Products of generalized functions

by

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In this paper we will consider products of the generalized functions 
$(x \pm \delta)^n$ which are defined for integer values of $n$ by 

$$(x \pm \delta)^n = x^n$$

for $n = 0, 1, 2, \ldots$ and

$$(x \pm \delta)^n = x^{-n} \pm \frac{\ln(-1)^n}{(n-1)!} \delta^{n-1}$$

for $n = 1, 2, \ldots$, see [1]. It follows immediately that

$$\frac{d}{dx} (x \pm \delta)^n = n (x \pm \delta)^{n-1}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

First of all if $r, s \geq 0$ we obviously have

$$(x \pm \delta)^r (x \pm \delta)^s = (x \pm \delta)^{r+s}$$

and we have

$$(x \pm \delta)^r (x \pm \delta)^{-r} = x^r \left\{ x^{-r} \pm \frac{\ln(-1)^r}{(r-1)!} \delta^{r-1} \right\}$$

$$= x^{n-r} \pm \frac{\ln(-1)^r}{(r-1)!} x^r \delta^{r-1}$$

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Now let us consider the square of $(x \pm \delta)^{-1}$. Formally, we have

$$(x \pm \delta)^{-1} (x \pm \delta)^{-1} = (x^{-1} \mp \delta)^2 = \left( (x^{-1})^2 - x^2 \delta^2 \right) \mp 2i \delta x^{-1}.$$
Although \((x^{-1})^2\) and \(\delta x\) do not exist on their own, Mikusiński [2] has shown that \(\{(x^{-1})^2 - \delta x\}^r\) exists if considered as a single entity and gives the result
\[(x^{-1})^2 - \pi^2 \delta x = x^{-1}.
\]
At the same time he gives the result
\[\delta x^{-1} = -\frac{1}{2} \delta x.
\]
Thus, using these results, we have
\[(x \pm i0)^{-1} (x \pm i0)^{-1} = (x \pm i0)^{-2}.
\]

We define further products by formal differentiation to obtain the result that
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = (x \pm i0)^{-n-r},
\]
for \(n, r = 1, 2, \ldots\). We prove this result by induction since assuming this result for \(n + r \leq m\) we have
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = (x \pm i0)^{-m}.
\]
Formal differentiation gives
\[m((x \pm i0)^{-1})^{m-1} \frac{d}{dx} (x \pm i0)^{-1} = -m(x \pm i0)^{-m-1},
\]
and hence we define
\[(x \pm i0)^{-n} (x \pm i0)^{-1} = (x \pm i0)^{-n-1}.
\]
Formal differentiation of the equation
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = (x \pm i0)^{-m}
\]
gives
\[(m - r)(x \pm i0)^{-m-1}(x \pm i0)^{-r} + s(x \pm i0)^{-n-r} = m(x \pm i0)^{-m-1}.
\]
Assuming the result that
\[(x \pm i0)^{-m-1} (x \pm i0)^{-r} = (x \pm i0)^{-m-1}
\]
for some \(s\) it follows that
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = (x \pm i0)^{-m-1}.
\]
Since the assumption is true for \(s = r\) it follows by induction that the result is true for \(s = 2, \ldots, m\). But the case \(s = m\) is equivalent to the case \(s = 1\) and so we have
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = (x \pm i0)^{-n-1}.
\]

for \(s = 1, 2, \ldots, m\) and by our first assumption we have
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = (x \pm i0)^{-n-r}
\]
for \(n + r \leq m + 1\). Our result now follows by induction. Thus
\[(x \pm i0)^n (x \pm i0)^r = (x \pm i0)^{n+r}
\]
for \(n, r = 0, 1, 2, \ldots\).

Thus considering negative integers only we have
\[(x \pm i0)^{-n} (x \pm i0)^{-r} = \frac{x^{-n}}{(n-1)!} \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{(n-k-1)!} \delta^{n-k} (x \pm i0)^{-r+k}
\]
\[= \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{(n-k-1)!} \delta^{n-k} (x \pm i0)^{-r+k}
\]
\[= (x \pm i0)^{-n-r} + \frac{i\pi}{(n+r-1)!} (x \pm i0)^{-n-r}.
\]
Equating real and imaginary parts we get
\[x^{-n} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{(n-k-1)!} \delta^{n-k} (x \pm i0)^{-r+k}
\]
\[= \frac{(-1)^{n-r} \delta^{n-r} (x \pm i0)^{-r}}{(n-1)!} + \frac{(-1)^r \delta^{n-r} (x \pm i0)^{-n}}{(n-1)!}
\]
for \(n, r = 1, 2, \ldots\).

In particular, when \(r = n\) we have
\[(x \pm i0)^{-n} - \frac{n^{-n}}{(n-1)!} \delta^{n-1} (x \pm i0)^{-n} = x^{-n},
\]
\[\delta^{n-1} (x \pm i0)^{-n} = \frac{(-1)^n (n-1)!}{2(n-1)!} \delta^{n-1} (x \pm i0)^{-n}.
\]
It follows that
\[x^{-n} (x \pm i0)^{-n} - \frac{n^{-n}}{(n-1)!} \delta^{n-1} (x \pm i0)^{-n} = \frac{i\pi}{2(n-1)!} \delta^{n-1} (x \pm i0)^{-n},
\]
\[\delta^{n-1} (x \pm i0)^{-n} = \frac{(-1)^n (n-1)!}{2(n-1)!} \delta^{n-1} (x \pm i0)^{-n}.
\]
Formal expansion of the product \((x+iy)^{-n}(x-iy)^{-r}\) gives
\[
\left\{ \begin{array}{l}
w^n + \frac{2\pi i}{n+1} \left\{ -\frac{\pi^2}{(n-1)!} \right\}^{\frac{1}{2}} \frac{\pi^{n-1} g^{n-1} g^{-(n-1)}}{r-1} \\
= \left\{ \begin{array}{l} w^n x^{-r} \\
\frac{x^n}{n-1} \frac{\pi^{n-1} g^{n-1} g^{-(n-1)}}{r-1}
\end{array} \right\}
\end{array} \right.
\]
and so both real and imaginary parts are divergent except when \(n = r\) and in this case the imaginary part is zero. We will, however, have
\[
2w^n x^{-r} + \frac{2\pi i}{n+1} \frac{\pi^{n-1} g^{n-1} g^{-(n-1)}}{r-1}
\]
and in particular when \(n = r\)
\[
2(w^n x - (x+iy)^{-n}(x-iy)^{-r}) = w^{n-x}
\]

References


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Two renorming constructions related to a question of Anselone

by

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To Professors S. Mazur and W. Orlicz on the fortieth anniversary of their scientific research

INTRODUCTION

Let \(X\) denote a normed linear space and \(X^*\) its conjugate space. For any point \(x\) of \(X\) let \(x^*\) denote the set of all points of \(X^*\) conjugate to \(x^*\); that is, \(y \in x^*\) if and only if \(y \cdot x^* + \langle y, x \rangle = \|x\|^2\), and \(\langle x, y \rangle = \|x\|^2\).

Let us say that \(X\) has the A-property provided that for each compact subset \(T\) of \(X\), the restriction of \(e\) to \(T\) admits a selection with totally bounded range; that is, there is a function \(s\) on \(T\) to \(X^*\) such that \(s(t) \in T\) for all \(t \in T\) and the set \(s(T)\) is totally bounded. This property was introduced by Anselone [1] in studying the total boundedness of sets of linear operators into \(X\). Clearly, every finite-dimensional \(X\) has the A-property. Anselone [1] noted that \(X\) has the A-property if \(X^*\) is uniformly rotund and asked whether all normed spaces have the A-property.

Here the question is resolved with the aid of an adaptation of a construction of Mazur and Sternbach [4] by showing that

Every infinite-dimensional Banach space can be renormed so as to lack the A-property.

On the other hand, the following problem is unsettled:

Can every Banach space (or at least every separable one) be renormed so as to have the A-property?

When \(X\) is complete the closure of any totally bounded subset of \(X\) is compact. For the A-property it then suffices to assume that the function \(e\) is single-valued and continuous or, equivalently, that the unit sphere \(S = \{x : \|x\| = 1\}\) is Fréchet-smooth at each point. This is weaker than uniform rotundity of \(X^*\), which is equivalent to uniform Fréchet-smooth-