)}\delta^{(r-1)}\right\}+ + i\pi\left\{\frac{(-1)^n}{(n-1)!}\delta^{(n-1)}x^{-r}-\frac{(-1)^r}{(r-1)!}\delta^{(r-1)}x^{-n}\right\}$$

and so both real and imaginary parts are divergent except when $n=r$ and in this case the imaginary part is zero. We will, however, have

$$2x^{-n}x^{-r}+\frac{2i\pi(-1)^n}{(n-1)!}\delta^{(n-1)}x^{-r}-(x+i0)^{-n}(x-i0)^{-r} = x^{-n-r}+\frac{i\pi(-1)^{n+r}}{(n+r-1)!}\delta^{(n+r-1)}$$

and in particular when $n=r$

$$2(x^{-n})^2-(x+i0)^{-n}(x-i0)^{-n} = x^{-2n}.$$

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Two renorming constructions related to a question of Anselone

by

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To Professors S. Mazur and W. Orlicz
on the fortieth anniversary of their scientific research

INTRODUCTION

Let X denote a normed linear space and X^* its conjugate space. For any point x of X let x^c denote the set of all points of X^* conjugate to x; that is, $y\in x^c$ if and only if $y\in X^*$, $\|y\|=\|x\|$, and $\langle x, y\rangle = \|x\|^2$. Let us say that X has the *A-property* provided that for each totally bounded subset T of X, the restriction of c to T admits a selection with totally bounded range; that is, there is a function s on T to X^* such that $s(t)\in t^c$ for all $t\in T$ and the set sT is totally bounded. This property was introduced by Anselone [1] in studing the total boundedness of sets of linear operators into X. Plainly, every finite-dimensional X has the A-property. Anselone [1] noted that X has the A-property if X^* is uniformly rotund and asked whether all normed spaces have the A-property. Here the question is resolved with the aid of an adaptation of a construction of Mazur and Sternbach [4] by showing that

Every infinite-dimensional Banach space can be renormed so as to lack the A-property.

On the other hand, the following problem is unsettled:

Can every Banach space (or at least every separable one) be renormed so as to have the A-property?

When X is complete the closure of any totally bounded subset of X is compact. For the A-property it then suffices to assume that the function c is single-valued and continuous or, equivalently, that the unit sphere $S=\{x: \|x\|=1\}$ is Fréchet-smooth at each point. This is weaker than uniform rotundity of X^*, which is equivalent to uniform Fréchet-smooth-

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