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Ideals in group algebras *

by

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Throughout this paper G shall denote a locally compact abelian group. The ideal structure of $L_1(G)$ is still not fully known. For example, at a recent international symposium on functional analysis held at Sopot, Poland, the following questions were asked: (i) are there maximal non-closed ideals in $L_1(G)$ and (ii) what type of prime ideals are in $L_1(G)$? The purpose of this paper is to answer the afore mentioned questions. The crux of the matter lies in the following theorem on Banach algebras:

THEOREM 1. *Let R be a commutative Banach algebra with bounded approximate identity and continuous involution. Then every maximal ideal of R is regular and, consequently, closed.*

Proof. In as much as R has non-regular maximal ideals if, and only if $R^2 \subsetneqq R$ (cf. [3], p. 87-88), we shall assume $R^2 \subset M \subset R$, M a non-regular maximal ideal. Hence, there exists $x \in R - R^2$ such that $M + \{ax\} = R$ (cf. [3]). Define f such that $f(m) = 0$, $m \in M$, and $f(ax) = a$. f is a linear functional and, in fact, is positive since $f(x^* x) = 0$, for $x^* x \in M$ for every x. Varopoulos [5] has proved that f is continuous under the hypothesis on R. Hence, if $\{e_a\}$ denotes the approximate identity, then $f(x) = \lim_a f(e_a x) = 0$ since $e_a x \in M$; i.e. $f \equiv 0$. This contradiction establishes Theorem 1.

COROLLARY 1. *Every maximal ideal of $L_1(G)$ is regular and, therefore, closed.*

Proof. $L_1(G)$ has the properties described above for the ring R.

Remark. We could have proved the Corollary by using a result of Hewitts [2] which shows $(L_1(G))^2 = L_1(G)$ and applying the results cited in [3] (cf. [3], p. 96).

We shall now study the prime ideals of $L_1(G)$. Corollary 1 allows us to use spectral synthesis methods in such a study.

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LEMMA 1. *If I is an ideal in $L_1(G)$ such that I is contained in exactly one maximal ideal, say M, then $\bar{I} = M$.*

Proof. Suppose $\hat{G}$ denotes the dual group of G and γ the character corresponding to M. Hence, $\bar{I}$ is annihilated by γ and by hypothesis no other character annihilates $\bar{I}$. Hence $a(\bar{I}) = \{\varphi \mid \varphi \in L_\infty(G)\colon \bar{I}\cdot\varphi = 0\}$ consists of scalar multiples of γ (cf. [4], p. 185). Hence, $a(\bar{I}) = a(M)$ and, therefore, $\bar{I} = M$.

LEMMA 2. *If a prime ideal I of $L_1(G)$ is contained in a maximal ideal, then I is contained in only one maximal ideal.*

Proof. Suppose M_i $(i = 1, 2)$ are maximal ideals such that $I \subset M_i$ $(i = 1, 2)$. Let $\gamma_i \in \hat{G}$ correspond to M_i, W_i be an open neighborhood of γ_i in $\hat{G}$ with compact closure such that $\gamma_j \notin W_i$ for $i \neq j$, and $W_1 \cap W_2 = \emptyset$ There exist $f_i \in L_1(G)$ such that the Gelfond transform $\hat{f}_i$ has the property that $\hat{f}_i(\gamma_i) = 1$ and $\hat{f}_i(\gamma) = 0$ for $\gamma \notin W_i$ (cf. [4], p. 49). Therefore, $\hat{f}_1\hat{f}_2 = 0$ so that $f_1\cdot f_2 = 0$, where $f_1\cdot f_2$ denotes convolution multiplication. Hence $f_1\cdot f_2 \in I$. Since $f_1 \notin M_1$ and $f_2 \notin M_2$, this contradicts $I \subset M_1$ and $I \subset M_2$.

LEMMA 3. *If I is an ideal such that I is contained in no maximal ideal, then $\bar{I} = L_1(G)$.*

Proof. If $\bar{I} \subsetneqq L_1(G)$, then by the general Tauberian theorem, $\bar{I}$ would be contained in at least one maximal regular ideal.

LEMMA 4. *Let I be an ideal in $L_1(G)$ such that I is contained in no maximal ideal. If M is a maximal ideal of $L_1(G)$ and $J = I \cap M$, then $\bar{J} = M$.*

Proof. Suppose $\bar{J} \subsetneqq M$, $h \in M$, and $\varphi \in L_\infty$ such that $\int h\varphi \neq 0$ and $\int k\varphi = 0$ for $k \in J$, where the integration is with respect to Haar measure. Hence, $\int (f\cdot h)\varphi = 0$ for $f \in I$ where $f\cdot h$ denotes convolution multiplication, so by Lemma 3, $\int (f\cdot h)\varphi = 0$ for $f \in L_1(G)$. If $\{e_a\}$ denotes a bounded approximate identity for $L_1(G)$, then $\int h\varphi = \lim_a \int (e_a\cdot h)\varphi = 0$.

THEOREM 2. *If I is a proper prime ideal in $L_1(G)$, then I is a maximal ideal if, and only if, I is closed.*

Proof. In view of Corollary 1, if I is maximal, I is closed. Suppose now I is prime and closed. Hence, if I is in no maximal ideal, then by Lemma 3, $I = L_1(G)$. Therefore, there exists a maximal ideal M containing I and by Lemmas 2 and 1, $I = M$.

We shall continue to denote the dual of G by $\hat{G}$.

THEOREM 3. *$L_1(G)$ contains a non-closed prime ideal if, and only if, $\hat{G}$ contains an infinite set.*

Proof. Suppose $\hat{G}$ is not compact. Set $I_0 = \{f \mid f \in L_1(G)$ and $\hat{f}$ has compact support on $\hat{G}\}$. I_0 is an ideal in $L_1(G)$. Pick $h \notin I_0$ and set $H = \{h, h^2, \ldots\}$, where h^n is h convoluted with h^{n-1}. H is multiplicative (a semigroup in $L_1(G)$) and $I_0 \cap H = \emptyset$. Hence, there is an ideal I such that $I_0 \subset I$, $I \cap H = \emptyset$, I is maximal with respect to containing I_0 and missing H. Hence, I is prime (cf. [1], p. 6). Suppose M_1 is a maximal ideal in $L_1(G)$ corresponding to $\gamma_1 \in \hat{G}$. Construct f_1 and W_1 as in the proof of Lemma 2, so that $f_1 \in I_0 \subset I$ and $f_1 \notin M_1$. Hence I is in no maximal ideal and thus by Lemma 3 is not closed.

Suppose now $\hat{G}$ is compact. Pick $\gamma_n \in \hat{G}$ $(i = 1, 2, \ldots)$ and $\gamma_0 \in \hat{G}$ such that $\gamma_0 \neq \gamma_n$ $(n = 1, 2, \ldots)$ and $\{\gamma_n\}_n$ has γ_0 as a cluster point. Let $\gamma_n \in W_n \subset \hat{G}$ such that W_n is an open set with compact closure and $\gamma_0 \notin W_n$. Let $h_n \in L_1(G)$ such that $0 \leqslant \hat{h}_n \leqslant 1$, $\hat{h}_n(\gamma_n) = 1$, $\hat{h}_n = 0$ off W_n (cf. [4], p. 49). Set $h = \Sigma(2^{-n}[1+\|h_n\|])^{-1}h_n$, $H = \{h, h^2, \ldots\}$, $h^n = h\cdot h,^{n-1}$ and $I_0 = \{f \mid f \in L_1(G)$ and $\hat{f}$ vanishes on some neighborhood of $\gamma_0\}$. We now obtain a prime ideal I as before and note $I \subset M_0$, where M_0 is the maximal ideal corresponding to the character γ_0. But $h \in \bar{I}$ and $h \in M_0$ so by Lemma 2 and Theorem 2, I is not closed.

The only if is of course self evident. If $\hat{G}$ is finite, G is finite and $L_1(G)$ is algebraically isomorphic to $C(G)$.

THEOREM 4. *Each prime ideal of $L_1(G)$ is contained in a unique maximal ideal if, and only if, G is a discrete group.*

Proof. If G is discrete, then $L_1(G)$ has an identity; so if I is prime ideal, then I is contained in a maximal ideal and, by Lemma 2, the maximal ideal is uniquely determined by I. To prove the only if part, suppose G is not discrete. Then $\hat{G}$ is not compact and the prime ideal constructed in the first part of the proof Theorem 3 is contained in no maximal ideal.

COROLLARY 4. *Each prime ideal in $L_1(G)$ is maximal if, and only if, G is a finite group.*

Hence, we have three possible types of prime ideals in $L_1(G)$:

(1) closed prime ideals; i.e., maximal regular;

(2) non-closed prime ideals having the property that each such ideal is contained in a unique maximal; and

(3) non-closed prime ideals I such that $\bar{I} = L_1(G)$.

For finite groups only the first type exists. For infinite discrete groups the first two types co-exist. For infinite non-discrete groups all three types co-exist.

Added in proof. The fact that $(L_1(G))^2 = L_1(G)$ was first proved by P. J. Cohen (cf. [2] for a reference to Cohen's paper). Hence, Corollary 1 follows from Cohen's result and the results referred to in [3].

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Products of generalized functions

by

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In this paper we will consider products of the generalised functions $(x\pm i0)^n$ which are defined for integer values of n by

$$(x\pm i0)^n = x^n$$

for $n = 0, 1, 2, \ldots$ and

$$(x\pm i0)^{-n} = x^{-n} \pm \frac{i\pi(-1)^n}{(n-1)!}\,\delta^{(n-1)}$$

for $n = 1, 2, \ldots$, see [1]. It follows immediately that

$$\frac{d}{dx}(x\pm i0)^n = n(x\pm i0)^{n-1}$$

for $n = 0, \pm 1, \pm 2, \ldots$

First of all if $r, n \geqslant 0$ we obviously have

$$(x\pm i0)^n (x\pm i0)^r = (x\pm i0)^{n+r}$$

and we have

$$\begin{aligned}(x\pm i0)^n (x\pm i0)^{-r} &= x^n\left\{x^{-r} \pm \frac{i\pi(-1)^r}{(r-1)!}\,\delta^{(r-1)}\right\}\\ &= x^{n-r} \pm \frac{i\pi(-1)^r}{(r-1)!}\,x^n\,\delta^{(r-1)}\\ &= \begin{cases} x^{n-r} & \text{for } n \geqslant r\\ x^{n-r} \pm \dfrac{i\pi(-1)^r}{(r-n-1)!}\,\delta^{(r-n-1)} & \text{for } r > n\end{cases}\\ &= (x\pm i0)^{n-r}.\end{aligned}$$

Now let us consider the square of $(x\pm i0)^{-1}$. Formally, we have

$$(x\pm i0)^{-1}(x\pm i0)^{-1} = (x^{-1}\mp i\pi\delta)^2 = \{(x^{-1})^2 - \pi^2\delta^2\} \mp 2i\pi\delta x^{-1}.$$